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An explicit formula for the zeros of the Rankin-Selberg L-function

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Abstract

In this report, we describe one explicit formula for the zeros of the Rankin-Selberg L-function by using the projection of the C^{∞}-automorphic forms **[Noda, (Kodai. Math. J. 2008)]**. The projection was introduced by **[Sturm (Duke Math. J. 1981)]** in the study of the special values of automorphic L-functions. Combining the idea of **[Zagier (Springer, 1981, Proposition 3)]** and the integral transformation of the confluent hypergeometric function, we derive an explicit formula which correlates the zeros of the zeta-function and the Hecke eigenvalues. The main theorem contains the case of the symmetric square L-function, that first appeared in author's previous paper **[Noda, (Acta. Arith. 1995)]**.

1 Rankin-Selberg L-function

Let k and l ($k \leqq l$) be positive even integers and S_k (resp. S_l) be the space of cusp forms of weight k (resp. l) on $SL_2(\mathbb{Z})$. Let $f(z) \in S_k$ and $g(z) \in S_l$ be normalized Hecke eigenforms with the Fourier expansions $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ and $g(z) = \sum_{n=1}^{\infty} b(n)e^{2\pi inz}$. For each prime p, we take α_p and β_p such that $\alpha_p + \beta_p = a(p)$ and $\alpha_p\beta_p = p^{k-1}$, and define

$$M_p(f) = \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}.$$

The Rankin-Selberg L-function attached to $f(z)$ and $g(z)$ is defined by

$$L(s, f\otimes g) = \prod_{p\,:\,\text{prime}} \det\left(I_4 - M_p(f)\otimes M_p(g)p^{-s}\right)^{-1}.$$

Here the product is taken over all rational primes, and I_n is the unit matrix of size n.

2 Fundamental properties

1. **Dirichlet series**

$$L(s, f \otimes g) = \zeta(2s+2-k-l) \sum_{n=1}^{\infty} a(n)b(n)n^{-s}$$

2. **Inner product (Rankin, Selberg)**

$$L(s, f \otimes g)\zeta(2s+2-k-l)^{-1} = \frac{(4\pi)^s}{\Gamma(s)} \int_{SL_2(\mathbf{Z})\backslash H} f(z)\overline{g(z)}E_{l-k}(z, s-l+1)y^{l-2}dxdy$$

3. **Analytic continuation**

For $l > k$, $\Gamma(s)\Gamma(s-k+1)L(s, f \otimes g)$ is an **entire function** in s. The functional equation is also known.

4. **Others**

(1 The **critical strip** is $(k+l-2)/2 < \mathrm{Re}(s) < (k+l)/2$.

(2) For $l = k$, $(\Gamma\text{-}factor)\zeta(s-k+1)^{-1}L(s, f \otimes f)$ is an **entire function** in s (**Shimura, Zagier**).

3 Statement of the results

Theorem 1 *Let k and l be positive even integers such that k, l = 12, 16, 18, 20, 22, and 26 respectively. Suppose $k \leqq l$. Let $\Delta_k(z) = \sum_{n=1}^{\infty} \tau_k(n)e^{2\pi i n z} \in S_k$ be the unique normalized Hecke eigenform, and let ρ be a zero of $L(s-1+(k+l)/2, \Delta_k \otimes \Delta_l)$ in the critical strip $0 < Re(s) < 1$. Assume that $\zeta(2\rho) \neq 0$. Then for each positive integer n,*

$$-\tau_k(n)\left\{\frac{n^{1-2\rho}(-1)^{\frac{l-k}{2}}\zeta(2\rho)}{(2\pi)^{2\rho}\Gamma(-\rho+\frac{k+l}{2})} + \frac{\zeta(2\rho-1)\Gamma(2\rho-1)}{\Gamma(\rho-1+\frac{k+l}{2})\Gamma(\rho+\frac{k-l}{2})\Gamma(\rho-\frac{k-l}{2})}\right\}$$

$$= \frac{1}{\Gamma(k)\Gamma(\rho-\frac{k-l}{2})} \sum_{m=1}^{n-1} \tau_k(m)\sigma_{1-2\rho}(n-m)F\left(1-\rho+\tfrac{k-l}{2}, -\rho+\tfrac{k+l}{2}; k; \tfrac{m}{n}\right)$$

$$+\frac{1}{\Gamma(l)\Gamma(\rho+\frac{k-l}{2})} \sum_{m=n+1}^{\infty} \left(\frac{n}{m}\right)^{-\rho+\frac{k+l}{2}} \tau_k(m)\sigma_{1-2\rho}(m-n)$$

$$\times F\left(1-\rho-\tfrac{k-l}{2}, -\rho+\tfrac{k+l}{2}; l; \tfrac{n}{m}\right).$$

4 Corollary and Remarks

Corollary 1 *Let $T(n,\rho;k;l)$ be the right-hand side of the equality in Theorem 1. Then, the following equivalence holds:*

$$\mathrm{Re}(\rho)=\frac{1}{2} \iff T(n,\rho;k;l) \asymp \tau_k(n) \qquad (as\ n\to\infty).$$

Remark 1. By Shimura (1976, 77), it is known that the periods of the modular form for $L(s,f\otimes g)$ are dominated by the cusp form of large weight, whereas our theorem is expressed by using the Fourier coefficients of the cusp form of small weight.

Remark 2. The Theorem 1 includes the formula for the symmetric square L-function $L_2(s,f)$ and the Riemann zeta function $\zeta(s)$, that first appeared in author's previous paper [5].

5 Eisenstein series

Let $k \geqq 0$ be an even integer, Let i be the imaginary unit, s be a complex number whose real part σ (sigma) and imaginary part t. As usual, H is the upper half plane. The non-holomorphic Eisenstein series for $SL_2(\mathbb{Z})$ is defined by

$$E_k(z,s)=y^s\sum_{\{c,d\}}(cz+d)^{-k}|cz+d|^{-2s}. \qquad (1)$$

Here z is a point of H, s is a complex variable and the summation is taken over $\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)$, a complete system of representation of $\left\{\left(\begin{smallmatrix} * & * \\ 0 & *\end{smallmatrix}\right)\in SL_2(\mathbb{Z})\right\}\backslash SL_2(\mathbb{Z})$. The right-hand side of (1) converges absolutely and locally uniformly on $\{(z,s)|\ z\in H,\ \mathrm{Re}(s)>1-\frac{k}{2}\}$, and $E_k(z,s)$ has a meromorphic continuation to the whole s-plane. It is also well-known the functional equation:

$$\begin{aligned}&\pi^{-s}\Gamma(s)\zeta(2s)E_k(z,s)\\ =\ &\pi^{-1+s+k}\Gamma(1-s-k)\zeta(2-2s-2k)E_k(z,1-s-k).\end{aligned}$$

6 Projection to the space of cusp forms

The C^∞-automorphic forms of bounded growth are introduced by Sturm in the study of zeta-functions of Rankin type. The function F is called a **C^∞-modular form** of weight k, if F satisfies the following conditions:

(A.1) F is a C^∞-function from H to $\mathbb{C}$,

(A.2) $F((az+b)(cz+d)^{-1}) = (cz+d)^k F(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$.

We denote by $\mathfrak{M}_k$ the set of all C^∞-modular forms of weight k. The function $F \in \mathfrak{M}_k$ is called **of bounded growth** if for every $\varepsilon > 0$

$$\int_0^1 \int_0^\infty |F(z)| y^{k-2} e^{-\varepsilon y} dy dx < \infty.$$

Let k be a positive even integer and S_k be the space of cusp forms of weight k on $SL_2(\mathbb{Z})$. For $F \in \mathfrak{M}_k$ and $f \in S_k$, we define the **Petersson inner product** as usual

$$(f,F) = \int_{SL_2(\mathbb{Z})\backslash H} f(z)\overline{F(z)} y^{k-2} dx dy.$$

The Poincaré series are defined by

$$P_m(z) = \sum_{\{c,d\}} \mathrm{e}\left(m \cdot \frac{az+b}{cz+d}\right)(cz+d)^{-k}$$

for $k \geqq 4$, $m \in \mathbb{Z}_{\geqq 0}$ and $z = x+iy \in H$. Here the summation is taken over as in the definition of the Eisenstein series. In 1981, Sturm constructed a certain kernel function by using Poincaré series, and showed the following theorem:

Theorem 2 *(Sturm 1981) Assume that $k > 2$. Let $F \in \mathfrak{M}_k$ be of bounded growth with the Fourier expansion $F(z) = \sum_{n=-\infty}^{\infty} a(n,y) e^{2\pi i n x}$. Let*

$$c(n) = (2\pi n)^{k-1} \Gamma(k-1)^{-1} \int_0^\infty a(n,y) e^{-2\pi n y} y^{k-2} dy.$$

Then $h(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z} \in S_k$ and

$$(g,F) = (g,h)$$

for all $g \in S_k$.

7 Fourier expansion of the Eisenstein series

Let $e(u) := \exp(2\pi i u)$ for $u \in \mathbb{C}$. For $z \in H$ and $\mathrm{Re}(s) > 1 - \frac{k}{2}$, $E_k(z,s)$ has an expansion:

$$E_k(z,s) = y^s + a_0(s) y^{1-k-s} + \frac{y^s}{\zeta(k+2s)} \sum_{m \neq 0} \sigma_{1-k-2s}(m) a_m(y,s) e(mx), \quad (2)$$

where

$$\begin{aligned} a_0(s) &= (-1)^{\frac{k}{2}} 2\pi \cdot 2^{1-k-2s} \frac{\zeta(k+2s-1)}{\zeta(k+2s)} \frac{\Gamma(k+2s-1)}{\Gamma(s)\Gamma(k+s)}, \\ \sigma_s(m) &= \sum_{d|m,\, d>0} d^s, \end{aligned}$$

$$a_m(y,s) = \int_{-\infty}^{\infty} e(-mu)(u+iy)^{-k} |u+iy|^{-2s} du. \quad (3)$$

and

$$a_m(y,s) = \begin{cases} \dfrac{(-1)^{\frac{k}{2}} (2\pi)^{k+2s} m^{k+2s-1}}{\Gamma(k+s)} e^{-2\pi y m} \Psi(s, k+2s; 4\pi y m) & (m > 0), \\ \dfrac{(-1)^{\frac{k}{2}} (2\pi)^{k+2s} |m|^{k+2s-1}}{\Gamma(s)} e^{-2\pi y |m|} \Psi(k+s, k+2s; 4\pi y |m|) & (m < 0). \end{cases}$$

Here $\Psi(\alpha, \beta; z)$ is the confluent hypergeometric function defined for $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(\alpha) > 0$ by the following

$$\Psi(\alpha, \beta; z) := \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-zu} u^{\alpha-1} (1+u)^{\beta-\alpha-1} du.$$

We call the first two terms of (2) are the constant term of $E(z,s)$. The integral (3) is entire function in s and of exponential decay in $y|m|$. This fact gives the meromorphical continuation and the y-aspect of $E(z,s)$ when y tends to ∞. Namely, there exist positive constants A_1 and A_2 depending only on k and s such that

$$|E_k(z,s)| \leqq A_1 y^{\mathrm{Re}(s)} + A_2 y^{1-\mathrm{Re}(s)-k} \qquad (y \to \infty),$$

except on the poles. Further, the modularity for $y^{\frac{k}{2}} E_k(z,s)$ gives the following:

Proposition 1 *Assume $E_k(z,s)$ is holomorphic at $s \in \mathbb{C}$. Then, there exist positive constants A_1 and A_2 depending only on k and s such that*

$$|E_k(z,s)| \leqq \begin{cases} A_1 (y^{-\mathrm{Re}(s)-k} + y^{\mathrm{Re}(s)}) & (\mathrm{Re}(s) > \frac{1-k}{2}) \\ A_2 (y^{-1+\mathrm{Re}(s)} + y^{1-\mathrm{Re}(s)-k}) & (\mathrm{Re}(s) \leqq \frac{1-k}{2}) \end{cases}$$

for every $y > 0$.

8 Proof of Theorem 1

By Proposition 1, it is easy to see the Eisenstein series $E_k(z,s)$ is a C^∞-modular form of weight k, and of bounded growth for $2-k < \mathrm{Re}(s) < -1$ except on the poles. Therefore

Lemma 1 *For $f(z) \in S_k$ and $s \in \mathbb{C}$ in $k/2 - l + 2 < \mathrm{Re}(s) < k/2 - 1$, $f(z)E_{l-k}(z,s)$ is a C^∞-***modular form** *of weight l and* **of bounded growth**.

We have also the following;

Lemma 2 *Let $f(z) \in S_k$ and $g(z) \in S_l$ be normalized Hecke eigenforms. Let ρ be a zero of $L(s-1+(k+l)/2, f \otimes g)$ in the critical strip $0 < Re(s) < 1$. Assume $\zeta(2\rho) \neq 0$. Then*

$$\langle f(z)E_{l-k}(z,\ \rho + \tfrac{k-l}{2}),\ g(z)\rangle = 0.$$

To evaluate the **Laplace-Mellin transform** of the Fourier coefficient of the product of the Eisenstein series and the Hecke eigenform, we use the following proposition.

Proposition 2 *The integral transform*

$$\int_0^\infty \Psi(a,c;y)y^{b-1}e^{-uy}dy = \frac{\Gamma(b)\Gamma(b-c+1)}{\Gamma(a+b-c+1)}u^{-b} \times F\left(a,b;a+b-c+1;1-\tfrac{1}{u}\right)$$

is valid when $\mathrm{Re}(u) > 0$ *and* $\mathrm{Re}(b-a) - M - N > 0$. *Here M and N are non-negative integers so as* $\mathrm{Re}(a+M) > 0$ *and* $\mathrm{Re}(c-a) \leqq N+1$ *respectively.*

Proof of Theorem 1 Let $\Delta_k(z)$ be the unique normalized Hecke eigenform for $k = 12, 16, 18, 20, 22$, and 26. We write the Fourier expansion as follows:

$$\Delta_k(z) \cdot E_{l-k}(z,s) = \sum_{n=-\infty}^{\infty} b(n,y,s)e^{2\pi inx}.$$

Using the notation $a_0(s)$ and $a_n(y,s)$ defined by (2) and (3),

$$b(n,y,s) = \{y^s + a_0(s)y^{1-l+k-s}\}\tau_k(n)e^{-2\pi ny} + \frac{y^s}{\zeta(2s+l-k)} \sum_{\substack{m=1,\\ m\neq n}}^{\infty} \tau_k(m)\sigma_{1-l+k-2s}(n-m)a_{n-m}(y,s)e^{-2\pi my}.$$

Here we regard $\tau_k(m)$ as 0 if $m \leqq 0$.

By Lemma 1 and Theorem 2, there exists $h(z,s) = \sum_{n=1}^{\infty} c(n,s)e^{2\pi i n z} \in S_l$ such that $\langle f(z) \cdot E_{l-k}(z,s),\ g(z)\rangle = \langle h(z,s),\ g(z)\rangle$ for all $g(z) \in S_l$ in the region $k/2 - l + 2 < \mathrm{Re}(s) < k/2 - 1$. The Fourier coefficients of $h(z,s)$ are given by

$$c(n,s) = (2\pi n)^{l-1}\Gamma(l-1)^{-1}\int_0^{\infty} b(n,y,s)e^{-2\pi n y}y^{l-2}dy,$$

for $n > 0$. We put $\gamma(n,l) = (2\pi n)^{l-1}\Gamma(l-1)^{-1}$. Then we have

$$\begin{aligned} c(n,s) = {} & \frac{\gamma(n,l)}{\zeta(2s+l-k)} \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \tau_k(m)\sigma_{1-l+k-2s}(n-m) \\ & \times \int_0^{\infty} a_{n-m}(y,s)y^{s+l-2}e^{-2\pi(m+n)y}dy \\ & + \text{(transformed constant terms)}. \end{aligned}$$

Combining Lemma 2 and Proposition 2, we obtain the equation in the Theorem 1. □

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