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Algebraic independence results for Ramanujan q-series

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1 Introduction

In 1916, Ramanujan [8] defined the function

$$S_{2j+1}(x)=\frac{1}{2}\zeta(-2j-1)+\sum_{n=1}^{\infty}\frac{n^{2j+1}x^{n}}{1-x^{n}} \quad (j=0,1,2,\ldots),$$

where $\zeta(s)$ is the Riemann zeta function, in particular

$$P(x)=-24S_{1}(x), \quad Q(x)=240S_{3}(x), \quad R(x)=-540S_{5}(x),$$

and proved various formulas for these functions.

Nesterenko [7] proved that, for an algebraic number x with $0<|x|<1$, the three numbers $P(x)$, $Q(x)$, $R(x)$ are algebraically independent.

In this paper we study, for a given algebraic number x with $0<|x|<1$, the algebraic independence of the numbers $S_{2j+1}(x)$ $(j=0,1,2,\ldots)$, or equivalently that of the numbers

$$A_{2j+1}=A_{2j+1}(q)=\sum_{n=1}^{\infty}\frac{n^{2j+1}q^{2n}}{1-q^{2n}} \quad (j=0,1,2,\ldots) \tag{1}$$

for $q^{2}=x$ (using the symbol in [10]). We remark that the q-series identity

$$A_{7}(q)=A_{3}(q)+120A_{3}(q)^{2} \tag{2}$$

follows from Table I in [8]. Our result is stated as follows:

Theorem 1. *The q-series $A_{2i+1}(q)$ and $A_{2j+1}(q)$ with $1\leq i<j$ are algebraically dependent over $\overline{\mathbb{Q}}(q)$ if and only if $(i,j)=(1,3)$.*

Theorem 2. *Let q be an algebraic number with $0<|q|<1$. Then the three numbers $A_{1}(q)$, $A_{2i+1}(q)$ and $A_{2j+1}(q)$ with $1\leq i<j$ and $(i,j)\neq(1,3)$ are algebraically independent.*

As an immediate corollary of Theorem 2 we have

Corollary 1. *The q-series $A_{1}(q)$, $A_{2i+1}(q)$ and $A_{2j+1}(q)$ with $1\leq i<j$ and $(i,j)\neq(1,3)$ are algebraically independent over $\overline{\mathbb{Q}}(q)$.*

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It is easy to see that, for any positive integers i, j, l, the q-series $A_{2i+1}(q)$, $A_{2j+1}(q)$, $A_{2l+1}(q)$ are algebraically dependent over $\mathbb{Q}$ (cf. Lemmas 2 and 3 in Section 2). Our proof of the theorem depends on three basic tools; namely, a corollary of Nesterenko's theorem (see Lemma 1), expressions of $A_{2j+1}(q)$ in terms of K/π, E/π and k given by Ramanujan [8], Zucker [10] and the authors [2], where K and E are the complete elliptic integrals of the first and second kind with the modulus k (see Lemmas 2 and 3), and an algebraic independence criterion (see Lemma 6). Our method of this paper can be applied to certain sequences of q-series studied by Zucker [10] and the authors [2], [3], [4].

2 Lemmas

Let K and E denote the complete elliptic integrals of the first and second kind

$$K = K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad E = E(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt,$$

with the modulus k such that $k^2 \in \mathbb{C} \setminus (\{0\} \cup [1, +\infty))$. The branch of each integrand is chosen so that it tends to 1 as $t \to 0$. Moreover set $K' = K(k')$, $k^2 + (k')^2 = 1$. For any $q \in \mathbb{C}$ with $0 < |q| < 1$, choose $K'(k)/K(k)$ so that $q = e^{-\pi K'/K}$. Then by Nesterenko's theorem ([7], see also [2, §3]) we have

Lemma 1. *If $q \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$, then k, K/π, and E/π are algebraically independent.*

The Jacobian elliptic function $w = \operatorname{sn} z = \operatorname{sn}(z, k)$ may be defined by

$$z = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

and we write

$$\operatorname{ns} z = \operatorname{ns}(z,k) = \frac{1}{\operatorname{sn}(z,k)}, \quad \operatorname{cs} z = \operatorname{cs}(z,k) = \frac{\operatorname{cn}(z,k)}{\operatorname{sn}(z,k)}, \quad \operatorname{cn}(z,k) = \sqrt{1-\operatorname{sn}^2(z,k)}.$$

Let a_j and $c_j = c_j(k)$ ($j \ge 0$) denote the coefficients of the series expansions

$$\operatorname{cosec}^2 z = z^{-2} + \sum_{j=0}^{\infty} a_j z^{2j}, \qquad \operatorname{ns}^2(z,k) = z^{-2} + \sum_{j=0}^{\infty} c_j(k) z^{2j}, \tag{3}$$

respectively; in particular

$$a_j := \frac{(-1)^j (2j+1) 2^{2j+2} B_{2j+2}}{(2j+2)!} \qquad (j \ge 0) \tag{4}$$

with the Bernoulli numbers $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, ... etc. The following evaluations of $A_{2j+1}(q)$ in terms of k, K/π and E/π were given by Ramanujan [8] for $j = 1, 2, \dots, 15$ and completed by Zucker [10].

Lemma 2. *For $q = e^{-\pi K'/K}$, the q-series $A_{2j+1}(q)$ is expressed in the form*

$$A_1(q) = \frac{1}{24} - \frac{1}{24}\left(\frac{2K}{\pi}\right)^2 \left(\frac{3E}{K} - 2 + k^2\right),$$

$$A_{2j+1}(q) = (-1)^j \frac{(2j)!}{2^{2j+3}} \left(a_j - \left(\frac{2K}{\pi}\right)^{2j+2} c_j(k)\right) \qquad (j \ge 1).$$

The coefficients of the series for $\operatorname{ns}^2(z,k)$ are determined recursively ([2, Lemma 2]):

Lemma 3. *The coefficients $c_j = c_j(k)$ are given by*

$$c_0 = \frac{1}{3}(1+k^2), \quad c_1 = \frac{1}{15}(1-k^2+k^4), \quad c_2 = \frac{1}{189}(1+k^2)(1-2k^2)(2-k^2),$$
$$(j-2)(2j+3)c_j = 3\sum_{i=1}^{j-2} c_i c_{j-i-1} \qquad (j \geq 3).$$

It follows from this lemma that $c_j = c_j(k) \in \mathbb{Q}[k^2]$ with $\deg c_j(k) = 2(j+1)$. Put

$$c_j(k) = \alpha_{j,0} + \alpha_{j,1}k^2 + \cdots + \alpha_{j,j+1}k^{2j+2} \qquad (j \geq 1). \tag{5}$$

It is easy to see that $z^2\mathrm{ns}^2(z,k)$ is analytic around $(z,k) = (0,0)$, and that $z^2\mathrm{ns}^2(z,k) = 1 + O(|z|^2 + |k|^2)$. The following lemma gives a detailed expression.

Lemma 4. *Around $(z,k) = (0,0)$ we have*

$$\mathrm{ns}^2(z,k) = \mathrm{cosec}^2 z + f(z)k^2 + g(z)k^4 + O(k^6),$$

with

$$f(z) = \frac{z}{4}(\cot z)'' + \frac{1}{2}(\cot z)' + \frac{1}{2} = \frac{1}{2} - \sum_{j=0}^{\infty} \frac{(-1)^{j+1}2^{2j}B_{2j+2}}{(2j)!} z^{2j},$$

$$g(z) = -\frac{z^2}{32}(\cot z)^{(3)} - \frac{3z}{64}(\cot z)'' + \frac{3}{32}(\cot z)' + \frac{1}{16} - \frac{1}{32}\cos 2z$$
$$= \frac{1}{16} + \sum_{j=0}^{\infty} \frac{(-1)^j 2^{2j-5}}{(2j)!}\left(2(4j-3)B_{2j+2} - 1\right) z^{2j}.$$

From (3), (4), (5) and Lemma 4 we deduce the following corollary.

Corollary 2. *For $j \geq 1$ we have*

$$\alpha_{j,0} = \frac{(-1)^j(2j+1)2^{2j+2}B_{2j+2}}{(2j+2)!},$$
$$\alpha_{j,1} = \frac{(-1)^{j+1}2^{2j}B_{2j+2}}{(2j)!} \quad (j \geq 1), \qquad \alpha_{0,1} = \frac{1}{3},$$
$$\alpha_{j,2} = \frac{(-1)^j 2^{2j-5}}{(2j)!}\left(2(4j-3)B_{2j+2} - 1\right) \quad (j \geq 1), \qquad \alpha_{0,2} = 0.$$

Lemma 5. *All Bernoulli numbers B_{2n} are distinct with the only exception $B_4 = B_8 = -1/30$.*

Lemma 6. (cf. [5]) *Let $x_1, \ldots, x_n \in \mathbb{C}$ be algebraically independent and let $y_j := U_j(x_1, \ldots, x_n)$, where $U_j(X_1, \ldots, X_n) \in \mathbb{Q}[X_1, \ldots, X_n]$ $(j = 1, \ldots, n)$. Assume that*

$$\det\left(\frac{\partial U_j}{\partial X_i}(x_1, \ldots, x_n)\right) \neq 0. \tag{6}$$

Then the numbers $y_1, \ldots, y_n$ are algebraically independent.

3 Sketch of the proof of Theorem 1

We assume that two q-series $A_{2i+1}(q)$ and $A_{2j+1}(q)$ with $1 \leq i < j$ are algebraically dependent over $\overline{\mathbb{Q}}(q)$ and deduce the case $(i,j) = (1,3)$. There exists an algebraic number q_0 with $0 < |q_0| < 1$ such that the two numbers $A_{2i+1}(q_0)$ and $A_{2j+1}(q_0)$ are algebraically dependent. Choose k so that $q_0 = e^{-\pi K'(k)/K(k)}$. We apply Lemma 6 with Lemma 1 and 2 by putting $x_1 = k$, $x_2 = K/\pi$, $y_\nu = U_\nu(x_1, x_2)$ $(\nu = 1, 2)$, where $U_1(X_1, X_2) = X_2^{2i+2}c_i(X_1)$, $U_2(X_1, X_2) = X_2^{2j+2}c_j(X_1)$. Then the vanishing of the determinant implies

$$(j+1)c_i'(k)c_j(k) - (i+1)c_j'(k)c_i(k) = 0$$

as a polynomial of k, or equivalently $c_j(k)^{i+1} = rc_i(k)^{j+1}$ for some $r \in \mathbb{Q} \setminus \{0\}$. Substituting (5) into both sides of the last identity and comparing the coefficients of k^2 and k^4, we get

$$\frac{2\alpha_{j,0}\alpha_{j,2} + i\alpha_{j,1}^2}{\alpha_{j,0}\alpha_{j,1}} = \frac{2\alpha_{i,0}\alpha_{i,2} + j\alpha_{i,1}^2}{\alpha_{i,0}\alpha_{i,1}}.$$

This with the explicit values of $\alpha_{i,j}$ given in Corollary 2 leads to

$$2(4j-3) - 1/B_{2j+2} + 8i(j+1) = 2(4i-3) - 1/B_{2i+2} + 8j(i+1),$$

that is $B_{2i+2} = B_{2j+2}$. By Lemma 5, we see that $(i,j) = (1,3)$. The 'if part' follows from the formula (2), and the proof is completed.

Similarly, we can prove Theorem 2, from which Corollary 1 follows.

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