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ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR BCF MODEL DESCRIBING CRYSTAL SURFACE GROWTH

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ABSTRACT. This paper treats the initial-boundary value problem for a nonlinear parabolic equation of forth order which was presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr [13] in order to describe the interesting phenomena of crystal surface growth under molecular beam epitaxy(MBE). First, we construct a dynamical system determined from the initial-boundary value problem of the model equation. Second, we study asymptotic behavior of solutions. Third, we investigate stability or instability of homogeneous stationary solution. Finally, we show some numerical results .

1. INTRODUCTION

We study the initial-boundary value problem for a nonlinear parabolic equation of fourth order

$$\begin{cases} \dfrac{\partial u}{\partial t} = -a\Delta^2 u - \mu\nabla\cdot\left(\dfrac{\nabla u}{1+|\nabla u|^2}\right) & \text{in} \quad \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial}{\partial n}\Delta u = 0 & \text{on} \quad \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega \end{cases} \tag{1.1}$$

in a two-dimensional bounded domain $\Omega \subset \mathbb{R}^2$. Such a problem was presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr [13] in order to describe the process growing of crystal surface by a mathematical model. Here, $u = u(x,t)$ denotes a displacement of height of surface from the standard level at a position $x \in \Omega$.

By physical experiments one can observe very interesting phenomena of crystal growth on the growing surface under the molecular beam epitaxy (MBE), see [27]. To understand their mechanisms Johnson et al. [13] presented the model given in (1.1) on the basis of the BCF theory due to Burton-Cabrera-Frank [4] (cf. also [12, 17, 19, 20]).

The term $-a\Delta^2 u$ in the equation of (1.1) denotes a surface diffusion which is caused by the difference of the chemical potential proportional to the curvature of the surface. Therefore the adatoms have tendency to migrate from the positions of a large curvature to those of a small one. The macroscopic representation of the surface diffusion by $-a\Delta^2 u$ is due to Mullins [15], where $a > 0$ is a surface diffusion constant. In the meantime, $-\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)$ denotes the effect of surface roughening. Such roughening is caused by Schwoebel barriers [7, 21] (cf. also [27]) which prevent adatoms from hopping from the upper terraces to lower ones. As a consequence, non-equilibrium uphill current is induced. The macroscopic representation of the roughening by $-\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)$ is formulated in the paper Johnson et al. [13], where $\mu > 0$ is a coefficient of surface roughening. Some numerical simulations for one or two-dimensional model of (1.1) were performed by the papers [12, 17, 19, 20].

This paper is devoted to studying (1.1) by mathematical analysis. On the first stage, our goal is to construct global solutions and furthermore a dynamical system determined from (1.1). First we shall show the local existence and uniqueness of solutions for initial functions $u_0 \in H^1(\Omega)$ by using the theory of abstract parabolic evolution equations (see [14, 23]). More precisely, we will apply the result due to [16] for semilinear abstract parabolic evolution equations. Second we shall obtain *a priori* estimates concerning the H^1-norm for local solutions to show the global existence. Owing to the techniques of abstract evolution equations, one can easily verify continuous dependence of the global solutions with respect to the initial functions. This shows that a continuous semigroup $S(t)$ is determined from the global solutions of (1.1) in the L^2-norm. We shall then be able to construct a dynamical system (see [2, 25]) in universal space $L^2(\Omega)$ the phase space of which is $H^1(\Omega)$.

In the section 6, we will handle (1.1) in the underlying space $L^2(\Omega)$. In the preceding paper [8], we have already constructed a global solution in $L^2(\Omega)$ for each initial function $u_0 \in H^1_m(\Omega)$, $H^1_m(\Omega)$ being a closed subspace of $H^1(\Omega)$ consisting of functions $u \in H^1(\Omega)$ with null mean, i.e., $|\Omega|^{-1}\int_\Omega u\,dx = 0$. And, by showing continuity of the global solutions with respect to the initial functions, we have constructed a dynamical system $(S(t), H^1_m(\Omega), L^2(\Omega))$ determined from (1.1) with the phase space $H^1_m(\Omega)$ in the universal space $L^2(\Omega)$. In this paper, we will proceed to investigate the structure of $(S(t), H^1_m(\Omega), L^2(\Omega))$. First, we shall construct exponential attractors. The notion of exponential attractor was presented by Eden et al. [6] as a new attractor set which is a positively invariant set of $S(t)$ with finite fractal dimension and attracts every trajectory at an exponential rate. The authors of the paper [6] presented also the squeezing property of semigroup $S(t)$ by which one can easily construct exponential attractors. We shall then show that our semigroup determined from (1.1) actually enjoy the squeezing property. Second, we shall present a Lyapunov function the value of which decreases monotonically along every trajectory of $(S(t), H^1_m(\Omega), L^2(\Omega))$. Finally, using this fact, we shall prove that the ω-limit set $\omega(u_0)$ of any initial value $u_0 \in H^1_m(\Omega)$ consists of equilibria of $S(t)$.

In the section 9, we are concerned with stability or instability of homogeneous stationary solution. Clearly, $\overline{u} = 0$ is a unique homogeneous stationary solution of (1.1) satisfying $m(\overline{u}) = 0$, namely, 0 is a unique homogeneous equilibrium of $(S(t), H^1_m(\Omega), L^2_m(\Omega))$. We will appeal to the linearized principle invented by Bavin-Vishik [2] and Temam [25] in the theory of infinite-dimensional dynamical system. Application of the principle to the dynamical systems determined from semilinear abstract parabolic evolution equations was described in [1], for this we will make a brief review in the previous sections. In fact, we shall prove that 0 is stable if the parameter μ is smaller than $a\lambda_1$, where $\lambda_1 > 0$ is the minimal eigenvalue of a realization of $-\Delta$ in $L^2_m(\Omega)$ under the homogeneous Neumann boundary conditions on $\partial\Omega$. To the contrary, 0 becomes unstable if the μ is larger than $a\lambda_1$ and the instability dimension is given by $\#\{\lambda_k;\ \mu > a\lambda_k\}$, where $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$ denote the eigenvalues of $-\Delta$ in $L^2_m(\Omega)$ under the Neumann boundary conditions. Using the instability dimension, we can give a lower dimension estimate for exponential attractors $\mathcal{M}$ of $(S(t), H^1_m(\Omega), L^2_m(\Omega))$. By definition, $\mathcal{M}$'s are finite-dimensional compact subsets of $L^2_m(\Omega)$.

In the last section, we will present some numerical results of (1.1). We investigate the long time behavior and the structure of stationary solution of (1.1). In this time, we examine the relation between the structure of stationary solution and the rising coefficient of surface roughening μ.

There may be several possibilities for choosing boundary conditions of u on $\partial\Omega$. In the present paper, we will take the homogeneous Neumann type boundary conditions. Since the equation is of forth order, we have to impose the Neumann conditions on Δu, too. These boundary conditions imply that, if $\int_\Omega u_0(x)dx = 0$, then $\int_\Omega u(x,t)dx = 0$ for every $0 < t < \infty$, i.e., the total mean of displacements is invariant in time. It is however possible to prove similar analytical results even for other types of boundary conditions like the homogeneous Dirichlet boundary conditions, periodic boundary conditions and so on.

Throughout the paper, Ω is a bounded domain of $\mathcal{C}^4$ class in $\mathbb{R}^2$. According to [11], the Poisson problem $-\Delta u = f$ in Ω under the homogeneous Neumann boundary conditions $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ enjoys the shift property that if $f \in H^2(\Omega)$, then $u \in H^4(\Omega)$.

2. Preliminary

We shall first recall the known results on semilinear evolution equations studied in [16]. Consider the initial value problem

$$\begin{cases} \dfrac{du}{dt} + Au = F(u), & 0 < t \le T, \\ u(0) = u_0 \end{cases} \tag{2.1}$$

in a Banach space X. Here, A is a closed linear operator of X the spectral set of which is contained in a sectorial domain $\Sigma = \{\lambda \in \mathbb{C}; \, |\arg \lambda| < \omega\}$ with some angle $0 < \omega < \frac{\pi}{2}$, and the resolvent satisfies the estimate

$$\|(\lambda - A)^{-1}\|_{\cdot(X)} \le \frac{M}{|\lambda| + 1}, \qquad \lambda \notin \Sigma \tag{2.2}$$

with some constant $M > 0$. Therefore, $-A$ generates an analytic semigroup e^{-tA} on X. U_0 is an initial value in $\mathcal{D}(A^\alpha)$ with the estimate

$$\|A^\alpha u_0\| \le R, \tag{2.3}$$

here α is some exponent such that $0 \le \alpha < 1$ and $R > 0$ is a constant. $F(\cdot)$ is a nonlinear mapping from $\mathcal{D}(A^\eta)$ to X with $\alpha \le \eta < 1$ and is assumed to satisfy a Lipschitz condition of the form

$$\begin{aligned} \|F(u) - F(v)\| \le{}& \varphi(\|A^\alpha u\| + \|A^\alpha v\|) \\ &\times [\|A^\eta(u - v)\| + (\|A^\eta u\| + \|A^\eta v\|)\|A^\alpha(u - v)\|], \qquad u, v \in \mathcal{D}(A^\eta), \end{aligned} \tag{2.4}$$

where $\varphi(\cdot)$ is some increasing continuous function. Then the following theorem is known.

Theorem 2.1 ([16, Theorem 3.1]). *Let $0 \le \alpha \le \eta < 1$ and let (2.2), (2.3) and (2.4) be satisfied. Then (1.1) possesses a unique local solution in the function space:*

$$\begin{cases} u \in \mathcal{C}([0, T_R]; \mathcal{D}(A^\alpha)) \cap \mathcal{C}^1((0, T_R]; X) \cap \mathcal{C}((0, T_R]; \mathcal{D}(A)), \\ t^{1-\alpha} u \in \mathcal{B}((0, T_R]; \mathcal{D}(A)), \end{cases}$$

where $T_R > 0$ being determined by R. Moreover, the estimate

$$t^{1-\alpha}\|Au(t)\| + t^{\eta-\alpha}\|A^\eta u(t)\| + \|A^\alpha u(t)\| \leq C_R, \qquad 0 < t \leq T_R$$

holds with some constant $C_R > 0$ determined by R alone.

We shall next list well-known results in the theory of function spaces and of linear operators. Let Ω be a bounded $\mathcal{C}^4$ domain in $\mathbb{R}^2$. For $0 \leq s \leq 4$, $H^s(\Omega)$ denotes the Sobolev space of order s, its norm being denoted by $\|\cdot\|_{H^s}$ (see [11, Chap. 1] and [26]). For $0 \leq s_0 \leq s \leq s_1 \leq 4$, $H^s(\Omega)$ coincides with the complex interpolation space $[H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta$, where $s = (1-\theta)s_0 + \theta s_1$, and the estimate

$$\|\cdot\|_{H^s} \leq C\|\cdot\|_{H^{s_0}}^{1-\theta}\|\cdot\|_{H^{s_1}}^{\theta} \tag{2.5}$$

holds. When $0 \leq s < 1$, $H^s(\Omega) \subset L^p(\Omega)$, where $\frac{1}{p} = \frac{1-s}{2}$, with continuous embedding

$$\|\cdot\|_{L^p} \leq C\|\cdot\|_{H^s}. \tag{2.6}$$

When $s = 1$, $H^1(\Omega) \subset L^q(\Omega)$ for any finite $2 \leq q < \infty$ with the estimate

$$\|\cdot\|_{L^p} \leq C\|\cdot\|_{H^1}^{1-\frac{p}{q}}\|\cdot\|_{L^p}^{\frac{p}{q}}, \tag{2.7}$$

where $1 \leq p < q < \infty$. When $s > 1$, $H^s(\Omega) \subset \mathcal{C}(\overline{\Omega})$ with continuous embedding

$$\|\cdot\|_{\cdot} \leq C\|\cdot\|_{H^s}. \tag{2.8}$$

Consider a sesquilinear form given by

$$a(u,v) = d\int_\Omega \nabla u \cdot \nabla \overline{v}\, dx + c\int_\Omega u\overline{v}\, dx, \qquad u,\ v \in H^1(\Omega)$$

on the space $H^1(\Omega)$, where $d > 0$ and $c > 0$ are positive constants. From this form we can define realization Λ of the Laplace operator $-d\Delta + c$ in $L^2(\Omega)$ under the homogeneous Neumann boundary conditions on $\partial\Omega$ (see [5, Chap. VI]). The realization $\Lambda \geq c$ is a positive definite self-adjoint operator of $L^2(\Omega)$ and its domain is characterized by

$$\mathcal{D}(\Lambda) = H^2_N(\Omega) = \{u \in H^2(\Omega);\ \tfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}. \tag{2.9}$$

For $0 \leq \theta \leq 1$, the fractional powers Λ^θ of Λ are defined and are also positive definite self-adjoint operators of $L^2(\Omega)$. As shown in [28], we can characterize for $0 \leq \theta \leq 1$, their domains in the form

$$\mathcal{D}(\Lambda^\theta) = \begin{cases} H^{2\theta}(\Omega), & \text{when } 0 \leq \theta < \frac{3}{4}, \\ H^{2\theta}_N(\Omega) = \{u \in H^{2\theta}(\Omega);\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}, & \text{when } \frac{3}{4} < \theta \leq 1. \end{cases} \tag{2.10}$$

In addition, it is verified that the following estimates

$$C_\theta^{-1}\|\Lambda^\theta \cdot\|_{L^2} \leq \|\cdot\|_{H^{2\theta}} \leq C_\theta\|\Lambda^\theta \cdot\|_{L^2}, \qquad 0 \leq \theta \leq 1,\ \theta \neq \tfrac{3}{4} \tag{2.11}$$

hold with some constants $C_\theta \geq 1$.

We remark that, even when $\theta = \frac{3}{4}$, it is true that $\mathcal{D}(\Lambda^{\frac{3}{4}}) \subset H^{\frac{3}{2}}(\Omega)$ continuously.

We shall finally consider realization of $-d\Delta$ in $L^2(\Omega)$ under the homogeneous Neumann boundary conditions. The operator $-d\Delta$ is a nonnegative self-adjoint operator of $L^2(\Omega)$ with the same domain $\mathcal{D}(-d\Delta) = H^2_N(\Omega)$ as Λ. Clearly, the constants functions are an eigenfunction of the eigenvalue 0 of $-d\Delta$. Consider the orthogonal complement of the space of constant functions, namely,

$$L^2_m(\Omega) = \{u \in L^2(\Omega);\ m(u) = 0\},$$

where $m(u)$ be the integral mean

$$m(u) = \frac{1}{|\Omega|}\int_\Omega u\,dx, \qquad u \in L^2(\Omega). \tag{2.12}$$

Then $-d\Delta$ is a self-adjoint operator of $L^2_m(\Omega)$ with domain $H^2_N(\Omega)\cap L^2_m(\Omega)$. On account of the Poincaré-Wirtinger inequality

$$\|u - m(u)\|_{L^2} \le C\|\nabla u\|_{L^2}, \qquad u \in H^1(\Omega)$$

(cf. [3, p. 194]), we verify that

$$(-d\Delta u, u) = d\|\nabla u\|^2_{L^2} \ge \delta\|u\|^2_{L^2}, \qquad u \in H^2_N(\Omega)\cap L^2_m(\Omega)$$

with some $\delta > 0$. This means that $-d\Delta$ is positive definite in $L^2_m(\Omega)$ with the estimate

$$\|-d\Delta u\|_{L^2} \ge \delta\|u\|_{L^2}, \qquad u \in H^2_N(\Omega)\cap L^2_m(\Omega). \tag{2.13}$$

3. Local solutions

We shall construct local solution to our problem (1.1) by handling it as an abstract equation of the form (2.1). The underlying space X is set as $X = L^2(\Omega)$.

The linear operator A is defined by $A = \Lambda^2$, where Λ is the realization of $-\sqrt{a}\Delta + 1$ in $L^2(\Omega)$ under the homogeneous Neumann boundary conditions, i.e., $d = \sqrt{a}$, $c = 1$. Clearly, $A \ge 1$ is also a positive definite self-adjoint operator of X. Consequently, A is a sectorial operator of X. In addition, we can verify the following properties.

Proposition 3.1. [8, Propositon 3.1] *For $0 \le \theta \le 1$, $\theta \ne \frac{3}{8}, \frac{7}{8}$, we have*

$$\begin{cases} \mathcal{D}(A^\theta) = H^{4\theta}(\Omega), & \text{when } 0 \le \theta < \frac{3}{8}, \\ \mathcal{D}(A^\theta) = H^{4\theta}_N(\Omega) = \{u \in H^{4\theta}(\Omega);\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}, & \text{when } \frac{3}{8} < \theta < \frac{7}{8}, \\ \mathcal{D}(A^\theta) = H^{4\theta}_{N^2}(\Omega) = \{u \in H^{4\theta}(\Omega);\ \frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\Delta u = 0 \text{ on } \partial\Omega\}, & \text{when } \frac{7}{8} < \theta \le 1. \end{cases} \tag{3.1}$$

Moreover,

$$D_\theta^{-1}\|A^\theta \cdot\|_{L^2} \le \|\cdot\|_{H^{4\theta}} \le D_\theta\|A^\theta \cdot\|_{L^2}, \qquad 0 \le \theta \le 1,\ \theta \ne \tfrac{3}{8}, \tfrac{7}{8} \tag{3.2}$$

with some constants $D_\theta \ge 1$.

We remark that, even when $\theta = \frac{3}{8}, \frac{7}{8}$, it is true that $\mathcal{D}(A^{\frac{3}{8}}) \subset H^{\frac{3}{2}}(\Omega)$ and $\mathcal{D}(A^{\frac{7}{8}}) \subset H^{\frac{7}{2}}(\Omega)$, respectively, with continuous embedding.

Fix two exponents α and η in such a way that $\alpha = \frac{1}{4}$ and $\eta = \frac{7}{8}$. In view of

$$-A = -(-\sqrt{a}\Delta + 1)^2 = -a\Delta^2 + 2\sqrt{a}\Delta - 1,$$

the nonlinear operator F is defined by

$$F(u) = -\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right) - 2\sqrt{a}\Delta u + u, \qquad u \in \mathcal{D}(A^{\frac{7}{8}}) \subset H^{\frac{7}{2}}(\Omega). \tag{3.3}$$

Then, we verify the following Lipschitz condition on F.

Proposition 3.2. [8, Proposition 3.2] *The operator F satisfies*

$$(3.4)\quad \|F(u)-F(v)\| \le C[\|A^{\frac{1}{2}}(u-v)\| + (\|A^{\frac{7}{8}}u\| + \|A^{\frac{7}{8}}v\|)\|A^{\frac{1}{4}}(u-v)\|], \qquad u,\, v \in \mathcal{D}(A^{\frac{7}{8}}).$$

As is obvious, (3.4) means that F fulfils (2.4) with $\alpha = \frac{1}{4}$ and $\eta = \frac{7}{8}$. Theorem 2.1 then provides the following local existence of solution.

Theorem 3.1. [8, Theorem 3.1] *For any $u_0 \in \mathcal{D}(A^{\frac{1}{4}}) = H^1(\Omega)$, there exists a unique solution to* (1.1) *in the function space:*

$$\begin{cases} u \in \mathcal{C}([0,T_0]; H^1(\Omega)) \cap \mathcal{C}^1((0,T_0]; L^2(\Omega)) \cap \mathcal{C}((0,T_0]; H^4_{N^2}(\Omega)), \\ t^{\frac{3}{4}} u \in \mathcal{B}((0,T_0]; H^4(\Omega)). \end{cases}$$

Here, $T_0 > 0$ is determined by the norm $\|u_0\|_{H^1}$ alone. Moreover,

$$(3.5)\qquad t^{\frac{3}{4}}\|u(t)\|_{H^4} + t^{\frac{5}{8}}\|u(t)\|_{H^{\frac{7}{2}}} + \|u(t)\|_{H^1} \le C_0, \qquad 0 < t \le T_0,$$

$C_0 > 0$ being determined by $\|u_0\|_{H^1}$ alone.

4. Global solutions

We shall establish *a priori* estimates for the local solutions.

Let $u_0 \in H^1(\Omega)$ and let u be any local solution of (1.1) on interval $[0, T_u]$ in the solution space:

$$(4.1)\qquad u \in \mathcal{C}([0,T_u]; H^1(\Omega)) \cap \mathcal{C}^1((0,T_u]; L^2(\Omega)) \cap \mathcal{C}((0,T_u]; H^4_{N^2}(\Omega)).$$

Proposition 4.1. [8, Proposition 4.1] *There exists a constant $C > 0$ independent of u_0 such that the estimate*

$$(4.2)\qquad \|u(t)\|_{H^1} \le C(\|u_0\|_{H^1} + 1), \qquad 0 \le t \le T_u$$

holds for any local solution u in the space (4.1).

The estimates [8, (4.5)] and [8, (4.7)] show the following result.

Corollary 4.1. [8, Corollary 4.1] *If an initial function $u_0 \in H^1(\Omega)$ satisfies $m(u_0) = 0$, then any local solution of* (1.1) *also satisfies $m(u(t)) = 0$ for every $0 \le t \le T_u$. Furthermore there exist an exponent $\rho > 0$ and a constant $C_\rho > 0$ which are independent of u_0 such that*

$$(4.3)\qquad \|u(t)\|^2_{H^1} \le C_\rho[e^{-\rho t}\|u_0\|^2_{H^1} + 1], \qquad 0 \le t \le T_u.$$

As an immediate consequence of a priori estimates, we can prove the global existence of solution.

Theorem 4.1. [8, Theorem 4.1] *Let $u_0 \in H^1(\Omega)$. Then,* (1.1) *possesses a unique global solution in the function space:*

$$(4.4)\qquad u \in \mathcal{C}([0,\infty); H^1(\Omega)) \cap \mathcal{C}^1((0,\infty); L^2(\Omega)) \cap \mathcal{C}((0,\infty); H^4_{N^2}(\Omega)).$$

By Proposition 4.1 we clearly verify that the global solution also satisfies the estimate

$$\|u(t)\|_{H^1} \le C(\|u_0\|_{H^1} + 1), \qquad 0 \le t < \infty, \tag{4.5}$$

where $C > 0$ is the same constant as in (4.2).

Moreover we can extend the estimate (3.5) to the global solutions.

Proposition 4.2. [8, Proposition 4.2] *There exist increasing functions $p(\cdot)$ such that, for any global solution with initial function $u_0 \in H^1(\Omega)$, it holds that*

$$\|u(t)\|_{H^4} \le (1 + t^{-\frac{3}{4}})p(\|u_0\|_{H^1}), \qquad 0 < t < \infty, \tag{4.6}$$

$$\|u(t)\|_{H^{\frac{7}{2}}} \le (1 + t^{-\frac{5}{8}})p(\|u_0\|_{H^1}), \qquad 0 < t < \infty. \tag{4.7}$$

We will conclude this section by verifying the Lipschitz continuity of solutions with respect to initial functions. Let B be a closed ball of initial functions

$$B = \{u_0 \in H^1(\Omega);\ \|u_0\|_{H^1} \le R\}$$

with arbitrarily fixed radius $R > 0$. By Theorem 4.1, there exists a unique global solution to (1.1) for each $u_0 \in B$.

Proposition 4.3. [8, Proposition 4.3] *Let u (resp. v) be the solution to* (1.1) *with initial function $u_0 \in B$ (resp. $v_0 \in B$). Then, for each $T > 0$ fixed, there exists some constant $C_{R,T} > 0$ depending on R and T alone such that*

$$\begin{aligned} t^{\frac{7}{8}}\|u(t) - v(t)\|_{H^{\frac{7}{2}}} + t^{\frac{1}{4}}\|u(t) - v(t)\|_{H^1} \\ + \|u(t) - v(t)\|_{L^2} \le C_{R,T}\|u_0 - v_0\|_{L^2}, \qquad 0 \le t \le T. \end{aligned} \tag{4.8}$$

5. Dynamical system

We already know by Corollary 4.1 that, if $u_0 \in H^1(\Omega)$ satisfies $m(u_0) = 0$, then the global solution $u(t; u_0)$ of (1.1) also satisfies the same condition for every $0 < t < \infty$ and in addition satisfies a dissipative estimate

$$\|u(t; u_0)\|^2_{H^1} \le C_\rho[e^{-\rho t}\|u_0\|^2_{H^1} + 1], \qquad 0 \le t < \infty \tag{5.1}$$

with the same ρ and C_ρ as in (4.3). In view of this fact, we set a phase space

$$H^1_m(\Omega) = \{u \in H^1(\Omega);\ m(u_0) = 0\}.$$

For $u_0 \in H^1_m(\Omega)$, set $S(t)u_0 = u(t; u_0)$, $0 \le t < \infty$. Then, $S(t)$ defines a nonlinear semigroup acting on $H^1_m(\Omega)$.

For each $0 < R < \infty$, let B_R be a ball of $H^1_m(\Omega)$ such that

$$B_R = \{u \in H^1_m(\Omega);\ \|u\|_{H^1} \le R\}.$$

We then put

$$K_R = \bigcup_{0 \le t < \infty} S(t)B_R. \tag{5.2}$$

In view of (5.1), we observe that $B_R \subset K_R \subset B_{\sqrt{C_\rho(R^2+1)}}$. Clearly, K_R is an invariant set of $S(t)$, i.e., $S(t)K_R \subset K_R$ for every $0 \le t < \infty$. Moreover, by Proposition 4.3, $S(t)$ is continuous in $B_{\sqrt{C_\rho(R^2+1)}}$ with respect to the L^2-norm, i.e., $(t, u_0) \mapsto S(t)u_0$ is continuous from $[0,\infty) \times B_{\sqrt{C_\rho(R^2+1)}}$ into $L^2(\Omega)$ with respect to the L^2-norm. Of course, the correspondence is continuous from $[0,\infty) \times K_R$ into K_R, too, with respect to the L^2-norm. Hence we have verified the following theorem.

Theorem 5.1. [8, Theorem 5.1] *For each $0 < R < \infty$, $(S(t), K_R, L^2(\Omega))$ is a dynamical system.*

Put $\widetilde{C} = \sqrt{2C_\rho}$, where C_ρ is the constant appearing in (5.1). Then we observe that, for any K_R, there exists a time t_R such that

$$S(t)K_R \subset B_{\widetilde{C}} \qquad \text{for all } t \in [t_R, \infty).$$

In this sense $B_{\widetilde{C}}$ is an absorbing set. Furthermore, in this sense, every dynamical system $(S(t), K_R, L^2(\Omega)$ is reduced to the dynamical system $(S(t), K_{\widetilde{C}}, L^2(\Omega))$ as $t \to \infty$, where $K_{\widetilde{C}}$ is the space given by (5.2) with $R = \widetilde{C}$.

We finally put

$$\mathcal{K} = S(1)K_{\widetilde{C}} \subset K_{\widetilde{C}}.$$

Then, $\mathcal{K}$ is an invariant set of $S(t)$. In addition, (3.5) yields that

$$\|u\|_{H^4} = \|S(1)u_0\|_{H^4} \le C\|u_0\|_{H^1}, \qquad u = S(1)u_0 \in \mathcal{K},\ u_0 \in K_{\widetilde{C}},$$

which shows that $\mathcal{K}$ is a bounded subset of $H^4(\Omega)$. We have thus arrive at the following theorem.

Theorem 5.2. [8, Theorem 5.2] *There is a dynamical system $(S(t), \mathcal{K}, L^2(\Omega))$ the phase space of which is a bounded subset of $H^4(\Omega)$. In addition, for any phase space $K_R \subset H^1_m(\Omega)$, there exists a time $t_R > 0$ such that $S(t)K_R \subset \mathcal{K}$ for all $t \in [t_R, \infty)$.*

We can verify that $S(t)$ defines also a dynamical system in the Sobolev space $H^\theta(\Omega)$ for $0 < \theta < 4$.

Corollary 5.1. [8, Corollary 5.1] *For each $0 < \theta < 4$, $(S(t), \mathcal{K}, H^\theta(\Omega))$ defines a dynamical system.*

6. Exponential attractors

In this section, we shall construct exponential attractors for the dynamical system $(S(t), \mathcal{K}, L^2(\Omega))$.

Let us first recall the definition of exponential attractor presented by Eden et al. [6]. Consider a dynamical system $(S(t), \mathcal{K}, X)$ in a universal space X (cf. [2, 25]), X being a Banach space. We assume that the phase space $\mathcal{K}$ is a compact subset of X and that the nonlinear semigroup $S(t)$ is continuous in the sense that a mapping $G(t,u) = S(t)u$ is continuous from $[0,\infty) \times \mathcal{K}$ into $\mathcal{K}$.

Let $\mathcal{A} = \bigcap_{0 \le t < \infty} S(t)\mathcal{K}$. Then, $\mathcal{A}$ is a nonempty compact set of X and is the global attractor of $(S(t), \mathcal{K}, X)$, namely, it holds that

$$\lim_{t\to\infty} h(S(t)\mathcal{K}, \mathcal{A}) = 0.$$

In what follows, $h(B_1, B_2)$ denotes the Hausdorff pseudo-distance

$$h(B_1, B_2) = \sup_{u \in B_1} \inf_{v \in B_2} \|u - v\|_X$$

for any two subsets B_1 and B_2 of $\mathcal{K}$.

A subset $\mathcal{M}$ of $\mathcal{K}$ is called an exponential attractor of $(S(t), \mathcal{K}, X)$ if $\mathcal{M}$ satisfies the following conditions:

(1) $\mathcal{M}$ is a compact subset of X containing the global attractor $\mathcal{A}$ (i.e., $\mathcal{A} \subset \mathcal{M} \subset \mathcal{K}$) and has a finite fractal dimension $d_F(\mathcal{M}) < \infty$;
(2) $\mathcal{M}$ is an invariant set of $S(t)$, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for every $t > 0$;
(3) $\mathcal{M}$ attracts $\mathcal{K}$ at an exponential rate

$$h(S(t)\mathcal{K}, \mathcal{M}) \leq Ce^{-\delta t}, \qquad 0 \leq t < \infty$$

with some exponent $\delta > 0$ and a constant $C > 0$.

Let X be a Hilbert space. In the paper [6], the authors presented also a sufficient condition for the semigroup $S(t)$ in order that $(S(t), \mathcal{K}, X)$ enjoys the exponential attractor. Assume that there exists time $0 < t^* < \infty$ which satisfies the following conditions:

(1) There exist some exponent $0 \leq \delta < \frac{1}{4}$ and an orthogonal projection P of finite rank N such that, for each pair u, v of vectors of $\mathcal{K}$, either

$$\|S^*u - S^*v\| \leq \delta\|u - v\| \tag{6.1}$$

or

$$\|(I - P)(S^*u - S^*v)\| \leq \|P(S^*u - S^*v)\| \tag{6.2}$$

holds, where $S^* = S(t^*)$;

(2) The mapping $G(t, u) = S(t)u$ is Lipschitz continuous on $[0, t^*] \times \mathcal{K}$, i.e.,

$$\|G(t, u) - G(s, v)\| \leq L\big(|t - s| + \|u - v\|\big), \qquad t,\ s \in [0, t^*];\ u,\ v \in \mathcal{K}. \tag{6.3}$$

Condition (6.1)-(6.2) is called the squeezing property of $S(t^*)$. According to [6, Theorem 3.1], the squeezing property (6.1)-(6.2) together with (6.3) in fact enables us to construct an exponential attractor $\mathcal{M}$ of $(S(t), \mathcal{K}, X)$ with fractal dimension

$$d_F(\mathcal{M}) \leq N \max\left\{1, \frac{\log(\frac{2L}{\delta} + 1)}{\log(\frac{1}{4\delta})}\right\} + 1. \tag{6.4}$$

When a dynamical system $(S(t), \mathcal{K}, X)$ is determined from the Cauchy problem of an abstract evolution equation like (2.1), the authors of [6] showed also some convenient method for verifying the squeezing properties of $S(t)$. Consider (2.1) in a Hilbert space X in which the linear operator A is a positive definite self-adjoint operator of X. Let the problem determine a dynamical system $(S(t), \mathcal{K}, X)$ with some compact phase space $\mathcal{K}$. We assume that the nonlinear operator $F(u)$ satisfies a Lipschitz condition of the form

$$\|F(u) - F(v)\| \leq C\|A^{\frac{1}{2}}(u - v)\|, \qquad u,\ v \in \mathcal{K}. \tag{6.5}$$

Then, it is possible to conclude that, for any $0 < t^* < \infty$, the nonlinear operator $S(t^*)$ fulfils (6.1)-(6.2) with a suitable exponent $0 \leq \delta < \frac{1}{4}$ and a projection P of finite rank N. Indeed, see [6, Proposition 3.1].

In the second half of this section, let us apply the general method to our dynamical system $(S(t), \mathcal{K}, L^2(\Omega))$ which was reviewed in the preceding section. To this end, it now suffices to verify that (6.3) and (6.5) are fulfilled.

Write

$$G(t, u) - G(s, v) = [S(t)u - S(s)u] + [S(s)u - S(s)v], \qquad u,\ v \in \mathcal{K}.$$

Since $\mathcal{K}$ is a bounded subset of $\mathcal{D}(A)$, it follows that

$$\|S(t)u - S(s)u\| = \left\| \int_s^t \frac{dS(\tau)}{d\tau} u \, d\tau \right\| = \left\| \int_s^t [-AS(\tau)u + F(S(\tau)u)] d\tau \right\|$$
$$\leq \sup_{w \in \bullet} \| -Aw + F(w)\| \, |t-s| \leq L_1 |t-s|.$$

In the meantime, let $0 < t^* < \infty$ be arbitrarily fixed. We already established that

$$\|S(s)u - S(s)v\| \leq L_2 \|u - v\|, \qquad 0 \leq s \leq t^*; \ u, \, v \in \mathcal{K}$$

due to [8, (4.13)]. Therefore, (6.3) is fulfilled. (6.5) has already been verified by (3.5).
We hence establish the following theorem.

Theorem 6.1. [9, Theorem 3.1] *The dynamical system* $(S(t), \mathcal{K}, L^2(\Omega))$ *enjoys an exponential attractor* $\mathcal{M}$ *with dimension given by* (6.4).

It is possible to substitute any Sobolev space $H^\theta(\Omega)$, where $0 < \theta < 4$, for the present universal space $L^2(\Omega)$. As an analogy of [8, Corollary 5.1], we can show the following result.

Corollary 6.1. [9, Corollary 3.1] *For each* $0 < \theta < 4$, *the exponential attractor* $\mathcal{M}$ *constructed above for* $(S(t), \mathcal{K}, L^2(\Omega))$ *is an exponential attractor of* $(S(t), \mathcal{K}, H^\theta(\Omega))$, *too.*

7. Lyapunov function

In this section, we shall construct a Lyapunov function $\Psi(u)$ for the dynamical system $(S(t), \mathcal{K}, L^2(\Omega))$.

Let $u_0 \in \mathcal{K}$ and let $S(t)u_0 = u(t; u_0) = u(t)$ be the global solution to (1.1) with initial function u_0. Multiply the equation of (1.1) by $\frac{\partial \overline{u}}{\partial t}$ and integrate the product in Ω. Then,

$$\int_\Omega \left| \frac{\partial u}{\partial t} \right|^2 dx = -a \int_\Omega \Delta^2 u \cdot \frac{\partial \overline{u}}{\partial t} \, dx - \mu \int_\Omega \left[\nabla \cdot \left(\frac{\nabla u}{1 + |\nabla u|^2} \right) \right] \frac{\partial \overline{u}}{\partial t} \, dx.$$

Since $\frac{\partial u}{\partial n} = \frac{\partial}{\partial n} \Delta u = 0$ on $\partial \Omega$, we have

$$\int_\Omega \Delta^2 u \cdot \frac{\partial \overline{u}}{\partial t} \, dx = \int_\Omega \Delta u \cdot \frac{\partial}{\partial t} \Delta \overline{u} \, dx.$$

Furthermore, taking the real parts of both hand sides, we have

$$\operatorname{Re} \int_\Omega \Delta^2 u \cdot \frac{\partial \overline{u}}{\partial t} \, dx = \int_\Omega \frac{1}{2} \left(\frac{\partial}{\partial t} \Delta u \cdot \Delta \overline{u} + \Delta u \cdot \frac{\partial}{\partial t} \Delta \overline{u} \right) dx = \frac{1}{2} \frac{d}{dt} \int_\Omega |\Delta u|^2 dx.$$

In the meantime, it is seen that

$$\int_\Omega \left[\nabla \cdot \left(\frac{\nabla u}{1 + |\nabla u|^2} \right) \right] \frac{\partial \overline{u}}{\partial t} \, dx = - \int_\Omega \frac{\nabla u}{1 + |\nabla u|^2} \cdot \nabla \frac{\partial \overline{u}}{\partial t} \, dx.$$

Therefore,

$$
\begin{aligned}
\operatorname{Re} \int_{\Omega} \left[\nabla \cdot \left(\frac{\nabla u}{1 + |\nabla u|^2} \right) \right] \frac{\partial \overline{u}}{\partial t}\, dx
&= - \int_{\Omega} \frac{1}{1 + |\nabla u|^2} \frac{1}{2} \left(\nabla \frac{\partial u}{\partial t} \cdot \nabla \overline{u} + \nabla u \cdot \nabla \frac{\partial \overline{u}}{\partial t} \right) dx \\
&= -\frac{1}{2} \int_{\Omega} \frac{1}{1 + |\nabla u|^2} \frac{\partial}{\partial t} |\nabla u|^2 dx = -\frac{1}{2} \frac{d}{dt} \int_{\Omega} \log (1 + |\nabla u|^2)\, dx.
\end{aligned}
$$

Hence, we obtain that

$$
\frac{d}{dt} \int_{\Omega} [a|\Delta u|^2 - \mu \log (1 + |\nabla u|^2)]\, dx = -2 \int_{\Omega} \left| \frac{\partial u}{\partial t} \right|^2 dx \leq 0, \qquad 0 < t < \infty. \tag{7.1}
$$

This indeed shows that the functional

$$
\Psi(u) = \int_{\Omega} [a|\Delta u|^2 - \mu \log (1 + |\nabla u|^2)]\, dx, \qquad u \in H^2(\Omega) \tag{7.2}
$$

is a Lyapunov function for the dynamical system $(S(t), \mathcal{K}, L^2(\Omega))$.

Theorem 7.1. [9, Theorem 4.1] *Along any trajectory $S(\cdot)u_0$, where $u_0 \in \mathcal{K}$, the function $\Psi(S(t)u_0)$ is monotonically decreasing and has a limit as $t \to \infty$. For $u_0 \in \mathcal{K}$ and $0 < t_0 < \infty$, $\overline{u} = S(t_0)u_0$ is an equilibrium if and only if $[\frac{d}{dt}\Psi(S(t)u_0)]_{|t=t_0} = 0$.*

8. ω-LIMIT SETS

We shall investigate asymptotic behavior of the trajectory $S(\cdot)u_0$ for each $u_0 \in \mathcal{K}$. For $u_0 \in \mathcal{K}$, the ω-limit set $\omega(u_0)$ of $S(\cdot)u_0$ is defined by

$$
\omega(u_0) = \bigcap_{t \geq 0} \overline{\{S(\tau)u_0;\ t \leq \tau < \infty\}} \qquad \text{(closure in the topology of } L^2(\Omega)),
$$

namely, $\overline{u} \in \omega(u_0)$ if and only if there exists a time sequence $\{t_n\}$ tending to ∞ such that $S(t_n)u_0 \to \overline{u}$ in $L^2(\Omega)$. Since

$$
\overline{\{S(t)u_0;\ 0 \leq t < \infty\}} \subset \mathcal{K}
$$

and $\mathcal{K}$ is a compact set, $\omega(u_0)$ is nonempty set. Moreover, we easily verify that $\omega(u_0)$ is a strictly invariant set of $S(t)$, i.e.,

$$
S(t)\big(\omega(u_0)\big) = \omega(u_0) \qquad \text{for every} \quad 0 < t < \infty. \tag{8.1}
$$

We prove that the ω-limit set consists of equilibria.

Theorem 8.1. [9, Theorem 5.1] *For any $u_0 \in \mathcal{K}$, $\omega(u_0)$ consists of equilibria of the dynamical system $(S(t), \mathcal{K}, L^2(\Omega))$.*

9. General framework

Consider the Cauchy problem for a semilinear abstract parabolic evolution equation

$$
\begin{cases} \dfrac{du}{dt} + Au = F(u), & 0 < t < \infty, \\ u(0) = u_0 \end{cases} \tag{9.1}
$$

in a Banach space X. Here, A is a closed linear operator of X the spectral set of which is contained in a sectorial domain $\Sigma = \{\lambda \in \mathbb{C};\ |\arg \lambda| < \omega\}$ with angle $0 < \omega < \frac{\pi}{2}$ and the resolvent of A satisfies [8, (2.2)]. We assume that the nonlinear operator $F(u)$ satisfies the Lipschitz condition [8, (2.4)] with some exponents $0 \le \alpha \le \eta < 1$. Then, as noticed by [8, Theorem 2.1], (9.1) has a unique local solution for any initial value $u_0 \in \mathcal{D}(A^\alpha)$ satisfying [8, (2.3)], i.e., $\|A^\alpha u_0\| \le R$. The local solution exists at least on an interval $[0, T_R]$, where $T_R > 0$ is determined by R alone.

For $u_0 \in \mathcal{D}(A^\alpha)$, let $u(\cdot; u_0)$ denote any local solution of (9.1). We assume that the a priori estimate

$$
\|A^\alpha u(t; u_0)\| \le p(\|A^\alpha u_0\|), \qquad u_0 \in \mathcal{D}(A^\alpha) \tag{9.2}
$$

holds for any local solution with some specifically fixed continuous increasing function $p(\cdot)$. By the standard arguments, we can conclude under (9.2) that (9.1) has a global solution on the whole interval $[0, \infty)$.

Let $u(\cdot; u_0)$ denote the global solution of (9.1). We then set $S(t)u_0 = u(t; u_0)$ for $u_0 \in \mathcal{D}(A^\alpha)$. Then, $S(t)$ is a continuous nonlinear semigroup acting on $\mathcal{D}(A^\alpha)$ and $(S(t), \mathcal{D}_\alpha, \mathcal{D}_\alpha)$ defines a dynamical system with phase space $\mathcal{D}_\alpha$ in the universal space $\mathcal{D}_\alpha$, $\mathcal{D}(A^\alpha)$ being abbreviated by $\mathcal{D}_\alpha$.

Let $\overline{u} \in \mathcal{D}(A)$ be a stationary solution of (9.1), i.e., $A\overline{u} = F(\overline{u})$. Clearly, $\overline{u}$ is an equilibrium of $(S(t), \mathcal{D}_\alpha, \mathcal{D}_\alpha)$. We are concerned with investigating stability or instability of $\overline{u}$.

To this end, we assume that $F\colon \mathcal{D}(A^\eta) \to X$ is of class $\mathcal{C}^{1,1}$ in a neighborhood of $\overline{u}$. That is, F is Fréchet differentiable from $\mathcal{D}(A^\eta)$ to X in a neighborhood of $\overline{u}$ in the topology of $\mathcal{D}_\alpha$ and the derivative satisfies

$$
\|[F'(u) - F'(v)]h\| \le C\|A^\alpha(u - v)\|\|A^\eta h\|, \qquad u, v \in \mathcal{O}(\overline{u});\ h \in \mathcal{D}(A^\eta), \tag{9.3}
$$

where $\mathcal{O}(\overline{u})$ is a neighborhood of $\overline{u}$.

These assumptions in fact imply that the semigroup $S(t)\colon \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is Fréchet differentiable; in addition, $S(t)$ is of class $\mathcal{C}^{1,1}$ in a neighborhood $\mathcal{O}'(\overline{u})$ of $\overline{u}$ in $\mathcal{D}_\alpha$, i.e.,

$$
\|S(t)'u - S(t)'v\|_{\cdot\,(\cdot\,_\alpha, \cdot\,_\alpha)} \le C\|A^\alpha(u - v)\|, \qquad u, v \in \mathcal{O}'(\overline{u});\ 0 \le t \le t^*, \tag{9.4}
$$

$t^* > 0$ being arbitrarily fixed time. For detail, see the proof of [1, Theorem 5.1].

We further assume a spectral separation condition for $\sigma(A - F'(\overline{u}))$ of the form

$$
\sigma(A - F'(\overline{u})) \cap \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda = 0\} = \emptyset. \tag{9.5}
$$

Then, since $S(t)'\overline{u} = e^{-t\overline{A}}$, where $\overline{A} = A - F'(\overline{u})$, we have the spectral separation for $S(t)'\overline{u}$, i.e.,

$$
\sigma^{\cdot\ \alpha}(S(t)'\overline{u}) \cap \{\lambda \in \mathbb{C};\ |\lambda| = 1\} = \emptyset. \tag{9.6}
$$

According to [25, Chapter VII, Theorem 3.1], under (9.4) and (9.6), there exists a smooth local unstable manifold $\mathcal{M}_+(\overline{u}; \mathcal{O})$ in a neighborhood $\mathcal{O}$ of $\overline{u}$ in $\mathcal{D}_\alpha$.

When

$$\sigma(A - F'(\overline{u})) \subset \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > 0\}, \tag{9.7}$$

it actually follows that $\mathcal{M}_+(\overline{u}; \mathcal{O}) = \{\overline{u}\}$. Hence, under (9.7), $\overline{u}$ is a stable stationary solution. In the meantime, when

$$\sigma(A - F'(\overline{u})) \cap \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda < 0\} \neq \emptyset, \tag{9.8}$$

$\mathcal{M}_+(\overline{u}; \mathcal{O})$ is not trivial and $\overline{u}$ is an unstable stationary solution.

10. Differentiability of $F(u)$

Let us apply the general results explained in the preceding section by setting $X_m = L^2_m(\Omega)$ and $A_m = (-\sqrt{a}\Delta + 1)^2$ is considered in $L^2_m(\Omega)$. So we have

$$\mathcal{D}(A_m) = \{u \in H^4_{N^2}(\Omega);\ m(u) = 0\}.$$

The nonlinear operator $F_m : \mathcal{D}(A_m^{\frac{7}{8}}) \to X_m$ is given by (3.3) again. We take as $\overline{u}$ the zero solution which is a unique homogeneous stationary solution to (1.1) in the space X_m.

We can entirely follow the arguments reviewed in the previous sections in order to construct a dynamical system $(S(t), H^1_m(\Omega), H^1_m(\Omega))$ as well as $(S(t), \mathcal{D}(A^\alpha_m), \mathcal{D}(A^\alpha_m))$ for any exponent $\frac{1}{4} \leq \alpha < 1$. Proposition 10.2 which will be shown below suggests that it is natural to take $\alpha = \frac{1}{2}$. In view of (3.1), we have

$$\mathcal{D}(A_m^{\frac{1}{2}}) = H^2_{N,m}(\Omega) \equiv \{u \in H^2_N(\Omega);\ m(u) = 0\}.$$

In this section, we intend to verify Fréchet differentiability of F_m and the conditions (9.3) with $\alpha = \frac{1}{2}$.

Proposition 10.1. [10, Propostion 4.1] *$F_m : \mathcal{D}(A_m^{\frac{7}{8}}) \to X_m$ is Fréchet differentiable and the derivative is given by*

$$F'_m(u)h = -\mu\nabla \cdot \left(\frac{\nabla h}{1 + |\nabla u|^2}\right) + 2\mu\nabla \cdot \left(\frac{(\nabla u \cdot \nabla h)\nabla u}{(1 + |\nabla u|^2)^2}\right) - 2\sqrt{a}\Delta h + h, \qquad u,\ h \in \mathcal{D}(A_m^{\frac{7}{8}}). \tag{10.1}$$

Proposition 10.2. [10, Propostion 4.2] *Let $u \in \mathcal{D}(A^\eta_m)$ varies in the ball $B^{\cdot\,(A_m^{\frac{1}{2}})}(0; 1)$. Then, $F'_m(u)$ satisfies the Lipschitz condition*

$$\|[F'_m(u) - F'_m(v)]h\|_{L^2} \leq C\|A_m^{\frac{1}{2}}(u - v)\|_{L^2}\|A_m^{\frac{7}{8}}h\|_{L^2}, \qquad u,\ v \in \mathcal{D}(A_m^{\frac{7}{8}}) \cap B^{\cdot\,(A_m^{\frac{1}{2}})}(0; 1);\ h \in \mathcal{D}(A_m^{\frac{7}{8}}).$$

11. Spectral separation condition

Under the same situation as in Section 10, let us now verify the condition (9.5).

Let Λ denote the realization of $-\Delta$ in $L^2_m(\Omega)$ under the Neumann boundary conditions. The operator Λ possesses denumerable positive eigenvalues and the corresponding real eigenfunctions can constitute an orthonormal basis of $L^2_m(\Omega)$. So, let

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \quad \to \infty$$

be eigenvalues of Λ and let ϕ_1, ϕ_2, $\phi_3, \ldots$ be corresponding real eigenfunctions which constitute an orthonormal basis. For each $k = 1, 2, 3, \ldots$, let X_k be the eigenspace of λ_k which is a one-dimensional subspace of $L^2_m(\Omega)$. Any two subspaces X_k and X_ℓ are orthogonal if $k \ne \ell$, and $X_m = L^2_m(\Omega)$ is given by an infinite sum $X_m = \sum_{k=1}^{\infty} X_k$.

According to (10.1), we have

$$F'_m(0)h = -(\mu + 2\sqrt{a})\Delta h + h, \qquad h \in \mathcal{D}(A_m^{\frac{7}{8}}).$$

Therefore, the operator $\overline{A}_m = A_m - F'_m(0) = a\Delta^2 + \mu\Delta$ maps the subspace X_k into itself, namely, X_k is an invariant set of $\overline{A}_m$ for every k. Consequently, the operator $\overline{A}_m$ can also be decomposed as $\overline{A}_m = \sum_{k=1}^{\infty} \overline{A}_k$, where $\overline{A}_k$ is the part of $\overline{A}_m$ in X_k, i.e.,

$$\overline{A}_k\phi_k = (a\lambda_k^2 - \mu\lambda_k)\phi_k.$$

Hence, $\sigma(\overline{A}_k) = \{a\lambda_k^2 - \mu\lambda_k\}$.

Let $\lambda \in i\mathbb{R}$. Let k be sufficiently large so that $a\lambda_k > \mu$ holds. Then, $\lambda \in \rho(\overline{A}_k)$ and

$$\|(\lambda - \overline{A}_k)^{-1}\|_{\cdot(X_k)} \le \frac{1}{(a\lambda_k - \mu)\lambda_k}.$$

This means that $\lambda \in i\mathbb{R}$ belongs to $\rho(\overline{A})$ if and only if $\lambda \in \rho(\overline{A}_k)$ for every $k = 1, 2, 3, \ldots$. In other words, $\lambda \notin \sigma(\overline{A})$ if and only $\lambda \notin \sigma(\overline{A}_k) = \{a\lambda_k^2 - \mu\lambda_k\}$ for every k. In view of this fact, we will make the following assumption

$$\lambda_k \ne \frac{\mu}{a} \qquad \text{for every } k = 1, 2, 3, \ldots. \tag{11.1}$$

Under (11.1), it is true that $\sigma(\overline{A}) \cap i\mathbb{R} = \emptyset$, namely, the spectral separation condition (9.5) is fulfilled.

12. Stability or instability conditions

Let $\lambda \in \mathbb{C}$ satisfy $\operatorname{Re}\lambda \le 0$. By the same reason as before, we see that $\lambda \notin \sigma(\overline{A}_m)$ if and only if $\lambda \notin \sigma(\overline{A}_k)$ for every k. Therefore, if the condition

$$\mu < a\lambda_1 \tag{12.1}$$

is valid, then, as $\bigcup_{k=1}^{\infty} \sigma(\overline{A}_k) \subset \{\lambda;\ \operatorname{Re}\lambda > 0\}$, λ such that $\operatorname{Re}\lambda \le 0$ cannot belong to $\sigma(\overline{A}_m)$, namely, $\sigma(\overline{A}_m) \subset \{\lambda;\ \operatorname{Re}\lambda > 0\}$. Thus, under (12.1), (9.7) is fulfilled and 0 is a stable stationary solution of $(S(t), H^2_{N,m}(\Omega), H^2_{N,m}(\Omega))$.

On the other hand, if the condition

$$N = \#\{\lambda_k;\ \mu > a\lambda_k\} \ne 0 \tag{12.2}$$

is satisfied, then $\sigma(\overline{A}) \cap \{\lambda;\ \operatorname{Re}\lambda < 0\} \ne \emptyset$, namely, (9.8) is fulfilled. Thus, under (11.1) and (12.2), 0 has a nontrivial unstable manifold $\mathcal{M}_+(0)$ and is an unstable equilibrium of $(S(t), H^2_{N,m}(\Omega), H^2_{N,m}(\Omega))$.

We remark $\overline{A}_m$ has a real eigenfunction ϕ_k for each λ_k. This means that the unstable manifold $\mathcal{M}_+(0)$ is tangential to an N-dimensional subspace of $H^2_{N,m}(\Omega)$ whose basis is composed by real functions. In particular, it is deduced that

$$\dim \mathcal{M} \geq \dim \mathcal{M}_+(0) \geq N.$$

13. Numerical Simulation

We are concerned with the process of growing crystal surface during the incidence of molecular beam. To investigate the qualitative characteristics and the structures of stationary solutions of (1.1), we perform the numerical simulation of (1.1) by varying values of coefficient of surface roughening μ and initial functions. The domain considered here is $\Omega = \{(x,y) : 0 \leq x \leq 32,\ 0 \leq y \leq 32\} \subset \mathbb{R}^2$. The model equation (1.1) is calculated numerically on a 256×256 square lattice with homogeneous Neumann boundary conditions utilizing the general explicit difference scheme with time interval $\Delta t = 1.0 \times 10^{-5}$. The surface diffusion constant is fixed as $a = 1.0$. First, the numerical result of the case $\mu = 1.0$ is shown in Fig 1 with initial function

$$\tag{13.1} \widetilde{u}(x,y,0) = \begin{cases} 50\exp\Big\{-(x-8)^2/8-(y-8)^2/8\Big\} \\ \text{in}\quad \Omega_1 = \Big\{(x,y) : 0 \leq x \leq 16,\ 0 \leq y \leq 16\Big\} \subset \mathbb{R}^2, \\ 50\exp\Big\{-(x-8)^2/8-(y-24)^2/8\Big\} \\ \text{in}\quad \Omega_2 = \Big\{(x,y) : 0 \leq x \leq 16,\ 16 \leq y \leq 32\Big\} \subset \mathbb{R}^2, \\ 50\exp\Big\{-(x-24)^2/8-(y-8)^2/8\Big\} \\ \text{in}\quad \Omega_3 = \Big\{(x,y) : 16 \leq x \leq 32,\ 0 \leq y \leq 16\Big\} \subset \mathbb{R}^2, \\ 50\exp\Big\{-(x-24)^2/8-(y-24)^2/8\Big\} \\ \text{in}\quad \Omega_4 = \Big\{(x,y) : 16 \leq x \leq 32,\ 16 \leq y \leq 32\Big\} \subset \mathbb{R}^2. \end{cases}$$

And $u(x,y,0)$ is given by $u(x,y,0) = \widetilde{u}(x,y,0) - \frac{1}{|\Omega|}\int_\Omega \widetilde{u}(x,y,0)\,dxdy$. If we carry out this operation, mean integral (2.12) is always equal to 0 for any initial functions in the simulation.

In this case, 4 Gaussian distribution curved surface can be seen at $t = 0$ which is shown in Fig 1. We perform the numerical computation till $t = 6000$ and this result is shown in the Fig 2. In this result, we observe one mountain at point$(0,0,u(0,0))$, on the while, valleys at the points $(0,32,u(0,32)), (32,0,u(32,0))$ and $(32,32,u(32,32))$.

Next, we carry out the numerical simulation for the case $\mu = 100.0$ with the initial function (13.1). Of course, initial state is the same as in the case $\mu = 1.0$. In this case, we compute till $t = 3200$ and the result is shown in Fig 3. In this figure, mountain is formed at the point $(0,32,u(0,32))$, valley at the point $(0,0,u(0,0))$. Clearly, this result is different from the case $\mu = 1.0$ and also we can confirm that the amplitude of the surface is enhanced.

So far we perform numerical simulations for the above initial function and the moduli of the surface roughening. In the future's work, we intent on investigating the numerical results which show the complex pattern and interesting shape of crystal surface.

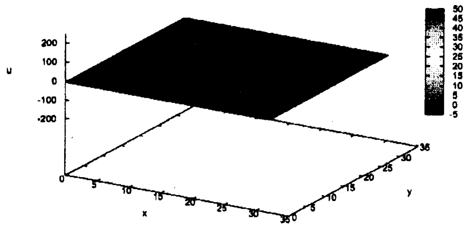

FIGURE 1. $t = 0$

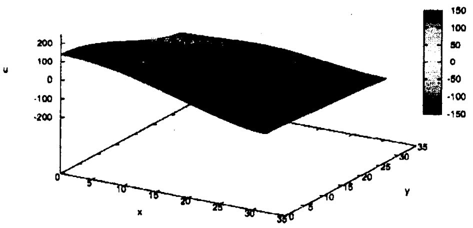

FIGURE 2. $\mu = 1.0, t = 6000$

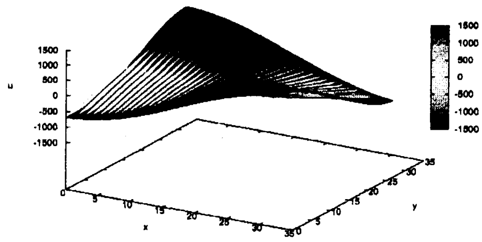

FIGURE 3. $\mu = 100.0, t = 3200$

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