

## 150 Years of Vortex Dynamics

HASSAN AREF<sup>1</sup>

*Center for Fluid Dynamics and Department of Physics  
Technical University of Denmark, Kgs. Lyngby, DK-2800, Denmark*  
and

*Department of Engineering Science and Mechanics, College of Engineering,  
Virginia Polytechnic Institute & State University, Blacksburg, VA 24061, USA*

### 1. Introduction

The subject of vortex dynamics can fairly be said to have been initiated by the seminal paper [23] of Hermann Ludwig Ferdinand Helmholtz 150 years ago. In this paper Helmholtz established his three laws of vortex motion in roughly the form they are found today in textbooks on fluid mechanics. One motivation seems to have been his interest in frictional phenomena, carried over from his interest in energetics; another was his growing awareness of the power of Green's theorem in hydrodynamics. In a speech [25] at a banquet on the occasion of his 70th birthday – an event that brought together 260 friends and admirers at Kaiserhof on November 2, 1891 – Helmholtz gave the following account:

I have also been in a position to solve several problems in mathematical physics, some of which the great mathematicians since the time of Euler had worked on in vain — for example, problems concerning vortex motion and the discontinuity of motion in fluids, the problem of the motion of sound waves at the open ends of organ pipes, and so on. But the pride which I might have felt about the final result of these investigations was considerably lessened by my knowledge that I had only succeeded in solving such problems, after many erroneous attempts, by the gradual generalization of favorable examples and by a series of fortunate guesses. I would compare myself to a mountain climber who, not knowing the way, ascends slowly and painfully and is often compelled to retrace his steps because he can go no farther; who, sometimes by reasoning and sometimes by accident, hits upon signs of a fresh path, which leads him a little farther; and who finally, when he has reached the summit, discovers to his annoyance a royal road on which he might have ridden up if he had been clever enough to find the right starting point at the beginning. In my papers and memoirs I have not, of course, given the reader an account of my wanderings, but have only described the beaten path along which one may reach the summit without trouble.

Until the appearance of Helmholtz's paper the integrals of the hydrodynamical equations had been determined almost exclusively on the assumption that the cartesian components of the velocity of each fluid particle are partial first derivatives of the *velocity potential*. Helmholtz eliminated this limitation, and took into account the effects of friction between different elements of the fluid or between the fluid and a solid boundary. At the time the effect of friction had not been fully understood mathematically. Helmholtz endeavored to identify aspects of the motion that frictional forces will produce in a fluid. Key among these is the spin-up of individual fluid particles, which is measured by the vector field known as the *vorticity*.

It is somewhat rare that a subject in a rather "mature" science such as fluid mechanics has so clear a starting date. Usually when this happens it is due to a seminal paper by a luminary of the field, a paper that is far ahead of anything else produced by his contemporaries, and a paper that is quickly embraced by the community and sets the stage for developments for decades to come. The early papers in the new field of vortex dynamics were scattered among many journals in many countries and were written in a multitude of languages, primarily English, French, German, Italian and Russian. This diversity of publication venue and language, unfortunately, often makes the

---

<sup>1</sup>Email address: haref@vt.edu

literature rather difficult to identify and access for the modern researcher. An attempt to assemble a comprehensive bibliography for the first century of vortex dynamics may be found in [47]. For additional background on Helmholtz and his work in hydrodynamics see [14].

## 2. Case studies

Some of the older papers collected in the bibliography [47] have maintained themselves into modern research while others have been long forgotten. For example, the thesis of Gröbli [19, 20] and the later paper by Synge [69] on the solution of the three-vortex problem were revived about 30 years ago through the independent rediscoveries by Novikov [53] and Aref [2]. For a review of the history of solution, neglect and re-discovery see [6]. While the three-vortex problem is very interesting of its own accord, the discovery of chaos in the four-vortex problem (cf. [3]) immediately propelled this kind of problem to the front lines of “modern science”. See also §2.2 below.

Another example of this kind may be found in the extensive series of works by Da Rios ([13] and several later papers) on vortex filament motion under the so-called *localized induction approximation*. In spite of having been done as a thesis under T. Levi-Civita, one of the most illustrious mathematicians of his day, this work, somehow, never “took”. It was not until the 1960’s when Arms & Hama [8] and Betchov [11] re-introduced this idea – and Batchelor included it in his well known text [9] – that it finally became a standard part of the subject. The beautiful transformation of Hasimoto [21], and the idea that vortex filaments can support soliton waves, also played a role in this “assimilation” into modern research. The history of Da Rios’ work has been reviewed by Ricca [58, 59].

### 2.1 Helmholtz’s paper

Helmholtz discovered a series of fundamental propositions in hydrodynamics that had entirely escaped his predecessors. He pointed out that already Euler had mentioned cases of fluid motion in which no velocity-potential exists, for example, the rotation of a fluid about an axis where every element has the same angular velocity. A minute sphere of fluid may move as a whole in a definite direction, and change its shape, all while spinning about an axis. This last motion is the distinguishing characteristic of vorticity. Helmholtz was the first to elucidate key properties of those portions of a fluid in which vorticity occurs. His investigation was restricted to a frictionless, incompressible fluid. He proved that in such an ideal substance vortex motion could neither be produced from irrotational flow nor be destroyed entirely by any natural forces that have a potential. If vorticity exists within a group of fluid particles, they are incapable of transmitting it to particles that have none. They cannot be entirely deprived of their vorticity themselves (although the vorticity of any individual particle may change in three-dimensional flow; in two-dimensional flow the vorticity of each particle is a constant of the motion). For an ideal fluid the laws of vortex motion establish a curious and indissoluble fellowship between fluid particles and their state of rotation.

In the Introduction to his paper Helmholtz states:

Hence it appeared to me to be of importance to investigate the species of motion for which there is no velocity-potential.

The following investigation shows that when there is a velocity-potential the elements of the fluid have no rotation, but that there is at least a portion of the fluid elements in rotation when there is no velocity-potential.

By *vortex-lines* (*Wirbellinien*) I denote lines drawn through the fluid so as at every point to coincide with the instantaneous axis of rotation of the corresponding fluid element.

By *vortex-filaments* (*Wirbelfäden*) I denote portions of the fluid bounded by vortex-lines drawn through

every point of the boundary of an infinitely small closed curve.

The investigation shows that, if all the forces which act on the fluid have a potential, —

1. No element of the fluid which was not originally in rotation is made to rotate.
2. The elements which at any time belong to one vortex-line, however they may be translated, remain on one vortex-line.
3. The product of the section and the angular velocity of an infinitely thin vortex-filament is constant throughout its whole length, and retains the same value during all displacements of the filament. Hence vortex-filaments must either be closed curves, or must have their ends in the bounding surface of the fluid.

According to Truesdell [75, p.58] the name *vorticity* was introduced by Lamb [35] for the vector,  $\boldsymbol{\omega}$ , whose Cartesian components,  $(\xi, \eta, \zeta)$ , are given in terms of the components  $(u, v, w)$  of the (Eulerian) velocity vector  $\boldsymbol{u}$  by

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \quad (1)$$

In modern vector notation

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}. \quad (2)$$

Helmholtz's result in §1 of his paper that an arbitrary instantaneous state of continuous motion of a deformable medium is at each point the superposition of a uniform velocity of translation, a motion of extension, a shearing motion, and a rigid rotation, precipitated an extended debate with the French academician Bertrand.

The third law contains two statements, *viz* that “vortex-filaments must either be closed curves”, or that they “must have their ends in the bounding surface of the fluid”. The first statement excludes the possibility of vortex lines that wander aperiodically and never close, as one finds, for example, in a chaotic, three-dimensional flow<sup>2</sup>. The second is, in principle, correct only for vortex lines, although an example of a thin vortex filament that ends at a point in the interior of the fluid has, so far as we are aware, never been given. The vorticity distribution in such a structure would be near-singular. See [18] for a modern perspective on this problem.

In §3 of his paper [23] Helmholtz addresses the inverse problem of finding the components of the velocity  $u, v, w$  from the components of vorticity  $\xi, \eta, \zeta$  (up to a potential flow that covers the boundary conditions). He independently obtains the representations of Stokes for the classical problem of vector analysis of determining a vector field of known divergence (“hydrodynamic integrals of the first class” in his terminology) and curl (“hydrodynamic integrals of the second class”). Determination of the velocity field for incompressible fluid leads to the *Biot-Savart law* of electromagnetism, which in the present case reads that each rotating element of fluid induces in every other element a velocity with direction perpendicular to the plane through the second element that contains the axis of the first element. The magnitude of this induced velocity is directly proportional to the volume of the first element, its angular velocity, and the sine of the angle between the line that joins the two elements and the axis of rotation, and is inversely proportional to the square of the distance between the two elements.

Helmholtz also establishes analogies between the induced velocity and the forces on magnetized particles. Most of these relations would today come under the heading of potential theory.

In §4 of his paper [23] Helmholtz derives an elegant expression for the conserved kinetic energy, “*vis viva*” in his terminology, of infinite fluid with a compact distribution of vorticity within it.

<sup>2</sup>The best known examples may be the *ABC flows* studied by several authors ever since their introduction in 1965-66 by Arnold and Hénon; see [4] for a brief description in the context of “chaotic advection”. There are many other instances where vortex lines do not close. Indeed, closed vortex lines are the exception.

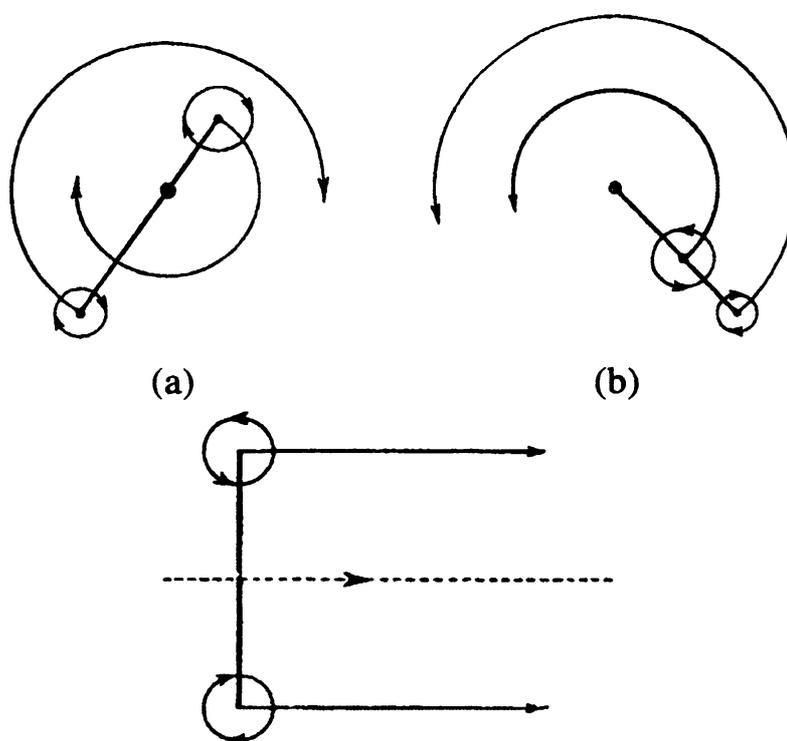


Figure 1: Motion of two parallel rectilinear vortices (or point vortices). Top: (a) circulations of the same sign; (b) circulations of opposite sign. Bottom: Motion of a vortex pair. From [27].

In §5, entitled “Straight parallel vortex-filaments”, Helmholtz studies certain simple cases in which the rotation of the elements occurs only in a set of parallel rectilinear vortex-filaments. In particular, he considers several infinitely thin, parallel vortex-filaments each of which carries a finite, limiting value,  $m$ , of the product of the cross-sectional area and the angular velocity. This is the now celebrated concept of a *point vortex*. Helmholtz considers simple cases of the dynamics of such vortices. He establishes the law of conservation of the *center of vorticity* of an assembly of point vortices. The discussion is phrased in terms of the “center of gravity” of the vortices (considering their values of  $m$  as the analog of “masses”): “The centre of gravity of the vortex-filaments remains stationary during their motions about one another, unless the sum of the masses be zero, in which case there is no centre of gravity.” Without further explanation Helmholtz notes the following two consequences:

1. If there be a single rectilinear vortex-filament of indefinitely small section in a fluid indefinite in all directions perpendicular to it, the motion of an element of the fluid at finite distance from it depends only on the product ( $\zeta da db = m$ ) of the velocity of rotation and the section, not on the form of that section. The elements of the fluid revolve about it with tangential velocity  $= \frac{m}{\pi r}$ , where  $r$  is the distance from the centre of gravity of the filament. The position of the centre of gravity, the angular velocity, the area of the section, and therefore, of course, the magnitude  $m$  remain unaltered, even if the form of the indefinitely small section may alter.
2. If there be two rectilinear vortex-filaments of indefinitely small section in an unlimited fluid, each will cause the other to move in a direction perpendicular to the line joining them. Thus the length of this joining line will not be altered. They will thus turn about their common centre of gravity at constant

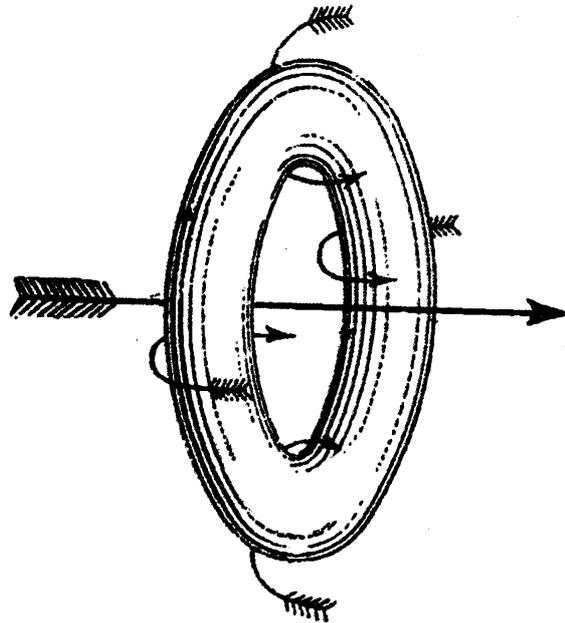


Figure 2: Self-induced forward motion of a vortex ring. From [27].

distances from it. If the rotation be in the same direction for both (that is, of the same sign) their centre of gravity lies between them. If in opposite directions (that is, of different signs), the centre of gravity lies in the line joining them produced. And if, in addition, the product of the velocity and the section be the same for both, so that the centre of gravity is at an infinite distance, they travel forwards with equal velocity, and in parallel directions perpendicular to the line joining them.

See Fig. 1 for later illustrations of these motions. In addition to introducing this notion of a “vortex pair” Helmholtz describes the motion of a single vortex-filament near an infinite plane to which it is parallel. He states that the boundary condition will be fulfilled if instead of the plane there is an infinite mass of fluid with another vortex-filament as the image (with respect to the plane) of the first, and concludes: “From this it follows that the vortex-filament moves parallel to the plane in the direction in which the elements of the fluid between it and the plane move, and with one-fourth of the velocity which the elements at the foot of a perpendicular from the filament on the plane have.”

In §6, entitled “Circular vortex-filaments”, Helmholtz studies the axisymmetric motion of several circular vortex-filaments whose planes are parallel to the  $xy$ -plane, and whose centers are on the  $z$ -axis. Here he considers the problem in full detail and arrives at the conclusion that “in a circular vortex-filament of very small section in an indefinitely extended fluid, the centre of gravity of the section has, from the commencement, an approximately constant and very great velocity parallel to the axis of the vortex-ring, and this is directed towards the side to which the fluid flows through the ring.” (See Fig.2 for a later illustration.)

When two such rings of infinitesimal cross-section have a common axis and the same direction of rotation, they travel in the same direction. As they approach, the first ring widens and travels more slowly, the second contracts and travels faster. Finally, if their velocities are not too different, the second ring overtakes the first and travels through it. This same process of “leapfrogging” is

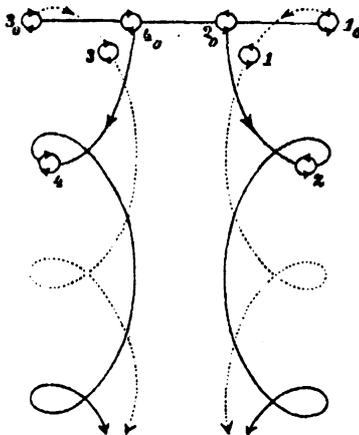


Figure 3: Illustration of “leapfrogging” by two vortex pairs. The induced velocities are indicated by arrows. From [27] based upon Gröbli’s calculations [19, 20].

then repeated indefinitely (in principle – in reality the finite cores of the rings and the effects of viscosity will only allow one or two cycles of this motion). If two vortex rings have equal radii and opposite angular velocities, they will approach each other and widen one another; and when they are very near to one another, their velocity of approach becomes smaller and smaller, and their rate of widening faster and faster. Just as in the case of the straight vortex filament near the plane wall, this motion is similar to the motion of a single vortex ring running up against a plane wall. The image of the ring in the wall is another similar ring with the opposite sense of circulation.

Lanchester saw this type of motion involving several vortices to be relevant to vortex formation behind a wing of finite span. He wrote [36, p. 122]:

Groups of filaments or rings behave in a similar manner to pairs: thus a group of rings may play “leapfrog” collectively so long as the total number of rings does not exceed a certain maximum; congregations of vortex filaments likewise by their mutual interaction move as a part of a concentrated system, like waltzers in a ball-room; when the number exceeds a certain maximum the whole system consists of a number of lesser groups.

Only in rare cases does a single paper put forward so many profound ideas and open so many avenues for further investigation. Almost fifty years later, in 1906, Lord Kelvin, who had himself conducted many great studies developing vortex dynamics further, wrote in the preface to a book about Helmholtz [33] that “his admirable theory of vortex rings is one of the most beautiful of all the beautiful pieces of mathematical work hitherto done in the dynamics of incompressible fluids.” Surprisingly Helmholtz never continued his investigations of the topic established in his groundbreaking paper [23]. Instead he wrote another remarkable paper [24] on discontinuous motion of an inviscid fluid in which he used the notion of a vortex sheet from [23].

## 2.2 Point vortices

A vast area of research started by Helmholtz’s paper is the study of the motion of straight, parallel, infinitely thin vortex filaments (rectilinear vortices) in incompressible inviscid fluid or, equivalently, the two-dimensional problem of point vortices on a plane. Through pioneering work of Rosenhead [62] and Westwater [76] in the 1930’s the discretization of two-dimensional hydrody-

namics provided by such vortex elements became the foundation for an entire family of numerical methods for flow simulation today collectively known as *vortex methods*.

The problem of  $N$  interacting point vortices on the unbounded  $xy$ -plane, with vortex  $\alpha = 1, \dots, N$  having strength  $\Gamma_\alpha$  (which is constant according to Helmholtz's theorems) and position  $(x_\alpha, y_\alpha)$ , consists in solving the following system of  $2N$  first-order, nonlinear, ordinary differential equations

$$\frac{dx_\alpha}{dt} = -\frac{1}{2\pi} \sum_{\beta=1}^N \Gamma_\beta \frac{y_\alpha - y_\beta}{l_{\alpha\beta}^2}, \quad \frac{dy_\alpha}{dt} = \frac{1}{2\pi} \sum_{\beta=1}^N \Gamma_\beta \frac{x_\alpha - x_\beta}{l_{\alpha\beta}^2}, \quad (3)$$

where  $\alpha = 1, 2, \dots, N$ ,  $l_{\alpha\beta} = \sqrt{(x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2}$  is the distance between vortices  $\alpha$  and  $\beta$ , and the prime on the summation indicates omission of the singular term  $\beta = \alpha$ . Typically, an initial value problem is addressed with the initial positions of the vortices and their strengths given so as to capture or model some flow situation of interest.

The system (3) can also be written as  $N$  ODEs for  $N$  complex coordinates  $z_\alpha = x_\alpha + iy_\alpha$

$$\frac{dz_\alpha^*}{dt} = \frac{1}{2\pi i} \sum_{\beta=1}^N \Gamma_\beta \frac{1}{z_\alpha - z_\beta}, \quad \alpha = 1, 2, \dots, N, \quad (4)$$

where the asterisk denotes complex conjugation.

In his lectures [32, Lecture 20] Kirchhoff demonstrated that the system (3) can be cast in Hamilton's canonical form<sup>3</sup>:

$$\Gamma_\alpha \frac{dx_\alpha}{dt} = \frac{\partial H}{\partial y_\alpha}, \quad \Gamma_\alpha \frac{dy_\alpha}{dt} = -\frac{\partial H}{\partial x_\alpha}, \quad \alpha = 1, 2, \dots, N, \quad (5)$$

where the Hamiltonian,

$$H = -\frac{1}{4\pi} \sum_{\alpha, \beta=1}^N \Gamma_\alpha \Gamma_\beta \log l_{\alpha\beta}, \quad (6)$$

is conserved during the motion of the point vortices. (Here and in what follows  $\log$  denotes the natural logarithm.)

In addition to  $H$  the Hamiltonian system (5) has three independent first integrals:

$$Q = \sum_{\alpha=1}^N \Gamma_\alpha x_\alpha, \quad P = \sum_{\alpha=1}^N \Gamma_\alpha y_\alpha, \quad I = \sum_{\alpha=1}^N \Gamma_\alpha (x_\alpha^2 + y_\alpha^2). \quad (7)$$

Regardless of the values of the vortex strengths, the integrals  $H$ ,  $I$ , and  $P^2 + Q^2$  are *in involution*, that is, the Poisson bracket between any two of them is zero; see the review paper [3] or the monograph [50]<sup>4</sup>. According to Liouville's theorem in analytical dynamics the Hamiltonian system (5) for  $N = 3$  is then integrable regardless of the values of the vortex strengths. A terse general statement to this effect was included by Poincaré in his lectures [56, §77].

An extensive analytical study of integrability and of several special cases of three-vortex motion had already been performed by Gröbli in his noteworthy 1877 Göttingen dissertation [19] (later

<sup>3</sup>A complete correspondence follows by setting the "generalized coordinates"  $q_\alpha = x_\alpha$  and the "generalized momenta"  $p_\alpha = \Gamma_\alpha y_\alpha$ . This results in the remarkable insight that the "phase space" – in the sense of Hamiltonian dynamics – for a point vortex system is, in essence, its configuration space, a fact later exploited by Onsager in a seminal paper [54] on the statistical mechanics of a system of point vortices.

<sup>4</sup>This recent monograph also contains a very useful bibliography connecting vortex dynamics and dynamical systems theory.

also published as an extensive paper [20]) that must rightly be considered a classic of the vortex dynamics literature. An account of the life, scientific achievements and tragic death of the Swiss scientist and mathematician Walter Gröbli (1852-1903) may be found in [6].

The solution of the three-vortex problem and the dissertation itself were mentioned in footnotes by Kirchhoff in the third (1883) edition of his lectures [32, Lecture 20, §3] and in the fundamental treatise by Lamb [35, §155] (although in a way that does not fully reveal the comprehensive nature of Gröbli's investigations). Based on these cursory citations it is not difficult to understand that almost a century later Batchelor would write in his important text [9] that the details of motion of three point vortices "are not evident". A lengthy excerpt (in English translation) from Gröbli's dissertation is given in [6].

The Hamiltonian (6) depends only on the mutual distances  $l_{\alpha\beta}$  between the vortices which suggests that one can write equations of motion that involve only these distances. Such equations were obtained by Gröbli and later by Laura [37] who also expounded on the canonical formalism. They are

$$\frac{dl_{\alpha\beta}^2}{dt} = \frac{2}{\pi} \sum_{\lambda=1}^N \prime\prime \Gamma_{\lambda} \epsilon_{\alpha\beta\lambda} A_{\alpha\beta\lambda} \left( \frac{1}{l_{\beta\lambda}^2} - \frac{1}{l_{\lambda\alpha}^2} \right), \quad \alpha, \beta = 1, 2, \dots, N, \quad (8)$$

where the two primes on the summation sign now mean that  $\lambda \neq \alpha, \beta$ . The quantity  $\epsilon_{\alpha\beta\lambda} = +1$  if vortices  $\alpha, \beta$  and  $\lambda$  appear counterclockwise in the plane, and  $\epsilon_{\alpha\beta\lambda} = -1$  if they appear clockwise. Finally,  $A_{\alpha\beta\lambda}$  is the area of the vortex triangle  $\alpha\beta\lambda$  which can, in turn, be expressed in terms of the three vortex separations (the sides of the vortex triangle) by Hero's formula. Interestingly, Eqs.(8) were re-discovered independently at least twice: by Sygne [69] in 1949 and by Novikov [53] in 1975. For  $N$  vortices one has  $\frac{1}{2}N(N-1)$  quantities  $l_{\alpha\beta}$  and, thus,  $\frac{1}{2}N(N-1)$  equations of the form (8). However, only  $2N-3$  of these are independent.

It can be shown that

$$\frac{1}{2} \sum_{\alpha, \beta=1}^N \Gamma_{\alpha} \Gamma_{\beta} l_{\alpha\beta}^2 = \left( \sum_{\alpha=1}^N \Gamma_{\alpha} \right) I - P^2 - Q^2. \quad (9)$$

The equations (8), then, have two general first integrals, viz the Hamiltonian (6) and the quantity on the left hand side of (9). Using these two integrals the three ODEs for  $l_{12}$ ,  $l_{23}$  and  $l_{31}$  may be reduced to a single ODE that can be solved by quadrature, and this was, in essence, the solution method outlined by Gröbli in his dissertation [19, 20]. The case  $N=3$  thus appears as a critical one since for more vortices additional "scales of motion" appear without any obvious integrals to constrain them. One may, therefore, expect the problem to become non-integrable. Indeed, this is what happens and the connection to the recent interest in the emergence of chaos in nonlinear dynamics is established. The appearance of chaos in point vortex dynamics as one goes from three to four vortices is analogous to the appearance of chaos in the gravitational  $N$ -body problem of celestial mechanics as one goes from two to three bodies. For the case of point masses the appearance of chaos or the absence of integrability became part of the legacy of Poincaré. For inexplicable reasons the analogous discussion for point vortices had to wait for more than a century after the solution of the three-vortex problem. Both Gröbli [19, 20] and later Laura [37] outlined how to determine the "absolute motion" of the vortices provided the solution for the "relative motion" as given by equations (8) was already known.

The nature of the motion of two vortices had already been outlined by Helmholtz [23]. The motion of three vortices – both the relative and the absolute motion – with various intensities and initial conditions was extensively analyzed by Gröbli [19, 20]. The relative motion of three

arbitrary vortices, based upon Eqs.(8), was studied and classified by Synge [69] by introducing triangular coordinates in a “phase space” of the three distances between the vortices. Gröbli had actually found such a representation for the case of three identical vortices, and this construction was found independently a century later by Novikov [53]. Synge’s comprehensive analysis was re-discovered independently in [2]. Thus, today the three-vortex problem may be considered to have a rather complete solution. Gröbli [19,20] also discovered an unusual case where the three vortices converge on a point in a *finite time*. Except for Synge’s study [69], which was itself overlooked, this intriguing case of *vortex collapse* also went unnoticed for a century. It is admittedly a somewhat special case requiring both that the harmonic mean of the three vortex strengths be zero and that the integral of motion (9) vanish.

The integrable problem of four vortices arranged as two coaxial pairs has been addressed in many papers. Gröbli [19, 20] investigated the case of “leapfrogging” when all vortices have the same absolute strength, and obtained an analytical representation for the vortex trajectories, cf. Fig.3. His analysis was repeated independently by Love [40] and Hicks [26].

The case of uniform rotation of a regular polygon of  $N$  vortices was addressed in the Adams Prize essay of J. J. Thomson [71]. He proved that the regular  $N$ -gon is stable to infinitesimal perturbations for  $N = 2, 3, 4, 5, 6$  but becomes unstable for  $N > 7$ . (For  $N = 7$  the polygon is marginally stable to linear order and one must go to the next order to decide the stability issue.) This study was extended by Havelock [22] and others, and the problem continues to be addressed in the literature in various forms. See the recent review [5] and also the extension to “triple rings” by Aref & van Buren [7].

Helmholtz was also the first to address problems of point vortices interacting with rigid boundaries [23]. As we have seen, he considered the case of a point vortex in the space bounded by a plane wall. Using an “image” vortex of opposite strength situated at the reflection of the original vortex in the plane boundary he reduced the problem to that of the motion of a vortex pair on the unbounded plane. This use of the “method of images” has since been widely employed in various problems of the motion of a single point vortex in various bounded domains. A particular case of an equilibrium of a vortex pair behind a cylinder in a uniform potential flow is known as the “Föppl problem” after the seminal paper [17].

The general case of the motion of point vortices in an arbitrary domain was studied by Routh [63] using the theory of conformal mappings. The velocity of a point vortex in the transformed plane is not equal to the velocity obtained by simple substitution of the conformal mapping into the expression for the velocity in the original plane – one requires also the influence of the images which is captured by the so-called “Routh correction”. A complete mathematical theory was developed by Lin [39] who showed that the problem is always Hamiltonian with a Hamiltonian function that is a hybrid of Kirchhoff’s Hamiltonian (6) for the unbounded plane and the Hamiltonian that Routh found for motion of a single vortex in a bounded domain [63].

W. Thomson [72] was the first to show that a vortex pair in steady motion on the unbounded plane is accompanied by an “atmosphere”, i.e., a fixed, closed volume (area) of fluid particles that move forward with the vortex pair. The bounding curve of this “atmosphere” is today sometimes called the “Kelvin oval”. Figure 4 reproduces the original drawing from [72] where we find this description:

The diagram represents precisely the convex outline referred to, and the lines of motion of the interior fluid carried along by the vortex, for the case of a double vortex consisting of two infinitely long, parallel, straight vortices of equal rotations in opposite directions. The curves have been drawn by Mr. D. McFarlane, from calculations which he has performed by means of the equation of the system of curves, which

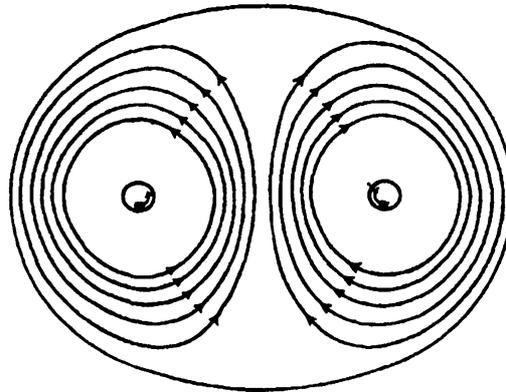


Figure 4: The “atmosphere” traveling with a vortex pair. From [72].

is

$$\frac{y^2}{a^2} = \frac{2x}{a} \frac{N+1}{N-1} - \left(1 + \frac{x^2}{a^2}\right), \text{ where } \log N = \frac{x+b}{a}.$$

The motion of the surrounding fluid must be precisely the same as it would be if the space within this surface were occupied by a smooth solid.

Each passive fluid particle may be considered “a point vortex of zero strength”, and the equations of motion for all particles advected by the translating vortex system are integrable. The deformation of a line of fluid connecting two vortices within the moving body was studied analytically by Riecke [60]; see [48] for additional illustrations.

### 2.3 Vortex atoms

In the 1860’s William Thomson, later Lord Kelvin, became very interested in vortex dynamics since he was convinced that atoms were to be modeled as vortex configurations in the aether. Tait made a complete English translation of Helmholtz’s paper [23] for his own use. He also devised some extremely clever experiments to illustrate the vortex theory using smoke vortex rings in air. Following completion of their famous *Treatise on Natural Philosophy*, referred to simply as “Thomson and Tait”, and the successful laying of the Atlantic cable in 1866 (for which Thomson was knighted and became Sir William Thomson), Thomson visited Tait in Edinburgh in mid-January 1867 and saw the smoke rings with his own eyes. Tait’s translation of Helmholtz’s paper was published that same year in *Philosophical Magazine*. One must imagine that Kelvin encouraged his friend and colleague to prepare this translation for publication.

Thomson’s prodigious talent produced several first rate studies of vortex dynamics which, although ultimately wrong-headed in terms of atomic physics, have had a lasting influence on fluid dynamics. The idea of *circulation*, for example, is from this period. The circulation is defined as the contour integral of the projection of the flow velocity on the tangent to the contour,

$$\Gamma = \oint_C \mathbf{V} \cdot d\mathbf{s}. \quad (10)$$

He showed that for any material contour moving according to Euler’s equation for incompressible flow, the circulation is an integral of the motion, a result known today as *Kelvin’s circulation theorem*. This theorem was considered by Einstein [15] among the most important scientific results of

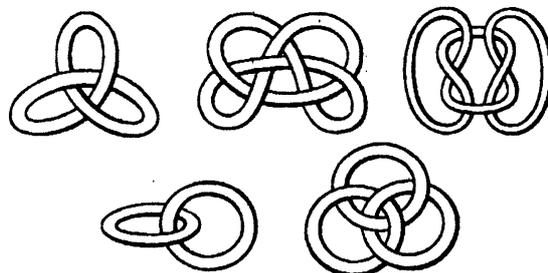


Figure 5: Tait's drawings, reproduced in [74], that capture Thomson's ideas on how atoms arise as vortex structures in the aether.

W. Thomson (Lord Kelvin). This profound insight has continued to exert an influence on the entire field of fluid mechanics, including in such areas as the assessment of the accuracy of numerical methods and in turbulence modeling. Circulation is a distinctly topological entity, independent of the shape of the vortex and measurable by integration along any circuit that loops around the vortex. In this sense, the notion of circulation may be taken as one of the earliest introductions of topological considerations into fluid mechanics. Tait's seminal work on the classification of knots on closed curves is a spin-off of his interest in vortex atoms. It has stood the test of time and is today recognized as an important contribution to topology, knot theory and graph theory. Maxwell was an important catalyst for Tait's work on knots, since he had also become interested in topological ideas. Today the intersection of fluid mechanics and topology, in its multiple forms, has matured into a subfield often referred to as topological fluid dynamics. The permanence of circulation in an ideal fluid was one of the cornerstones of vortex atom theory. Like atoms, vortices in the aether could neither be created nor destroyed.

Thomson's fascination with the floating magnet experiments by Mayer, e.g., [45, 46], and his role in the re-publication of these works in journals such as *Nature* and *Philosophical Magazine*, were also outgrowths of his conviction that vortices and atoms are intimately related. See Snelders' article [66] for a comprehensive historical review of this topic. The famous quote from Thomson that "Helmholtz's [vortex] rings are the only true atoms" summarizes the theme of this research thrust. Figure 5 depicts the kind of things he envisioned.

Although it ultimately faded, the vortex atom idea maintained itself for many years and through Kelvin's boundless energy and great influence spread widely in the scientific community. The extensive work by J. J. Thomson, discoverer of the electron, on vortex dynamics in was stimulated by vortex atom theory. Even in his great paper of 1897 entitled "Cathode Rays", in which the discovery of the electron is announced, we find these remarks: "If we regard the system of magnets as a model of an atom, the number of magnets being proportional to the atomic weight, ... we should have something quite analogous to the periodic law...", where by "periodic law" he means the periodic table of the elements. The reference to the floating magnets is to Mayer's experiments mentioned above. We see what a profound role these demonstration experiments played in the thinking of these great scientists. We should not forget that at the time analog experiments were one of the only ways of exploring solutions to nonlinear equations that did not easily yield to analytical methods. Computers and numerical solutions were still many years in the future.

The vortex atom theory received considerable attention from Maxwell [44] in his article on the

atom written for the 1878 edition of *Encyclopedia Britannica*. He provided a detailed description of properties of vortices in ideal fluid and strongly supported the idea of vortex atoms. Apparently, he was reminded of his own earlier articles in which his celebrated electromagnetic theory was initially formulated based upon a mechanical model that also made reference to Helmholtz's paper [23].

**2.5 Vortex rings** In spite of the great popularity of Tait's [70, pp. 291-294] smoke box for generating vortex rings in air, the first observation of vortex rings probably corresponds with the introduction of smoking tobacco! Northrup [51, p.211] writes:

It is not improbable that the first observer of vortex motions was Sir Walter Raleigh; if popular tradition may be credited regarding his use of tobacco, and probably few smokers since his day have failed to observe the curiously persistent forms of white rings of tobacco smoke which they delight to make. But some two hundred eighty years went by, after the romantic days of Raleigh and Sir Francis Drake, who made tobacco popular in England, before a scientific explanation of smoke rings was attempted.

Edwin Fitch Northrup (1866-1940) was a professor of physics at Princeton and author of a science fiction book entitled "Zero to Eighty: Being my Lifetime Doings, Reflections, and Inventions; also my Journey around the Moon." The book was published in Princeton in 1937 under the pseudonym Akkad Pseudoman. It gives a fictional account, supported by valid scientific data, of a Morris County resident's trip around the moon. It appears to have a sustained following in the world of science fiction.

By curious coincidence the first experimental observations of the generation of vortex rings in air were performed by Rogers [61] in the same year (1858) that Helmholtz published his seminal paper [23]. William Barton Rogers (1804-1882) will be better known today as the founder of MIT. Indeed, he was heavily engaged in this enterprise at about the time his paper on vortex rings was written.

The extensive study by Northrup [51, 52] should also be mentioned here. It contains a very detailed description of a "vortex gun", including all the parameters, together with beautiful photos of interacting vortex rings and vortex rings interacting with rigid obstacles, e.g., with a small watch chain. The modern reader may be intrigued to see in these near-century old papers an essentially contemporary elucidation of the interaction of two circular vortex rings tilted towards one another so as to interact after having propagated for some distance, cf. Fig.6.

Theoretical studies of the motion of a circular vortex ring of closed toroidal shape with core radius  $a$  and radius  $R$  of the center line of the torus, where  $a \ll R$ , in an ideal fluid led to a formula for the self-induced translational velocity  $V_{ring}$ , directed normally to the plane of the ring:

$$V_{ring} = \frac{\Gamma}{4\pi R} \left( \log \frac{8R}{a} - C \right) + O(a/R)$$

Here  $\Gamma$  is the (constant) intensity of the vortex ring, equal to the circulation along any closed path around the vortex core, and  $C$  is a constant. There was some disagreement in the literature concerning the value of  $C$ . The value  $C = \frac{1}{4}$  was given (without proof) by W. Thomson [73] and later by Hicks, Basset, Dyson and Gray. This corresponds to the case where the vorticity inside the core varies directly as the distance from the centerline of the ring. The value  $C = 1$  was given by Lewis [38], J. J. Thomson [71], Chree, Joukovskii, and Lichtenstein for a uniform distribution of vorticity within the core. For a hollow vortex core, or if one assumes the fluid inside the core is stagnant, the value  $C = \frac{1}{2}$  results [?]. The review by Shariff and Leonard [65] on vortex ring dynamics traces further evolution of this intriguing subject.

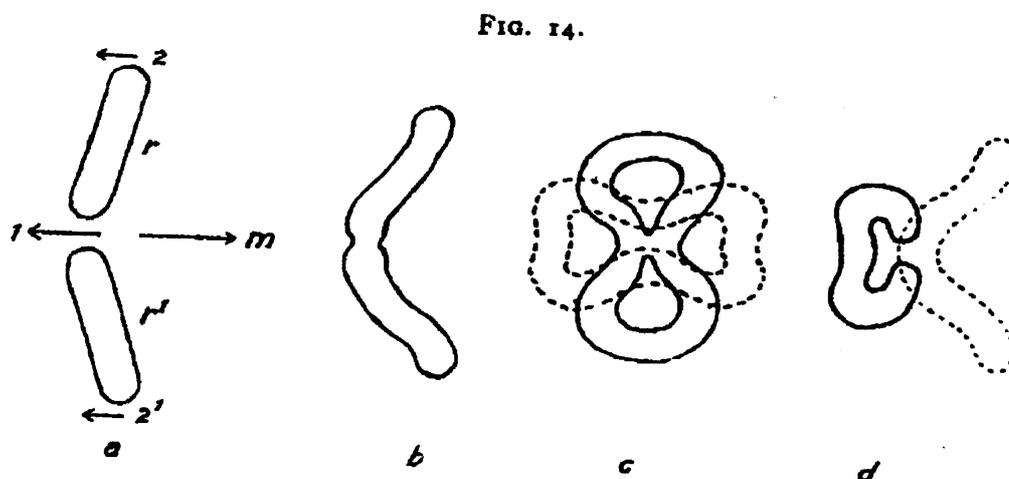


Figure 6: Sketch of interaction of two identical vortex rings launched on a collision course. From Northrup [52].

### 2.5 Vortex streets

Most students of fluid mechanics know that the common staggered array of vortices that forms in the wake of a cylinder (or any bluff body) is called the *Kármán vortex street*. The concept of the vortex street is among the best known in all of fluid mechanics, in the same “league” as *Reynolds number*, *Bernoulli’s equation* and the concept of the *boundary layer*. The formation and structure of vortex wakes downstream of bluff bodies had been studied extensively in experiments going back to Leonardo da Vinci, but von Kármán’s theory was the first real analysis of the phenomenon. In his charming book [30] he explains that his interest was aroused by an early picture of such vortices in a fresco in one of the churches in Bologna, Italy, where St. Christopher is shown carrying the child Jesus across a flowing stream. Alternating vortices are seen behind the saint’s foot; see [49] for a beautiful color picture of this fresco at the Church of St Dominic, entitled *Madonna con bambino tra I Santi Domenico, Pietro Martire e Critoforo*, painted by an unknown artist of the fourteenth century.

Alternating vortices in air were observed and imaged by the English scientist Mallock [41,42] while impressive photos of such vortices in water were obtained by the German scientist Ahlborn [1]. The French scientist Bénard [10] also observed the alternating formation of detached vortices on the two sides of a bluff obstacle in water and later in many viscous fluids and in colloidal solutions.

Analysis shows that only two such configurations will propagate in the streamwise direction: The vortices must either be arranged in a symmetric or in a staggered configuration. Numerically the intensities of the vortices,  $\Gamma$ , are all equal, but the vortices on the two horizontal rows have opposite signs. In remarkable theoretical investigations [28,29] von Kármán examined the question of stability of such processions in unbounded, incompressible, inviscid, two-dimensional flow with embedded point vortices. He became interested in this problem when he was appointed as a graduate assistant in Göttingen in Prandtl’s laboratory in 1911. Prandtl had a doctoral candidate, K. Hiemenz, to whom he had given the task of constructing a water channel in order to observe the separation of the flow behind a cylinder. Much to his surprise, Hiemenz found that the flow in his channel oscillated violently, and he failed to achieve symmetrical flow about a circular cylinder.

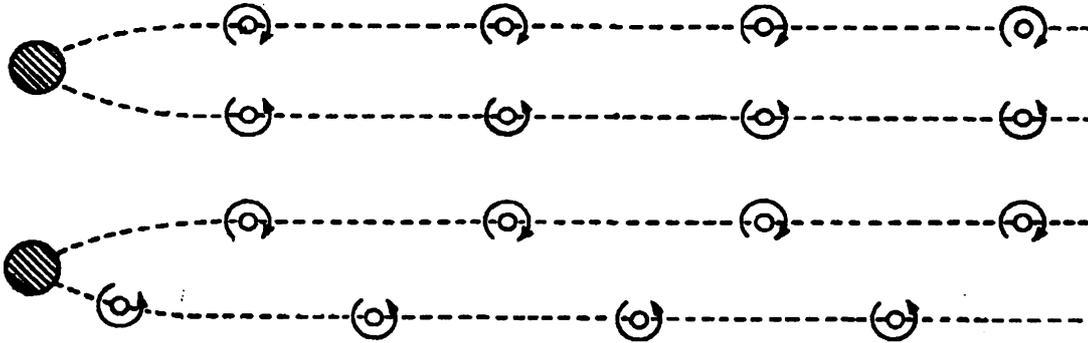


Figure 7: Schematic of a symmetric and a staggered vortex street downstream of a bluff body. From [30].

Von Kármán addressed the model problem of two infinite rows of point vortices and derived a criterion for when such a configuration is not unstable to *linearized* perturbations. He showed that the symmetric configuration, cf. Fig.7, is always unstable and that the staggered configuration is also unstable unless the spacing between successive vortices in either row and the distance between the rows has a definite ratio.

If the spacing between successive vortices in the same row is called  $l$ , and if the distance between the two parallel rows is called  $h$ , von Kármán's criterion [29] is

$$\cosh \frac{\pi h}{l} = \sqrt{2}, \quad \text{or} \quad \frac{h}{l} = 0.283. \quad (11)$$

The velocity,  $U$ , of horizontal translation of the infinite rows is found to be

$$U = \frac{\Gamma}{2l} \tanh \frac{\pi h}{l}. \quad (12)$$

This is today very well known. What is probably less well known is that in the original paper [28] von Kármán found the criterion (11) with  $\sqrt{3}$  (or  $h/l = 0.365$ ) on the right hand side rather than (the correct)  $\sqrt{2}$ , which was confirmed subsequently in [31, 64] (with reference to the then newly created theory of an infinite system of linear differential equations due to O. Toeplitz in 1907) and by Lord Rayleigh [57]. The original drawings [31] of the streamlines in a coordinate system moving steadily with the vortices are reproduced in Fig.8. (When the ratio  $h/l$  is given by Eq.(11), the propagation speed of the street,  $U$ , in Eq.(12) is  $\Gamma/l\sqrt{8}$ .)

The erroneous value was, for example, later used by Synge [68] in his re-derivation of the Kármán drag formula, although the analysis is easily corrected.

Von Kármán's analysis precipitated huge amounts of work, both experimental, analytical – and much later – numerical. On July 18, 1922, a young W. Heisenberg, then a student of A. Sommerfeld at the Institute for Theoretical Physics at University of München, submitted an article [?] in which he tried to define an absolute size of the Kármán vortex street behind a flat plate of width  $d$  placed perpendicularly to the oncoming flow of velocity  $U_\infty$  far upstream. Based on physical arguments he arrived at the numerical values  $l/d = 5.45$  and  $h/d = 1.54$ . These values fit von Kármán's second value for the ratio of width to intra-row spacing,  $h/l = 0.283$  and the ratio of the speed of propagation of the row relative to the flow speed at infinity,  $U/U_\infty = 0.229$ . Heisenberg's thoughts at this time were already turning to other topics, most notably the creation

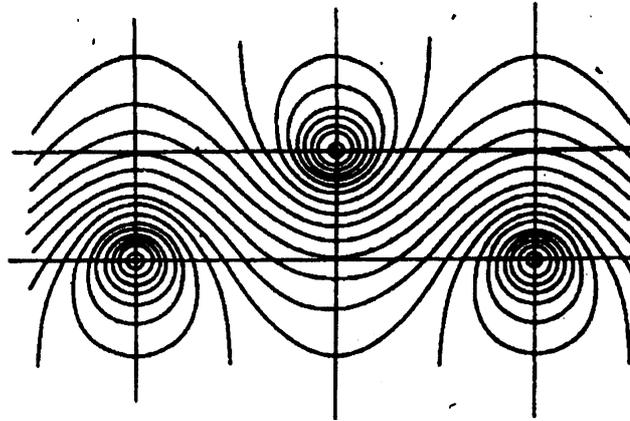


Figure 8: Streamlines of a vortex street in the co-moving frame based on the point vortex model [31].

of a new “matrix” version of quantum mechanics for which he was to receive the Nobel prize in 1933 together with E. Schrödinger and P. A. M. Dirac. Nevertheless, his doctoral dissertation, completed in July 1923, was on hydrodynamics, in particular stability theory and turbulence, and he would return briefly to the topic of fully developed turbulence in the period following World War II.

The necessary condition for absence of linear instability was generalized to vortex streets moving obliquely to the direction of the “free stream” by Dolaptschiew and Maue. While the paper by Maue [43] will probably be familiar, in part because this work was highlighted in the well-known lectures of Sommerfeld [67], the extensive work of Dolaptschiew is less well known than it ought to be. Insofar as assimilation into the literature in the West is concerned, the situation was not helped by several of Dolaptschiew’s papers being published in Bulgarian and Russian, albeit usually with an abstract or summary in German.

### 3. Conclusion

History, to paraphrase Leibnitz, is a useful thing, for its study not only gives to the researchers of the past their just due but also provides those of the present with a guide for the orientation of their own endeavors. While Helmholtz’s 1858 paper on vortex dynamics and vorticity is of great importance and spawned the new subfield of vortex dynamics, one must admit that in the greater scheme of things Helmholtz is today primarily remembered for other contributions to science. There are several individuals who would not today be immediately associated with the field of vortex dynamics, since they did work in other fields – often well outside fluid mechanics – that became of even greater importance. We may list Dirichlet, Friedmann, Hankel, Heisenberg, Klein, Lin, Love, J. J. Thomson, Zermelo and probably even Lord Kelvin.

In the “case studies” in Section II we have focused on what one may call the classical applications of Helmholtz’s vortex theory. It is the test of any significant advance that it elicits interest far beyond the boundaries anticipated by its creator. Thus, the importance of vortex dynamics was realized in meteorology and oceanography by such towering figures as Vilhelm Bjerknes whose seminal work [12] bears the title “On the dynamics of the circular vortex: with applications to the atmosphere and atmospheric vortex and wave motions.” At the other end of the size-scale spectrum we may cite the application of classical vortex dynamics to superfluid Helium [16], where

the famous footnote in [54] announced the quantization of circulation in this case. As one surveys the now vast literature in vortex dynamics some 150 years after Helmholtz's paper one is struck by the richness of the subject matter, and by how the various aspects enter different applications in almost infinitely varied ways. Küchemann's figurative characterization of vortices as "the sinews and muscles of fluid motions" [34] is no less apt today than it was when it was written 40 years ago.

**Acknowledgments** The author acknowledges support of a Niels Bohr Visiting Professorship at the Danish Technical University provided by the Danish National Research Foundation. This paper is largely based on the joint paper with V. V. Meleshko [47]. I thank Slava for sharing his many insights into the history and practice of vortex dynamics.

## References

- [1] F. Ahlborn 1902 Über den Mechanismus des hydrodynamischen Widerstandes, *Abh. Geb. Naturw. Hamburg* **17**, 1–57.
- [2] H. Aref 1979 Motion of three vortices. *Phys. Fluids* **22**, 393–400.
- [3] H. Aref 1983 Integrable, chaotic and turbulent vortex motion in two-dimensional flows. *Annu. Rev. Fluid Mech.* **15**, 345–389.
- [4] H. Aref 2002 The development of chaotic advection. *Phys. Fluids* **14**, 1315–1325.
- [5] H. Aref, P. K. Newton, M. A. Stremler, T. Tokieda & D. L. Vainchtein 2002 Vortex crystals, *Adv. Appl. Mech.* **39**, 1–79.
- [6] H. Aref, N. Rott & H. Thomann 1992 Gröbli's solution of the three-vortex problem. *Annu. Rev. Fluid Mech.* **24**, 1–20.
- [7] H. Aref & M. & Buren 2005 Vortex triple rings. *Physics of Fluids* **17**, 057104, 21pp.
- [8] R. J. Arms & F. R. Hama 1965 Localized-induction concept on a curved vortex and motion of an elliptic vortex ring. *Phys. Fluids* **8**, 553–559.
- [9] G. K. Batchelor 1967 *An Introduction to Fluid Dynamics*. (Cambridge University Press)
- [10] H. Bénard 1908 Formation de centres de giration à l'arrière d'un obstacle en mouvement, *C. R. Acad. Sci. Paris* **147**, 839–842, 970–972.
- [11] R. Betchov 1965 On the curvature and torsion of an isolated vortex filament. *J. Fluid Mech.* **22**, 471–479.
- [12] V. Bjerknes 1921 On the dynamics of the circular vortex: with applications to the atmosphere and atmospheric vortex and wave motions. *Geofys. Publ.* **2**, No 4.
- [13] L. S. Da Rios 1906 Sul moto d'un liquido indefinito con un filetto vorticoso di forma qualunque, *Rend. Circ. Mat. Palermo* **22**, 117–135.
- [14] O. Darrigol 1998 From organ pipes to atmospheric motion: Helmholtz on fluid mechanics, *Hist. Stud. Phys. Sci.* **29**, 1–51.
- [15] A. Einstein 1924 Zum hundertjährigen Gedenktage von Lord Kelvin's Geburt. (26. Juni 1824), *Naturwissenschaften* **12**, 601–602.
- [16] R. P. Feynman 1955 Application of quantum mechanics to liquid Helium, In: C.J. Gorter, ed., *Progress in Low Temperature Physics* (North Holland, Amsterdam) **1** 17–53.
- [17] L. Föppl 1913 Wirbelbewegung hinter einem Kreiszyylinder, *Sber. Bayer. Akad. Wiss.* **43**, 1–17.

## 150 Years of Vortex Dynamics

- [18] O. Velasco Fuentes 2007 On the topology of vortex lines and tubes. *J. Fluid Mech.* **584**, 147–156.
- [19] W. Gröbli, 1877a *Spezielle Probleme über die Bewegung geradliniger paralleler Wirbelfäden* (Zürcher und Furrer, Zürich).
- [20] W. Gröbli, 1877b *Spezielle Probleme über die Bewegung geradliniger paralleler Wirbelfäden*, *Vierteljschr. Naturf. Ges. Zürich* **22** 37–82, 129–168.
- [21] H. Hasimoto, A soliton on a vortex filament, *J. Fluid Mech.* **51** (1972) 477–485.
- [22] T. H. Havelock 1931 Stability of motion of rectilinear vortices in ring formation, *Phil. Mag. (ser. 7)* **11**, 617–633.
- [23] H. Helmholtz 1858 Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen. *J. Reine Angew. Math.* **55** (1858) 25–55. English translation by P. G. Tait: *Phil. Mag. (ser. 4)* **33**, 485–510.
- [24] H. Helmholtz, 1868 Ueber discontinuirliche Flüssigkeitsbewegungen. *Monatsber. Akad. Wiss. Berlin* (1868) 215–228. English translation: On discontinuous movements of fluids, *Phil. Mag. (ser. 4)* **36** (1868) 337–346.
- [25] H. von Helmholtz 1892 Autobiographisches Tischrede bei der Feier des 70. Geburtstages. In: *Ansprachen und Reden, gehalten bei der am 2. November 1891 zu Ehren von Hermann von Helmholtz veranstalteten Feier* (Hirschwald, Berlin) 46–59. English translation: An autobiographical sketch, In *Selected Writings of Hermann von Helmholtz*, R. Kahl, ed. (Wesleyan University Press, Middletown) 466–478.
- [26] W. M. Hicks 1922 On the mutual threading of vortex rings, *Proc. R. Soc. London A* **102**, 111–131.
- [27] N. E. Joukovskii 1892 [Helmholtz's] works on mechanics. In *Hermann von Helmholtz 1821-1891. Public lectures delivered at the Imperial Moscow University for the Helmholtz fund*. A.G. Stoletov, ed., (Moscow University Press, Moscow) 37–52 (in Russian).
- [28] Th. von Kármán 1911 Über den Mechanismus des Widerstandes, den ein bewegter Körper in einer Flüssigkeit erfährt. 1. Teil, *Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl.* 509–517.
- [29] Th. von Kármán 1912 Über den Mechanismus des Widerstandes, den ein bewegter Körper in einer Flüssigkeit erfährt. 2. Teil, *Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl.* 547–556.
- [30] Th. von Kármán 1954 *Aerodynamics. Selected Topics in the Light of their Historical Development* (Cornell University Press, Ithaca).
- [31] Th. von Kármán & H. Rubach 1912 Über den Mechanismus des Flüssigkeits- und Luftwiderstandes, *Phys. Z.* **13**, 49–59.
- [32] G. Kirchhoff 1876 *Vorlesungen über mathematische Physik. Mechanik*. (Teubner, Leipzig) Also: 2nd edn (1877); 3rd edn (1883); 4th edn (1897).
- [33] L. Koenigsberger 1906 *Hermann von Helmholtz* (Clarendon Press, Oxford; Reprinted by Dover Publications, New York, 1965)
- [34] D. Küchemann 1965 Report on the I.U.T.A.M. Symposium on concentrated vortex motions in fluids, *J. Fluid Mech.* **21**, 1–20.
- [35] H. Lamb 1932 *Hydrodynamics* 6th edn (Cambridge University Press)
- [36] F. W. Lanchester, *Aerodynamics. Constituting the first volume of a complete Work on Aerial Flight* (Constable & Co., London, 1907).
- [37] E. Laura 1905 Sulle equazioni differenziali canoniche del moto di un sistema di vortici elementari, rettilinei e paralleli, in un fluido incompressibile indefinito, *Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat.* **40**, 296–312.
- [38] T. C. Lewis, On the images of vortices in a spherical vessel, *Quart. J. Pure Appl. Math.* **16** (1879) 338–347.

- [39] C. C. Lin 1943 *On the Motion of Vortices in Two Dimensions* (University of Toronto Press, Toronto).
- [40] A. E. H. Love 1894 On the motion of paired vortices with a common axis, *Proc. Lond. Math. Soc.* **25** 185–194.
- [41] A. Mallock 1907 On the resistance of air, *Proc. R. Soc. London A* **79** 262–273.
- [42] A. Mallock 1911 Influence of viscosity on the stability of the flow of fluids, *Proc. R. Soc. London A* **84** 482–491.
- [43] A. W. Maue 1940 Zur Stabilität der Kármánschen Wirbelstraße, *Z. Angew. Math. Mech.* **20**, 129–137.
- [44] J. C. Maxwell 1878 Atom, In: *Encyclopædia Britannica* 9th edn (London) **3** 36–48.
- [45] A. M. Mayer 1878a A note on experiments with floating magnets; showing the motions and arrangements in a plane of freely moving bodies, acted on by forces of attraction and repulsion; and serving in the study of the directions and motions of lines of magnetic force, *Am. J. Sci.* (ser. 3) **15**, 276–277.
- [46] A. M. Mayer 1878b Floating magnets, *Nature* **17**, 487–488.
- [47] V. V. Meleshko & H. Aref 2007 A bibliography of vortex dynamics 1858–1956. *Adv. Appl. Mech.* **41**, 197–292.
- [48] V. V. Meleshko & G. J. F. van Heijst 1994 Interacting two-dimensional vortex structures: Point vortices, contour kinematics and stirring properties, In *Chaos Applied to Fluid Mixing*, H. Aref and M. S. El Naschie, eds, (Oxford, Pergamon) 233–267.
- [49] T. Mizota, M. Zdravkovich, K.-U. Graw & A. Leder 2000 St. Christopher and the vortex. *Nature* **404**, 226.
- [50] P. K. Newton 2001 *The N-vortex Problem: Analytical Techniques. Applied Mathematical Sciences, vol. 145.* (Springer, New York)
- [51] E. F. Northrup 1911 An experimental study of vortex motions in liquids. *J. Franklin Inst.* **172**, 211–226, 345–368. Also: *Scient. Am. Suppl.* **74** (1912) 204–205, 215, 238–239, 248–249.
- [52] E. F. Northrup 1912 A photographic study of vortex rings in liquids. *Nature* **88**, 463–468.
- [53] E. A. Novikov 1975 Dynamics and statistics of a system of vortices. *ZhETF* **68** 1868–1882. English translation in: *Soviet Phys.-JETP* **41** (1975) 937–943.
- [54] L. Onsager 1949 Statistical hydrodynamics. *Nuovo Cimento* **6** (Suppl.) 279–287.
- [55] H. C. Pocklington 1895 The configuration of a pair of equal and opposite hollow straight vortices, of finite cross-section, moving steadily through fluid. *Proc. Camb. Phil. Soc.* **8**, 178–187.
- [56] H. Poincaré 1893 *Théorie des tourbillons* (Carré, Paris).
- [57] Lord Rayleigh 1915 Æolian tones, *Phil. Mag.* (ser. 6) **29**, 433–444.
- [58] R. L. Ricca, Rediscovery of Da Rios equations, *Nature* **352** (1991) 561–562.
- [59] R. L. Ricca, The contributions of Da Rios and Levi-Civita to asymptotic potential theory and vortex filament dynamics, *Fluid Dyn. Res.* **18** (1996) 245–268.
- [60] E. Riecke 1888 Beiträge zur Hydrodynamik, *Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl.* (1888) 347–357.
- [61] W. B. Rogers 1858 On the formation of rotating rings by air and liquids under certain conditions of discharge, *Am. J. Sci.* (ser. 2) **26**, 246–258.
- [62] L. Rosenhead 1931 The formation of vortices from a surface of discontinuity, *Proc. R. Soc. London A* **134**, 170–192.
- [63] E. J. Routh 1881 Some applications of conjugate functions, *Proc. Lond. Math. Soc.* **12**, 73–89.
- [64] H. L. Rubach 1914 *Ueber die Entstehung und Fortbewegung des Wirbelpaares hinter zylindrischen Körpern* (dissertation, Göttingen).

## 150 Years of Vortex Dynamics

- [65] K. Shariff & A. Leonard 1992 Vortex rings, *Annu. Rev. Fluid Mech.* **24**, 235-279.
- [66] H. A. M. Snelders 1976 A. M. Mayer's experiments with floating magnets and their use in the atomic theories of matter, *Ann. Sci.* **33**, 67-80.
- [67] A. Sommerfeld 1943 *Vorlesungen über theoretische Physik. 2. Mechanik der deformierbaren Medien* (Becker & Erler, Leipzig, 1943). English translation: *Lectures on theoretical Physics, 2. Mechanics of Deformable Bodies* (Academic Press, New York).
- [68] J. L. Synge 1927 Mathematical investigation of the thrust experienced by a cylinder in a current, the motion being periodic, *Proc. R. Irish Acad.* **37**, 95-109.
- [69] J. L. Synge 1949 On the motion of three vortices, *Can. J. Math.* **1** 257-270.
- [70] P. G. Tait 1876 *Recent Advances in Physical Sciences* 2nd edn (Macmillan, London).
- [71] J. J. Thomson 1883 *A Treatise on the Motion of Vortex Rings: an essay to which the Adams prize was adjudged in 1882, in the University of Cambridge* (Macmillan, London). Reprinted: (Dawsons of Pall Mall, London, 1968).
- [72] W. Thomson 1867a On vortex atoms, *Proc. R. Soc. Edinb.* **6** (1867) 94-105. Also: *Phil. Mag. (ser. 4)* **34** (1867) 15-24. Also: *Proc. Glasgow Phil. Soc.* **6** (1868) 197-208.
- [73] W. Thomson 1867b The translatory velocity of a circular vortex ring, *Phil. Mag. (ser. 4)* **33** (1867) 511-512.
- [74] W. Thomson 1869 On vortex motion, *Trans. R. Soc. Edinb.* **25**, 217-260.
- [75] C. Truesdell 1954 *The Kinematics of Vorticity*. (Indiana University Press, Bloomington)
- [76] F. L. Westwater 1935 The rolling up of the surface of discontinuity behind an aerofoil of finite span. *Rep. Memo. Advis. Comm. Aeronaut.* (1935) No 1692. Also: *Tech. Rep. Advis. Comm. Aeronaut.* (1935/36) 116-131.