数理解析研究所講究録
第 1643 巻 2009 年 27-37

VECTOR-VALUED WEAKLY ALMOST PERIODIC FUNCTIONS AND MEAN ERGODIC THEOREMS IN BANACH SPACES

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1. INTRODUCTION

Let C be a closed and convex subset of a real Banach space. Then a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. In 1975, Baillon [3] originally proved the first nonlinear ergodic theorem in the framework of Hilbert spaces: Let C be a closed and convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, we have that P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and Px is contained in the closure of convex hull of $\{T^n x : n = 1, 2, \dots\}$ for each $x \in C$. We call such a retraction *"an ergodic retraction"*. In 1981, Takahashi [31, 33] proved the existence of ergodic retractions for amenable semigroups of nonexpansive mappings on Hilbert spaces. Rodé [26] also found a sequence of means on a semigroup, generalizing the Cesàro means, and extended Baillon's theorem. These results were extended to a uniformly convex Banach space whose norm is Fréchet differentiable in the case of commutative semigroups of nonexpansive mappings by Hirano, Kido and Takahashi [13]. In 1999, Lau, Shioji and Takahashi [18] extended Takahashi's result and Rodé's result to amenable semigroups of nonexpansive mappings in the Banach space.

By using Rodé's method, Kido and Takahashi [15] also proved a mean ergodic theorem for noncommutative semigroups of linear bounded operators in Banach spaces.

On the other hand, Edelstein [11] studied a nonlinear ergodic theorem for nonexpansive mappings on a compact and convex subset in a strictly convex Banach space: Let C be a compact and convex subset of a strictly convex Banach space, let T be a nonexpansive mapping of C into itself and let $\xi \in C$. Then, for each point x of the closure of convex hull of the ω-limit set $\omega(\xi)$ of ξ, the Cesàro means $1/n\sum_{k=0}^{n-1} T^k x$ converge to a fixed point of T, where the ω-limit set $\omega(\xi)$ of ξ is the set of cluster points of the sequence $\{T^n\xi : n = 1, 2, \dots\}$. By using results of Bruck [5], Atsushiba and Takahashi [1] proved a nonlinear ergodic theorem for nonexpansive mappings on a compact and convex subset of a strictly convex Banach space: Let C be a compact and convex subset of a strictly convex Banach space and let T be a nonexpansive mapping of C into itself. Then, for each $x \in C$, the Cesàro means $1/n\sum_{k=0}^{n-1} T^k x$ converge to a fixed point of T. This result was extended to commutative semigroups of nonexpansive mappings by Atsushiba, Lau and Takahashi [2]. Suzuki and Takahashi [30] constructed a nonexpansive mapping of a compact and convex subset C of a Banach space into itself such that for some $x \in C$, the Cesàro means $1/n\sum_{k=0}^{n-1} T^k x$ converge to a point of C, but the limit point is not a fixed point of T. Motivated by the example of Suzuki and Takahashi, Miyake and Takahashi [22] proved a nonlinear ergodic theorem for nonexpansive mappings on a compact and convex subset of a general Banach space: Let C be a compact and convex subset of a Banach space and let T be a nonexpansive mapping of C into itself. Then, for each $x \in C$, the Cesàro means $1/n\sum_{k=0}^{n-1} T^k x$ converge. They also proved a nonlinear ergodic theorem for semigroups of nonexpansive mappings on a compact and convex subset of a general Banach space.

Motivated by Kido and Takahashi [15], Hirano, Kido and Takahashi [13], Lau, Shioji and Takahashi [18], Atsushiba, Lau and Takahashi [2] and Miyake and Takahashi [22], Miyake and Takahashi [23] first proved weak and strong mean ergodic theorems for vector-valued weakly almost periodic functions (in the sense of Eberlein) which are defined on an abstract semigroup and take values in a Banach space. Using these results, they obtained well-known and new mean ergodic theorems for commutative and noncommutative semigroups of nonexpansive mappings, affine nonexpansive mappings and linear bounded operators in Banach spaces. In this paper, we summarize their results in [23] to show that mean ergodic theorems for vector-valued functions can be applied, in the systematic way, to obtain well-known and new mean ergodic theorems for semigroups of linear and non-linear operators in Banach spaces, by considering such semigroups of operators as vector-valued functions which are defined on a semigroup and take values in a Banach space.

2. Preliminaries

Throughout this paper, we denote by S a semigroup with identity and by E a real Banach space. Let $\langle E, F\rangle$ be the duality between vector spaces E and F. For each $y \in F$, we define a linear functional f_y on E by $f_y(x) = \langle x, y\rangle$. We denote by $\sigma(E, F)$ the weak topology on E generated by $\{f_y : y \in F\}$. If X is a Banach space, we denote by X^* the topological dual of X. We also denote by $\langle\cdot,\cdot\rangle$ the canonical bilinear form between E and E^*, that is, for $x \in E$ and $x^* \in E^*$, $\langle x, x^*\rangle$ is the value of x^* at x. If A is a subset of E, then the closure of convex hull of A is denoted by $\overline{\text{co}}A$.

We denote by $l^\infty(S)$ the Banach space of bounded real-valued functions on S with supremum norm. For each $s \in S$, we define operators $l(s)$ and $r(s)$ on $l^\infty(S)$ by

$$(l(s)f)(t) = f(st) \quad \text{and} \quad (r(s)f)(t) = f(ts)$$

for each $t \in S$ and $f \in l^\infty(S)$, respectively. Let X be a subspace of $l^\infty(S)$ which contains constants. Then, X is said to be *translation invariant* if $l(s)f \in X$ and $r(s)f \in X$ for each $s \in S$ and $f \in X$. A linear functional μ on X is said to be a *mean* on X if $\|\mu\| = \mu(e) = 1$, where $e(s) = 1$ for each $s \in S$. We often write $\mu_s f(s)$ instead of $\mu(f)$ for each $f \in X$. For $s \in S$, we can define a *point evaluation* δ_s by $\delta_s(f) = f(s)$ for each $f \in X$. A convex combination of point evaluations is called a *finite mean* on S. As is well known, μ is a mean on X if and only if

$$\inf_{s\in S} f(s) \le \mu(f) \le \sup_{s\in S} f(s)$$

for each $f \in X$; see [34] for more details. If X is translation invariant, then a mean μ on X is said to be *left invariant* (resp. *right invariant*) if $\mu(l(s)f) = \mu(f)$ (resp. $\mu(r(s)f) = \mu(f)$) for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant. If there exists an invariant mean on X, then X is said to be *amenable*. We know from [7] that if S is commutative, then X is amenable. Let $\{\mu_\alpha\}$ be a net of means on X. Then $\{\mu_\alpha\}$ is said to be *(strongly) asymptotically invariant* if for each $s \in S$, both $l(s)^*\mu_\alpha - \mu_\alpha$ and $r(s)^*\mu_\alpha - \mu_\alpha$ converge to 0 in the weak topology $\sigma(X^*, X)$ (the norm topology), where $l(s)^*$ and $r(s)^*$ are the adjoint operators of $l(s)$ and $r(s)$, respectively. Such nets were first studied by Day [7].

We denote by $l^\infty(S,E)$ the Banach space of vector-valued functions on S that take values in a Banach space E such that for each $f \in l^\infty(S,E)$, $f(S) \subset E$ is bounded. We also denote by $l_c^\infty(S,E)$ the subspace of those elements $f \in l^\infty(S,E)$ such that $f(S) = \{f(s) : s \in S\}$ is a relatively weakly compact subset of E. Let X be a subspace of $l^\infty(S)$ containing constants such that for each $f \in l_c^\infty(S,E)$ and $x^* \in E^*$, the function $s \mapsto \langle f(s), x^*\rangle$ is contained in X. Then, for each $\mu \in X^*$ and $f \in l_c^\infty(S,E)$, we define a bounded linear functional $\tau(\mu)f$ on E^* by

$$\tau(\mu)f : x^* \mapsto \mu\langle f(\cdot), x^*\rangle.$$

It follows from the bipolar theorem that $\tau(\mu)f$ is contained in E. We know that if μ is a mean on X, then $\tau(\mu)f$ is contained in the closure of convex hull of $\{f(s) : s \in S\}$. We also know that for each $\mu \in X^*$, $\tau(\mu)$ is a bounded linear mapping of $l_c^\infty(S,E)$ into E such that for each $f \in l_c^\infty(S,E)$, $\|\tau(\mu)f\| \leq \|\mu\|\|f\|$; see [14].

Let C be a closed and convex subset of E and let T be a mapping of C into itself. Then, T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$. Let $L(E)$, $A(C)$ and $N(C)$ be the semigroups of linear bounded operators on E, affine nonexpansive mappings and nonexpansive mappings of C into itself under operator multiplication, respectively. If $\mathcal{S}$ is a semigroup homomorphism of S into $L(E)$ ($A(C)$ or $N(C)$), then $\mathcal{S} = \{T(s) : s \in S\}$ is said to be a *representation* of S as linear bounded operators on E (as affine nonexpansive mappings on C or as nonexpansive mappings on C). A subspace X of $l^\infty(S)$ is said to be *admissible* if for each $x \in E$ (or C) and $x^* \in E^*$, the function $s \mapsto \langle T(s)x, x^*\rangle$ is contained in X. We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, that is, $F(\mathcal{S}) = \cap_{s\in S}\{x \in C : T(s)x = x\}$.

Let C be a closed and convex subset of a Banach space E and let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as linear bounded operators on E (as affine nonexpansive mappings on C or as nonexpansive mappings on C) such that $T(\cdot)x \in l_c^\infty(S,E)$ for some $x \in E$ (or C), let X be an admissible subspace of $l^\infty(S)$ which contains constants and let μ be a mean on X. Then, there exists a unique point x_0 of E such that $\mu\langle T(\cdot)x, x^*\rangle = \langle x_0, x^*\rangle$ for each $x^* \in E^*$. We denote such a point x_0 by $T(\mu)x$. Note that $T(\mu)x$ is contained in the closure of convex hull of $\{T(s)x : s \in S\}$ for each $x \in C$; see [31] and [13] for more details.

For each $s \in S$, we define the operators $R(s)$ and $L(s)$ on $l^\infty(S,E)$ by

$$(R(s)f)(t) = f(ts) \quad \text{and} \quad (L(s)f)(t) = f(st)$$

for each $t \in S$ and $f \in l^\infty(S,E)$, respectively. We denote by $\mathcal{LO}(f)$ (resp. $\mathcal{RO}(f)$) the set $\{L(s)f \in l^\infty(S,E) : s \in S\}$ of left translates of f (resp. the set $\{R(s)f \in l^\infty(S,E) : s \in S\}$ of right translates of f). A function $f \in l^\infty(S,E)$ is said to be *left (resp. right) almost periodic* if $\mathcal{LO}(f)$ (resp. $\mathcal{RO}(f)$) is relatively compact in $l^\infty(S,E)$; the notion of almost periodicity for real-valued functions on an abstract group is due to von Neumann [24]. A function $f \in l^\infty(S,E)$ is also said to be *left (resp. right) weakly almost periodic* if $\mathcal{LO}(f)$ (resp. $\mathcal{RO}(f)$) is relatively weakly compact in $l^\infty(S,E)$; the notion of weakly almost periodicity was introduced by Eberlein [10]. See also [9]. Note that every weakly almost periodic function $f \in l^\infty(S,E)$ is contained in $l_c^\infty(S,E)$.

3. Vector-valued weakly almost periodic functions

In 1934, von Neumann first proved the existence of the mean values for real-valued almost periodic functions which are defined on an abstract group. Later, Bochner and von Neumann extended von Neumann's result to vector-valued almost periodic functions which are defined on an abstract group and take values in a Banach space.

Theorem 1 (von Neumann [24]). *Let G be a group and let $AP(G)$ be the Banach space of real-valued almost periodic functions on G. Then, for each f in $AP(G)$, the closure of convex hull of $\mathcal{RO}(f)$ contains exactly one constant function c_f. In this case, putting $\mu(f) = c_f$, μ is a linear functional on $AP(G)$ such that the following are satisfied:*

(1) $\inf_{g\in G} f(g) \le \mu(f) \le \sup_{g\in G} f(g)$;
(2) $\mu(r(g)f) = \mu(f)$ *for each* $f \in AP(G)$ *and* $g \in G$;
(3) $\mu(l(g)f) = \mu(f)$ *for each* $f \in AP(G)$ *and* $g \in G$;
(4) $\mu_x(f(x^{-1})) = \mu_x f(x)$ *for each* $f \in AP(G)$.

Theorem 2 (Bochner and von Neumann [4]). *Let G be a group, let $AP(G)$ be the Banach space of real-valued almost periodic functions on G and let $AP(G,E)$ be the closed subspace of $l^\infty(S,E)$ whose elements are almost periodic. Then, for each $f \in AP(G,E)$, the closure of convex hull of $\mathcal{RO}(f)$ contains exactly one constant function c_f. In this case, putting $\tau(\mu)f = c_f$, $\tau(\mu)$ is a linear operator from $AP(G,E)$ into E such that the following are satisfied:*

(1) $\tau(\mu)c = c$ *for each constant* $c \in AP(G,E)$;
(2) $\tau(\mu)(R(g)f) = \tau(\mu)f$ *for each* $f \in AP(G,E)$ *and* $g \in G$;
(3) $\tau(\mu)(L(g)f) = \tau(\mu)f$ *for each* $f \in AP(G,E)$ *and* $g \in G$;
(4) $\tau(\mu)_x(f(x^{-1})) = \tau(\mu)f$ *for each* $f \in AP(G,E)$.

In 1949, Eberlein [10] introduced a notion of weak almost periodicity for real-valued bounded functions which are defined on a locally compact abelian group.

Theorem 3 (Eberlein [10]). *Let G be a locally compact abelian group and let $WAP(G)$ be the Banach space of real-valued weakly almost periodic functions on G. Then, for each $f \in WAP(G)$, the closure of convex hull of $\mathcal{RO}(f)$ contains exactly one constant function c_f. In this case, putting $\mu(f) = c_f$, μ is a linear functional on $WAP(G)$ such that the following are satisfied:*

(1) $\inf_{g \in G} f(g) \leq \mu(f) \leq \sup_{g \in G} f(g)$;
(2) $\mu(r(g)f) = \mu(f)$ *for each* $f \in WAP(G)$ *and* $g \in G$;
(3) $\mu(l(g)f) = \mu(f)$ *for each* $f \in WAP(G)$ *and* $g \in G$;
(4) $\mu_x f(x^{-1}) = \mu_x f(x)$ *for each* $f \in WAP(G)$.

Recently, Miyake and Takahashi [23] introduced a notion of weak almost periodicity in the sense of Eberlein for vector-valued functions which are defined on an abstract semigroup and take values in a Banach space, and also proved the existence of the mean values for vector-valued weakly almost periodic functions.

Theorem 4. *Let $f \in l^\infty(S, E)$ be a right weakly almost periodic function and let X be a closed and translation invariant subspace of $l^\infty(S)$ containing constants such that for each $x^* \in E^*$, the function $s \mapsto \langle f(s), x^* \rangle$ is contained in X. If X has a left invariant mean, then there exists a unique constant function in the closure K of convex hull of $\mathcal{RO}(f)$. In this case, the constant function is $\tau(l(\cdot)^*\mu)f = \tau(\mu)f$ for each left invariant mean μ on X. In particular, if μ and ν are left invariant means on X, then $\tau(\mu)f = \tau(\nu)f$.*

Remark 1. It is well-known that if a semigroup S is left (or right) reversible, that is, any two right ideals has non-empty intersection, then $WAP(S)$ has a left (or right) invariant mean; See DeLeeuw and Glicksberg [9]. In particular, $WAP(G)$ has an invariant mean.

They also showed that (ergodic) means are well-defined for vector-valued weakly almost periodic functions in the sense of Eberlein by using a notion of "vector-valued" means which was studied by Kada and Takahashi [14].

Lemma 1. *Let $f \in l^\infty(S, E)$ be a right weakly almost periodic function, let X be a closed and translation invariant subspace of $l^\infty(S)$ containing constants such that for each $x^* \in E^*$, the function $s \mapsto \langle f(s), x^* \rangle$ is contained in X and let μ be a mean on X. Then, the function*

$s \mapsto \tau(l(s)^*\mu)f$ *is contained in the closure* K *of convex hull of* $\mathcal{RO}(f)$ *in* $l^\infty(S, E)$.

Using two above results, weak and strong mean ergodic theorems were obtained for vector-valued weakly almost periodic functions in the sense of Eberlein.

Theorem 5. *Let* $f \in l^\infty(S, E)$ *be a right weakly almost periodic function in the sense of Eberlein, let* X *be a closed and translation invariant subspace of* $l^\infty(S)$ *containing constants such that for each* $x^* \in E^*$*, the function* $s \mapsto \langle f(s), x^* \rangle$ *is contained in* X *and let* $\{\mu_\alpha\}$ *be an asymptotically invariant net of means on* X. *Then,* $\{\tau(l(\cdot)^*\mu_\alpha)f\}$ *converges weakly to a constant function* p *in the closure* K *of convex hull of* $\mathcal{RO}(f)$. *In this case,* $p(\cdot) = \tau(\mu)f$ *in* E *for each invariant mean* μ *on* X.

Theorem 6. *Let* $f \in l^\infty(S, E)$ *be a right weakly almost periodic function in the sense of Eberlein, let* X *be a closed and translation invariant subspace of* $l^\infty(S)$ *containing constants such that for each* $x^* \in E^*$*, the function* $s \mapsto \langle f(s), x^* \rangle$ *is contained in* X *and let* $\{\mu_\alpha\}$ *be a strongly asymptotically invariant net of means on* X. *Then,* $\{\tau(l(\cdot)^*\mu_\alpha)f\}$ *converges strongly to a constant function* p *in the closure* K *of convex hull of* $\mathcal{RO}(f)$. *In this case,* $p(\cdot) = \tau(\mu)f$ *for each invariant mean* μ *on* X.

4. Mean ergodic theorems for semigroups of linear and non-linear operators

By considering semigroups of operators in Banach spaces as vector-valued functions which are defined on a semigroup and take values in a Banach space, mean ergodic theorems for such functions can be applied to obtain new and well-known mean ergodic theorems for semigroups of linear and non-linear operators in Banach spaces in the systematic way. See also Eberlein [10] and Ruess and Summers [27]. In this section, for the purpose of explaining our idea, we show complete proofs of well-known mean ergodic theorems for semigroups of linear operators and nonexpansive mappings in Banach spaces, respectively.

Theorem 7. *Let* $\mathcal{S} = \{T(s) : s \in S\}$ *of* S *be a representation of* S *as linear bounded operators on a Banach space* E *such that for* $s \in S$, $\|T(s)\| \leq M$ *and for each* $x \in E$, $\{T(s)x : s \in S\}$ *is relatively weakly compact, let* X *be a closed, translation invariant and admissible subspace of* $l^\infty(S)$ *containing constants and let* $\{\mu_\alpha\}$ *be a strongly asymptotically invariant net of means on* X. *Then, for each* $x \in E$,

$\{T(l(h)^*\mu_\alpha)x\}$ converges strongly to a common fixed point p of $\mathcal{S}$ uniformly in $h \in S$. In this case, $p = T(\mu)x$ and

$$\{T(\mu)x\} = \overline{co}\{T(s)x : s \in S\} \cap F(\mathcal{S})$$

for each invariant mean μ on X.

Proof. For each $x \in E$, we define a function $f_x \in l^\infty(S, E)$ by $f_x(s) = T(s)x$ for each $s \in S$. We show that for each $x \in E$, f_x is right weakly almost periodic. In fact, we have, for each $s \in S$,

$$(R(s)f_x)(t) = T(ts)x = T(t)T(s)x = f_{T(s)x}(t)$$

for each $t \in S$. Hence, $\mathcal{RO}(f_x)$ is contained in $\{f_y : y \in C\}$, where $C = \overline{co}\{T(s)x : s \in S\}$. We define a mapping Φ of E into $l^\infty(S, E)$ by $\Phi(x) = f_x$ for each $x \in E$. Then, Φ is a bounded linear mapping and hence is weak-to-weak continuous. Since C is weakly compact, $\mathcal{RO}(f_x)$ is contained in a weakly compact subset $\Phi(C)$ of $l^\infty(S, E)$. So, for each $x \in E$, $f_x \in l^\infty(S, E)$ is right weakly almost periodic.

It follows from Theorem 2 that $\{T(l(\cdot)^*\mu_\alpha)x\}$ converges strongly to a constant function q in $l^\infty(S, E)$. In this case, $q(\cdot) = T(\mu)x$ for each invariant mean μ on X. Hence, for each $x \in E$, $\{T(l(h)^*\mu_\alpha)x\}$ converges strongly to a point $T(\mu)x$ in C uniformly in $h \in S$ where μ is an invariant mean on X. Since, for each $s \in S$ and $x^* \in E^*$,

$$\begin{aligned}
\langle T(s)T(\mu)x, x^*\rangle &= \langle T(\mu)x, T(s)^*x^*\rangle = \mu\langle T(\cdot)x, T(s)^*x^*\rangle \\
&= \mu\langle T(s)T(\cdot)x, x^*\rangle = \mu\langle T(s\cdot)x, x^*\rangle \\
&= l(s)^*\mu\langle T(\cdot), x^*\rangle = \mu\langle T(\cdot), x^*\rangle \\
&= \langle T(\mu)x, x^*\rangle
\end{aligned}$$

where $T(s)^*$ is the adjoint operator of $T(s)$, we have $T(s)T(\mu)x = T(\mu)x$ for each $s \in S$.

It remains to show that $\{T(\mu)x\} = \overline{\text{co}}\{T(s)x : s \in S\} \cap F(\mathcal{S})$ for each $x \in C$. Since μ is an invariant mean on X, we have $T(\mu)x = T(r(s)^*\mu)x = T(\mu)T(s)x$ for each $s \in S$ and hence $T(\mu)x = T(\mu)y$ for each $y \in \overline{\text{co}}\{T(s)x : s \in S\}$. This completes the proof. □

Theorem 8 (Miyake and Takahashi [22]). *Let C be a compact and convex subset of a Banach space E, let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as nonexpansive mappings on C, let X be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constants and let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on X. Then, for each $x \in C$, $\{T(l(h)^*\mu_\alpha)x\}$ converges strongly to a point p uniformly in $h \in S$. In this case, $p = T(\mu)x$ for each invariant mean μ on X.*

Proof. For each $x \in C$, we define a function $f_x \in l^\infty(S, E)$ by $f_x(s) = T(s)x$ for each $s \in S$. We show that for each $x \in C$, f_x is right almost periodic. In fact, we have, for each $s \in S$,

$$(R(s)f_x)(t) = T(ts)x = T(t)T(s)x = f_{T(s)x}(t)$$

for each $t \in S$. Hence, $\mathcal{RO}(f_x)$ is contained in $\{f_y : y \in C\}$. We define a mapping Φ of C into $l^\infty(S, E)$ by $\Phi(x) = f_x$ for each $x \in C$. Then, we have, for each $x, y \in C$,

$$\begin{aligned} \|\Phi(x) - \Phi(y)\| &= \|f_x - f_y\| \\ &= \sup_{t \in S} \|f_x(t) - f_y(t)\| \\ &= \sup_{t \in S} \|T(t)x - T(t)y\| \\ &\leq \|x - y\| \end{aligned}$$

and hence Φ is norm-to-norm continuous. Since C is compact, $\mathcal{RO}(f_x)$ is contained in a compact subset $\Phi(C)$ of $l^\infty(S, E)$. So, for each $x \in C$, $f_x \in l^\infty(S, E)$ is right almost periodic.

It follows from Theorem 1 that $\{T(l(\cdot)^* \mu_\alpha)x\}$ converges strongly to a constant function q in $l^\infty(S, E)$. In this case, $q(\cdot) = T(\mu)x$ for each invariant mean μ on X. Hence, $\{T(l(h)^* \mu_\alpha)x\}$ converges strongly to a point $T(\mu)x$ uniformly in $h \in S$ where μ is an invariant mean on X. This completes the proof. $\square$

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