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ON FIRMLY NONEXPANSIVE TYPE MAPPINGS IN BANACH SPACES
(バナッハ空間における FIRMLY NONEXPANSIVE TYPE 写像について)

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ABSTRACT. In this paper, we state the recently obtained strong convergence theorem of Browder's type for firmly nonexpansive-type mappings in Banach spaces.

1. INTRODUCTION

The following is Browder's strong convergence theorem [5] for nonexpansive mappings in Hilbert spaces; see, for instance, Takahashi [24]:

Theorem 1.1 (Browder [5]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, T a nonexpansive mapping from C into itself such that $F(T)$ is nonempty, and $x \in C$. Then the following hold:*

(1) *For each $t \in (0,1)$, there exists a unique $u_t \in C$ such that*

$$u_t = tx + (1-t)Tu_t;$$

(2) *the net $\{u_t\}$ converges strongly to $P_{F(T)}(x)$ as $t \downarrow 0$, where $P_{F(T)}$ denotes the metric projection from H onto $F(T)$.*

This result was extended to accretive operators in Banach spaces by Reich [18] and Takahashi and Ueda [27].

Recently, the authors [13] proposed the class of *firmly nonexpansive-type* mappings in Banach spaces. It is a subclass of *D-firm* operators introduced by Bauschke, Borwein, and Combettes [3]. This class contains the classes of firmly nonexpansive mappings in Hilbert spaces and resolvents of maximal monotone operators in Banach spaces. In [14], the class of *nonspreading* mappings in Banach spaces was also introduced. Every firmly nonexpansive-type mapping is known to be nonspreading. Then fixed point theorems and convergence theorems for these nonlinear operators were investigated [13, 14].

In this paper, we state a strong convergence theorem [15] of Browder's type for firmly nonexpansive-type mappings in Banach spaces.

2. PRELIMINARIES

Throughout the paper, every linear space is real. The set of real numbers is denoted by $\mathbb{R}$. The conjugate space of a Banach space E is denoted by E^*. We denote $x^*(x)$ by $\langle x, x^*\rangle$ for all $(x, x^*) \in E \times E^*$. For a sequence $\{x_n\}$ of E, the strong and weak convergence of $\{x_n\}$ to $x \in E$ is denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

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Let E be a Banach space with norm $\|\cdot\|$ and let $S(E) = \{x \in E : \|x\| = 1\}$. Then the *duality mapping* J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \tag{2.1}$$

for all $x \in E$. It is known that $Jx \neq \emptyset$ for all $x \in E$. The space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exitsts for all $x, y \in S(E)$. In this case, the norm of E is said to be *Gâteaux differentiable.* The norm of E is also said to be *uniformly Gâteaux differentiable* (resp. *uniformly Fréchet differentiable*) if the limit (2.2) converges uniformly in $x \in S(E)$ for all $y \in S(E)$ (resp. uniformly in $x, y \in S(E)$). The space E is said to be *uniformly smooth* if the norm of E is uniformly Fréchet differentiable.

The space E is said to be *strictly convex* if $\|(x+y)/2\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is also said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x - y\| \geq \varepsilon$ and $x, y \in S(E)$ imply that $\|(x+y)/2\| \leq 1 - \delta$. The space E is said to have the *Kadec–Klee property* if $x_n \to x$ whenever $\{x_n\}$ is a sequence of E such that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. We know the following; see, for instance, [10, 24]:

(1) If E is smooth, then J is single-valued;
(2) if E is strictly convex, then $Jx \cap Jy \neq \emptyset$ implies that $x = y$;
(3) if E is reflexive, then J is onto;
(4) E is uniformly smooth if and only if E^* is uniformly convex;
(5) if E is uniformly convex, then E is a strictly convex and reflexive Banach space with the Kadec–Klee property.

Let E be a smooth Banach space. Following [1, 12], let ϕ be a mapping from $E \times E$ into $\mathbb{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \tag{2.3}$$

for all $x, y \in E$. It is obvious that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \tag{2.4}$$

for all $x, y \in E$. If C is a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, then for each $x \in E$, there exists a unique $z \in C$ (denoted by $\Pi_C x$) such that $\phi(z, x) = \min_{y \in C} \phi(y, x)$. The mapping Π_C is called the *generalized projection* [1] from E onto C. Similarly, for each $x \in E$, there exists a unique $z \in C$ (denoted by $P_C x$) such that $\|z - x\| = \min_{y \in C} \|y - x\|$. The mapping P_C is called the *metric projection* from E onto C. It is easy to see that

$$\Pi_C(0) = P_C(0). \tag{2.5}$$

If E is a Hilbert space, then $\Pi_C(x) = P_C(x)$ for all $x \in E$. For $(x, z) \in E \times C$, the following hold; see [1, 12, 24]:

(1) $z = \Pi_C(x)$ if and only if $\langle y - z, Jx - Jz \rangle \leq 0$ for all $y \in C$;
(2) $z = P_C(x)$ if and only if $\langle y - z, J(x - z) \rangle \leq 0$ for all $y \in C$.

Let E be a smooth Banach space, C a nonempty closed convex subset of E, and T a mapping from C into itself. The set of fixed points of T is denoted by $F(T)$. Then T is said to be of *firmly nonexpansive type* [13] if

$$\langle Tx - Ty, Jx - JTx - (Jy - JTy)\rangle \geq 0 \tag{2.6}$$

for all $x, y \in C$. If E is a Hilbert space, then $J = I$ (the identity operator on E) and hence T is of firmly nonexpansive type if and only if it is firmly nonexpansive in the classical sense, that is,

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle \tag{2.7}$$

for all $x, y \in C$; see, for example, [6, 8, 9, 11, 26]. It is easy to verify that the generalized projection operator Π_C is of firmly nonexpansive type and $F(\Pi_C) = C$. If $r > 0$, C is a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, and $A \subset E \times E^*$ is a monotone operator such that $D(A) \subset C \subset J^{-1}R(J + rA)$, then the *resolvent* Q_r of A defined by

$$Q_r x = (J + rA)^{-1}Jx \tag{2.8}$$

for all $x \in C$ is a firmly nonexpansive-type mapping from C into itself and $F(Q_r) = A^{-1}0$; see [13–15] for more details. The class of firmly nonexpansive-type mappings is included in the class of *D-firm operators* [3], where D stands for a Bregman distance. We also know that T is of firmly nonexpansive type if and only if

$$\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(Tx, x) + \phi(Ty, y) \leq \phi(Tx, y) + \phi(Ty, x) \tag{2.9}$$

for all $x, y \in C$; see [3, 13]. In particular, if a firmly nonexpansive-type mapping T has a fixed point, then

$$\phi(u, Tx) + \phi(Tx, x) \leq \phi(u, x) \tag{2.10}$$

for all $u \in F(T)$ and $x \in C$. The mapping T is also said to be *nonspreading* [14] if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) \tag{2.11}$$

for all $x, y \in C$. It is easy to see that every firmly nonexpansive-type mapping is nonspreading. A point $u \in C$ is said to be *asymptotic fixed point* [19] of T if there exists a sequence $\{x_n\}$ of C such that $x_n \rightharpoonup u$ and $\|x_n - Tx_n\| \to 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$. The mapping T is also said to be *relatively nonexpansive* [16, 17] if the following conditions are satisfied:

(1) $F(T)$ is nonempty;
(2) $\widehat{F}(T) = F(T)$;
(3) $\phi(u, Tx) \leq \phi(u, x)$ for all $u \in F(T)$ and $x \in C$.

We know the following lemmas:

Lemma 2.1 ([14]). *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and T a nonspreading mapping from C into itself. Then $\widehat{F}(T) = F(T)$. Further, if $F(T)$ is nonempty, then T is relatively nonexpansive.*

Lemma 2.2 ([17]). *Let E be a smooth and strictly convex Banach space, C a nonempty closed convex subset of E, and T a mapping from C into itself such that $F(T)$ is nonempty and $\phi(u, Tx) \leq \phi(u, x)$ for all $u \in F(T)$ and $x \in C$. Then $F(T)$ is closed and convex.*

Motivated by the technique in [23, 24], the following fixed point theorem for nonspreading mappings in Banach spaces was shown:

Theorem 2.3 ([14]). *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, and T a nonspreading mapping from C into itself. Then $F(T)$ is nonempty if and only if there exists $x \in C$ such that $\{T^n x\}$ is bounded.*

As a direct consequence of Theorem 2.3, we obtain the following:

Corollary 2.4 ([13]). *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, and T a firmly nonexpansive-type mapping from C into itself. Then $F(T)$ is nonempty if and only if there exists $x \in C$ such that $\{T^n x\}$ is bounded.*

The following lemma implies that the class of firmly nonexpansive-type mappings is coincident with that of resolvents of monotone operators:

Lemma 2.5 ([14]). *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, and T a mapping from C into itself. Then T is of firmly nonexpansive type if and only if there exists a monotone operator $A \subset E \times E^*$ such that $D(A) \subset C \subset J^{-1}R(J + A)$ and $Tx = (J + A)^{-1}Jx$ for all $x \in C$.*

3. Results

Using Lemmas 2.1, 2.2 and Corollary 2.4, we can show the following strong convergence theorem of Browder's type for firmly nonexpansive-type mappings in Banach spaces:

Theorem 3.1 ([15]). *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty bounded closed convex subset of E with $0 \in C$, and T a firmly nonexpansive-type mapping from C into itself. Then the following hold:*

(1) *For each $t \in (0, 1)$, there exists a unique $u_t \in C$ such that*

$$u_t = (1 - t)Tu_t;$$

(2) *if E has the Kadec–Klee property and the norm of E is uniformly Gâteaux differentiable, then the net $\{u_t\}$ converges strongly to $P_{F(T)}(0)$ as $t \downarrow 0$, where $P_{F(T)}$ denotes the metric projection from E onto $F(T)$.*

The following is a direct consequence of Theorem 3.1 and Lemma 2.5:

Theorem 3.2 ([15]). *Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty bounded closed convex subset of E with $0 \in C$. Let r be a positive real number, $A \subset E \times E^*$ a monotone operator such that $D(A) \subset C \subset J^{-1}R(J + rA)$, and $Q_r x = (J + rA)^{-1}Jx$ for all $x \in C$. Then the following hold:*

(1) *For each $t \in (0, 1)$, there exists a unique $u_t \in C$ such that*

$$u_t = (1 - t)Q_r u_t;$$

(2) *if E has the Kadec–Klee property and the norm of E is uniformly Gâteaux differentiable, then the net $\{u_t\}$ converges strongly to $P_{A^{-1}0}(0)$ as $t \downarrow 0$, where $P_{A^{-1}0}$ denotes the metric projection from E onto $A^{-1}0$.*

Corollary 3.3. *Let E be a smooth, strictly convex, and reflexive Banach space and $A \subset E \times E^*$ a maximal monotone operator such that $D(A)$ is bounded and $0 \in \overline{D(A)}$, where $\overline{D(A)}$ denotes the norm closure of $D(A)$. Let r be a positive real number and $Q_r x = (J + rA)^{-1}Jx$ for all $x \in \overline{D(A)}$. Then the following hold:*

(1) *For each* $t \in (0,1)$, *there exists a unique* $u_t \in \overline{D(A)}$ *such that*

$$u_t = (1-t)Q_r u_t;$$

(2) *if* E *has the Kadec–Klee property and the norm of* E *is uniformly Gâteaux differentiable, then the net* $\{u_t\}$ *converges strongly to* $P_{A^{-1}0}(0)$ *as* $t \downarrow 0$, *where* $P_{A^{-1}0}$ *denotes the metric projection from* E *onto* $A^{-1}0$.

Proof. We know that $\overline{D(A)}$ is closed and convex. In fact,

$$\lim_{t\downarrow 0} J_t x = x \tag{3.1}$$

for all $x \in \overline{\text{co}}\, D(A)$, where $\overline{\text{co}}\, D(A)$ denotes the closed convex hull of $D(A)$ and J_t is defined by $J_t = (I + tJ^{-1}A)^{-1}$ for all $t > 0$; see [2, 25] for more details. Thus we have $\overline{\text{co}}\, D(A) \subset \overline{D(A)}$. This implies that $\overline{\text{co}}\, D(A) = \overline{D(A)}$ and hence $\overline{D(A)}$ is closed and convex.

Since A is maximal monotone, we know that $R(J + rA) = E^*$ see [2, 7, 22, 25]. Putting $C = \overline{D(A)}$, we know that C is a bounded closed convex subset of E with $0 \in C$,

$$D(A) \subset C \subset E = J^{-1}E^* = J^{-1}R(J + rA), \tag{3.2}$$

and Q_r is a firmly nonexpansive-type mapping from C into itself. Thus, by Theorem 3.2, we obtain the conclusion. □

Let E be a Banach space and f a function from E into $(-\infty, \infty]$. Then f is said to be *proper* if the *effective domain* $D(f) = \{x \in E : f(x) \in \mathbb{R}\}$ of f is nonempty. It is said to be *convex* if

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y) \tag{3.3}$$

whenever $x, y \in E$ and $\alpha \in (0,1)$. It is also said to be *lower semicontinuous* if $\{x \in E : f(x) \le r\}$ is closed in E for all $r \in \mathbb{R}$. Let $x \in E$ be given. Then a point $x^* \in E^*$ is said to be a *subgradient* of f at x if

$$f(x) + \langle y - x, x^* \rangle \le f(y) \tag{3.4}$$

for all $y \in E$. The set of subgradients of f at x is said to be the *subdifferential* of f at x and denoted by $\partial f(x)$. The mapping $\partial f \subset E \times E^*$ is called the *subdifferential mapping* of f.

Using Corollary 3.3, we can also show the following corollary:

Corollary 3.4 ([15]). *Let* E *be a smooth, strictly convex, and reflexive Banach space,* r *a positive real number, and* f *a proper lower semicontinuous convex function from* E *into* $(-\infty, \infty]$ *such that* $D(f)$ *is bounded and* $0 \in \overline{D(f)}$. *Then the following hold:*

(1) *For each* $t \in (0,1)$, *there exists a unique* $u_t \in \overline{D(f)}$ *such that*

$$u_t = (1-t) \cdot \arg\min_{y \in E} \Big\{ f(y) + \frac{1}{2r}\phi(y, u_t) \Big\};$$

(2) *if* E *has the Kadec–Klee property and the norm of* E *is uniformly Gâteaux differentiable, then the net* $\{u_t\}$ *converges strongly to* $P(0)$ *as* $t \downarrow 0$, *where* P *denotes the metric projection from* E *onto* $\arg\min_{y\in E} f(y)$.

Proof. Brøndsted and Rockafellar's theorem [4] implies that $D(\partial f)$ is norm dense in $D(f)$, that is, $D(f) \subset \overline{D(\partial f)}$; see also [25]. This gives us that $\overline{D(\partial f)} = \overline{D(f)}$. Rockafellar's theorem [20, 21] also ensures that the subdifferential ∂f of f is maximal monotone; see also [25]. We also know that

$$Q_r x = \arg\min_{y \in E} \left\{ f(y) + \frac{1}{2r} \phi(y, x) \right\} \tag{3.5}$$

for all $x \in C = \overline{D(f)}$, where $Q_r x = (J + r\partial f)^{-1} J$ for all $x \in C$; see, for instance, [12, 25]. It is also known that $(\partial f)^{-1}(0) = \arg\min_{y \in E} f(y)$ and $D(\partial f) \subset D(f)$. Thus, by Corollary 3.3, we obtain the conclusion. □

We do not know the answers to the following problems:

Problem 3.5. Is it possible to prove Theorem 3.1 without assuming that C is bounded?

Problem 3.6. Is it possible to prove Theorem 3.1 for a net of the form: $x \in C$ and

$$u_t = tx + (1 - t) T u_t \tag{3.6}$$

for all $t \in (0, 1)$?

Problem 3.7. Is it possible to obtain an analogous result of Browder's strong convergence theorem for *nonspreading* mappings in Banach spaces?

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