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Differentiable Minty variational inequalities. A survey.

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Abstract

When a differentiable Minty variational inequality has a solution, that is also a solution to the primitive optimization problem. This may lead to argue that some relation between the existence of solution of the inequlity and the (generalized) convexity of the primitive function can be established. The paper review some results on this topic, exlporing the case of non differentiable primitive function and generalized variational inequality of Minty type. Finally the common convexity assumption of the feasible region is investigated. **Keywords:** Minty variational inequality, generalized convexity, star-shaped sets, existence of solutions.

1 Introduction

We restrict this survey to the case of real spaces. Some improvement to linear spaces can be found in [3]. The classical formulation of the Minty Variational Inequality (for short mvi) involves a function $F : K \subset \mathbb{R}^n \to \mathbb{R}^n$ and a nonempty feasible region K. A solution to mvi is any vector $x^* \in K$ such that:

$$MVI(F,K) \qquad \langle F(x), x^* - x\rangle \leq 0, \qquad \forall x \in K.$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product defined on $\mathbb{R}^n$. We also say $x^* \in MVI(F,K)$ to denote the solution set of the mvi.
When there exists a primitive, Fréchet differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ such that $F = f'$, the variational inequality $(MVI\,(f', K))$ is said differentiable and provides a very general and suitable mathematical model for a wide range of problems (see e.g. [1, 11, 14] and the references therein). Differentiable variational inequalities have been widely studied in conjunction to the minimization of the primitive function f over the feasible region K (see e.g. [11]). In [10] a vector extension of the variational inequality is introduced and related to vector optimization. Further development

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of vector variational inequalities can be found in [4, 6, 7, 8, 15]. It can be easily understood that the differentiable mvi states that the directional derivative of f at x, along the direction $x^* - x$ has to be non positive.

Without any specific assumption on f, but differentiability, we have that the (differentiable) mvi is a sufficient optimality condition over a convex feasible region. Inspired by this result we wondered if the result may lead to conclude that, when $MVI(f', K)$ is solvable, necessarily the primitive function f has some regularity condition. Supposedly this condition could be some generalized convexity. Indeed Section 2 proves that, for $n = 1$, the primitive function has to be quasiconvex. However an example shows that the same is no longer true for $n > 1$. We focus on this problem under a more general setting which involves Dini type derivatives and a generalized Minty type variational inequality in Section 3. We prove that f is an increasing along rays function, which is a generalization of quasiconvexity. Finally, in Section 4 we explore the role of convexity of the feasible region.

2 Preliminar results

The (primitive) optimization problem we refer to is:

$$P(f,K) \qquad\qquad \min f(x), \qquad\qquad x \in K \subseteq \mathbb{R}^n.$$

A point $x^* \in K$ is a (global) solution of $P(f,K)$ when $f(x) - f(x^*) \geq 0$, $\forall x \in K$. The solution is strict if $f(x) - f(x^*) > 0$, $\forall x \in K \backslash \{0\}$. We denote by $GM(f,K)$ the set of solutions to $P(f,K)$. Along this section we may assume that f is defferentiable over an open set containing K.

The following classical result, known as Minty variational principle (see [10]), motivates our interest on the problem.

Theorem 1. *Let $K \subset \mathbb{R}^n$ be convex. If $x^* \in K$ is a solution to $MVI(f', K)$, then x^* is a global minimizer of f over K*

Proof: Since x^* is a solution of the MVI and K is convex, then we have:

$$\langle f'\big(tx^* + (1-t)y\big), tx^* + (1-t)y - x^*\rangle \geq 0\ ,\ \forall t \in \big[0,1\big],\ \forall y \in K.$$

Hence, we obtain $\langle f'\big(tx^* + (1-t)y\big), y - x^*\rangle \geq 0$, $\forall t \in [0,1[$ and $\forall y \in K$. By a classical Mean Value Theorem, we have that, $\forall y \in K$, $\exists \hat{t} \in]0,1[$, such that:

$$f(y) - f(x^*) = \langle f'(\hat{t}x^* + (1-\hat{t})y), y - x^*\rangle.$$

This completes the proof. $\square$

Theorem 1 states that the mvi is a sufficient optimality condition for differentiable function f. We might be interest to check if, under the assumption the mvi is solvable, then f is quasiconvex. We first recall the definition and a few facts on this class of function.

Definition 1. *Let $K \subseteq \mathbb{R}^n$ be convex. A function $f : K \to \mathbb{R}$ is* quasi-convex *if and only if the (sub-)level sets:*

$$lev_{\leq c} f := \{x \in K \mid f(x) \leq c\}, \qquad \forall c \in \mathbb{R}$$

are convex.

Proposition 1 ([2]). *Let $K \subseteq \mathbb{R}^n$ be convex. A differentiable function $f : K \to \mathbb{R}$ is* quasi-convex *if and only if, for any couple x_1, $x_2 \in K$, such that $f(x_1) \leq f(x_2)$, we have:*

$$\langle f'(x_2), x_1 - x_2 \rangle \leq 0.$$

In the simple case of $f : K \subseteq \mathbb{R} \to \mathbb{R}$, our claim proves to be true.

Proposition 2. *Let $K \subseteq \mathbb{R}$ be convex and $f : \mathbb{R} \to \mathbb{R}$. If there exists a solution x^* of $MVI(f', K)$, then f is quasi-convex.*

Proof: From $MVI(f', K)$ we see that the function is nondecreasing on each halfline with origin at x^*, and hence is quasi-convex. □

However the same result is no longer true when $n > 1$.

Example 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x_1, x_2) = (x_1 x_2)^2$. The feasible region is $K = \mathbb{R}^2$, hence convex. Although $\mathbf{x}^* = (0,0)$ is a solution to mvi:*

$$\left\langle \begin{bmatrix} 2x_1 x_2^2 \\ 2x_1^2 x_2 \end{bmatrix}, \begin{bmatrix} 0 - x_1 \\ 0 - x_2 \end{bmatrix} \right\rangle \leq 0$$

the function f is clearly not quasiconvex.

Remark 1. One can notice that sublevel sets of f in Example 1 are not convex, but star-shaped.

3 A more general setting of the problem

First we try to consider also the case of non differentiable optimization problem $P(f, K)$. To do so, we recall that the lower Dini directional derivative of f at the point $x \in K$ in the direction $u \in \mathbb{R}^n$ is defined as an element of $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ by:

$$f'_{-}(x, u) = \liminf_{t \to +0} \frac{f(x + tu) - f(x)}{t}.$$

Now it is straightforward to introduce the following generalized mvi, for $x^* \in K$:

$$MVI(f'_{-}, K) \qquad\qquad f'_{-}(y, x^* - y) \leq 0, \qquad\qquad \forall y \in K.$$

Again, a solution to the inequality is denoted by $x^* \in MVI(f'_{-}, K)$.

In order to extend the result in Proposition 2, we look for some kind of generalized convexity of $f : \mathbb{R}^n \to \mathbb{R}$ which, for $n = 1$, may reduce to ordinary quasyconvexity. This is the case of the notion of Increasing Alogn Rays (IAR) functions, popularized, among others, by Rubinov (see e.g. [13]). As convex functions are defined over convex sets, IAR functions involves star-shaped (*st-sh* in the following) sets.

Definition 2. *A set K is said to be* star-shaped *at $x \in K$ if and only if, $\forall y \in K$ and $\forall t \in [0,1]$, we have $z := x + t(y-x) \in K$.*

Definition 3. *For any set $K \subseteq \mathbb{R}^n$ the* kernel *of K is defined as the set:*

$$ker\, K := \{x \in K \,:\, y \in K,\ t \in [0,1]) \Longrightarrow x + t(y-x) \in K\}.$$

Remark 2. i) A nonempty set K is *star-shaped* if $\ker K \neq \emptyset$.

ii) The set $ker\, K$ is convex for any arbitrary *st-sh* set K.

iii) We assume by definition that the empty set is *st-sh*.

Definition 4. *A function f defined on $\mathbb{R}^n$ is called increasing along rays at a point x^* (for short, $f \in IAR(x^*)$) if the restriction of this function on the ray $\mathbb{R}_{x^*,x} = \{x^* + \alpha x | \alpha \geq 0\}$ is increasing for each $x \in \mathbb{R}^n$. (A function g of one real variable is called increasing if $t_2 \geq t_1$ implies $g(t_2) \geq g(t_1)$.)*

Definition 5. *Let $K \subseteq \mathbb{R}^n$ be a st-sh set and $x^* \in \ker K$. A function f defined on K is called increasing along rays at x^* (for short, $f \in IAR(K, x^*)$), if the restriction of this function on the intersection $\mathbb{R}_{x^*,x} \cap K$ is increasing, for each $x \in K$.*

The following result characterize the class of IAR functions of one real variable. One can easily check that $K \subseteq \mathbb{R}$ is *st-sh* if and only if it is convex.

Proposition 3. *Let $f : K \subseteq \mathbb{R} \to \mathbb{R}$, K convex. $f \in IAR(K, x^*)$ if and only if it is quasi-convex with a global minimum over K at x^*.*

The function in Example 1 is easily seen to be $IAR\left(\mathbb{R}^2, (0,0)\right)$, although it is clearly not quasiconvex. Therefore, the class of IAR functions at x^* is broader then that of quasi-convex functions with a global minimum at x^*.

The next result gives some basic properties of functions which are increasing along rays.

Proposition 4. *Let $K \subseteq \mathbb{R}^n$ be a st-sh set, $x^* \in \ker K$ and $f \in IAR(K, x^*)$. Then:*

i) x^* is a solution of $P(f, K)$;

ii) No point $x \in K$, $x \neq x^*$, can be a strict local solution of $P(f, K)$.

iii) $x^* \in \ker \operatorname{argmin}(f, K)$.

Proof:

i) Let $x \in K$ and set $z(t) = x^* + t(x - x^*)$, $t \in [0,1]$. Since $x^* \in \ker K$, then $z(t) \in K$, $\forall t \in [0,1]$ and since $f \in IAR(K, x^*)$, we have $f(z(t)) \geq f(x^*) = f(z(0))$, $\forall t \in [0,1]$, and in particular $f(z(1)) = f(x) \geq f(x^*)$. Since $x \in K$ is arbitrary, then x^* is a global minimizer of f over K.

ii) Let x and $z(t)$ as above. Since $f \in IAR(K, x^*)$, it easily follows $f(z(t)) \leq f(x) = f(z(1))$, $\forall t \in [0,1]$. If U is an arbitrary neighborhood of x, then for t "near enough" to 1, we have $z(t) \in U$ and so x cannot be a strict local minimizer for f over K.

iii) Let $x \in \operatorname{argmin}(f, K)$, $x \neq x^*$. Since $z(t) \in K$, we have $f(z(t)) \leq f(x)$, $\forall t \in [0,1]$ and readily follows that for every $t \in [0,1]$, $z(t) \in \operatorname{argmin}(f, K)$.

□

Moreover we make use of a foundamental characterization of IAR functions proved in [16].

Proposition 5. *Let $K \subseteq \mathbb{R}^n$ be a st-sh set, $x^* \in \ker K$ and f be a function defined on K. Then $f \in IAR(K, x^*)$ if and only if for each $c \in \mathbb{R}$ with $c \geq f(x^*)$, we have $x^* \in \ker lev_{\leq c} f$.*

According to Definition 1, Proposition 5 presents the class of IAR functions as a straightforward generalization of quasiconvex functions (with a minimizer at x^*).

Finally we make use also of some continuity assumption. We denote rays starting at $x^* \in \mathbb{R}^n$ as the set $R_{x^*,x} := \{x(t) \in \mathbb{R}^n : x(t) = (1-t)x^* + tx,\ t \geq 0\}$, where $x \in \mathbb{R}^n$.

Definition 6. *Let $K \subseteq \mathbb{R}^n$, $x^* \in \ker K$ and let f be a function defined on an open set containing K. The function f is said to be radially lower semicontinuous in K along rays starting at x^*, if for each $x \in K$, the restriction of f on the interval $R_{x^*,x} \cap K$ is lower semicontinuous.*

We will use the abbreviation $f \in RLSC(K, x^*)$ to denote that f satisfies the previous definition.

The following result is proved in [5] and applies in other proofs.

Theorem 1 (Mean Value Theorem). *Let $x^* \in \ker K$, $f \in RLSC(K, x^*)$, $y \in K$, and $t > 0$ such that $y + t(x^* - y) \in K$. Then there exists a number $\alpha \in]0, t]$, such that:*

$$f(y + t(x^* - y)) - f(y) \leq t f'_-(y + \alpha(x^* - y), x^* - y).$$

Theorem 2. *Let $K \subseteq \mathbb{R}^n$ be a st-sh set and $x^* \in \ker K$.*

i) If x^* solves $MVI(f'_-, K)$ and $f \in RLSC(K, x^*)$, then $f \in IAR(K, x^*)$.

ii) Conversely, if $f \in IAR(K, x^*)$, then x^* is a solution of $MVI(f'_-, K)$.

Proof:

i) Let x^* be a solution of $MVI(f'_-, K)$, $y \in K$ and $y + t_2(x^* - y)$, $y + t_1(x^* - y)$ be points in $\mathbb{R}_{x^*,x} \cap K$ with $t_2 > t_1 \geq 0$. By Theorem 1:

$$f(y + t_2(x^* - y)) - f(y + t_1(x^* - y)) \leq (t_2 - t_1) f'_-(y + \alpha(x^* - y), x^* - y) \leq 0,$$

with $\alpha \in]t_1, t_2]$. Hence, it is proved $f \in IAR(K, x^*)$.

ii) Assume that $f \in IAR(K, x^*)$ and let $y \in K$. For every $t \in [0,1]$, we have: $f(y + t(x^* - y)) = f(x^* + (1-t)(y - x^*)) \le f(y)$ and hence:

$$\frac{f(y + t(x^* - y)) - f(y)}{t} \le 0.$$

Taking lim inf as $t \to +0$, we obtain that x^* solves $MVI(f'_-, K)$.

□

Any continuity of f was not explicitly assumed in Proposition 2, while it is necessary in Theorem 2, as the following example proves.

Example 2. *Let $K = \mathbb{R}$, $x^* = 0$ and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as:*

$$f(x) = \begin{cases} 1, & \text{if } x \neq 2 \\ 3, & \text{if } x = 2 \end{cases}$$

Then $f \notin RLSC(K, 0)$ and it holds $f'_-(y, -y) \le 0$, $\forall y \in \mathbb{R}$, but $f \notin IAR(K, 0)$.

Moreover, the classical Minty Variational Principle can be extended to the generalized mvi $MVI(F'_-, K)$ as a consequence of Theorem 2 and Proposition 4

Corollary 1. *Let $x^* \in \ker K$ and let $f \in RLSC(K, x^*)$. If x^* solves Minty $VI(f'_-, K)$, then x^* solves $P(f, K)$.*

4 The shape of the feasible region

In the previous section, Theorem 2 requires that the feasible region is star-shaped and the solution x^* belongs to its kernel. We now show that such requirements are quite natural for mvi. Indeed, when $f \in RLSC(f, K)$, a solution to $MVI(f'_-, K)$ exists only when K is *st-sh* and no solution can be outside *ker* K.

To prove this result, we find easier to extend our problem by involving the function $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ which is defined as $\bar{f}(x) = \begin{cases} f(x), & x \in K \\ +\infty, & x \in \mathbb{R}^n \setminus K \end{cases}$. It is quite obvious that f and $\bar{f}$ coincide over K. Moreover we can define the (generalized) mvi with respect to the Dini derivative of $\bar{f}$ as

$GMVI(\bar{f}', \mathbb{R}^n)$ $\qquad\qquad \bar{f}'_-(x, x^* - x) \le 0, \ \forall x \in \mathbb{R}^n .$

We say $x^* \in \mathbb{R}^n$ is a solution to $GMVI(\bar{f}', E)$ when the inequality is satisfied any time the directional derivative $\bar{f}'_-(x, x^* - x)$ has sense. Since in $\mathbb{R} \cup \{\pm\infty\}$ the operation $+\infty - (+\infty)$ is not defined, we accept that $\bar{f}'_-(x, u)$ has no sense when $f(x) = +\infty$ and there exists an interval $(x, x + \varepsilon u) = \{x + tu : 0 < t < \varepsilon\}$ with $\varepsilon > 0$ contained in $\mathbb{R}^n \setminus \operatorname{dom} \bar{f}$ (recall that $\operatorname{dom} \bar{f} = \{x \in \mathbb{R}^n : \bar{f}(x) \in \mathbb{R}\}$). We can also extend problem $P(f, K)$, by means of $\bar{f}$, to

$$\min \bar{f}(x), \quad \text{s.t.} \quad x \in \mathbb{R}^n . \tag{1}$$

Next proposition gives the relation between these pairs of problems. The proof can be found in [3].

Proposition 6. *Let $\bar{f} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a given function and let $K := \operatorname{dom} \bar{f}$ and $f := \bar{f}_{|K}$ (the restriction of $\bar{f}$ on K). Then $GMVI(\bar{f}', \mathbb{R}^n)$ has a solution $x^* \in K$ if and only if x^* solves $MVI(f', K)$. Similarly, problem (1) has a global (local) minimizer $x^* \in K$ if and only x^* is a global (local) minimizer for problem $P(f, K)$.*

Some properties can be defined on $\bar{f}$ from those of f.

Definition 7. *Let $\bar{f} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a given function. Then we say that $\bar{f}$ has the RLSC property ($\bar{f} \in RLSC$), if and only if $\bar{f} \in RLSC(\mathbb{R}^n, x^*)$ $\forall x^* \in \mathbb{R}^n$.*

Definition 8. *Let $K \subseteq \mathbb{R}^n$ and $x^* \in \mathbb{R}^n$. We say that K is radially closed along the rays starting at x^* when $K \cap R_{x^*,x}$ is closed in $R_{x^*,x}$, any ray starting at x^*. We say, that K is radially closed if and only if the previous property holds for every $x^* \in \mathbb{R}^n$.*

Proposition 7 ([3]). *Let $\bar{f} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a given function and $K := \operatorname{dom} \bar{f}$ be radially closed. Then $\bar{f} \in RLSC$ if and only if $f \in RLSC(K, x^*)$, $\forall x^* \in K$.*

In [3] we proved as well the Minty Variational Principle applied to $\bar{f}$.

Theorem 3. *Let $\bar{f} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a given function, and $K := \operatorname{dom} \bar{f}$ be radially closed. If $x^* \in \operatorname{dom} \bar{f}$ is a solution of $GMVI(\bar{f}'_-, \mathbb{R}^n)$ and $\bar{f}$ has the RLSC property, then $\bar{f} \in IAR(\mathbb{R}^n, x^*)$. Conversely, if $x^* \in \operatorname{dom} \bar{f}$ and $\bar{f} \in IAR(\mathbb{R}^n, x^*)$, then x^* is a solution of $GMVI(\bar{f}', \mathbb{R}^n)$.*

Hence we can state the following result, which prove that, when the generalized mvi has a solution, then the feasible region is star-shaped and the solution itself is in the kernel of K.

Theorem 4. *Let $\bar{f} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function and $K := \operatorname{dom} \bar{f}$ be closed. Assume that $GMVI(\bar{f}'_-, \mathbb{R}^n)$ has at least one solution $x^* \in K$. Then*

i) *all the level sets of $\bar{f}$ are* st-sh *and contain x^* in their kernels;*

ii) *the set $GM(\bar{f}, \mathbb{R}^n)$ of the global minimizers of (1) is* st-sh *with $x^* \in \ker GM(\bar{f}, \mathbb{R}^n)$;*

iii) *K is* st-sh *with $x^* \in \ker K$;*

iv) *the set of solutions $x^* \in K$ of $GMVI(\bar{f}', \mathbb{R}^n)$ is a convex and closed subset of $\ker GM(\bar{f}, \mathbb{R}^n)$.*

Examples in [3] suggest the inclusions described in Theorem 4 are strict. The coincidence can be proved assuming that f is quasiconvex.

Theorem 5. *Let $\bar{f} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a quasiconvex function having the RLSC property and $K := \operatorname{dom} \bar{f}$ be radially closed and convex. Then the set $GMVI(\bar{f}', \mathbb{R}^n)$ and the set $GM(\bar{f}, \mathbb{R}^n)$ of global minimizers of problem (1) coincide.*

Proof: Because of Theorem 4, we only need to prove that each $x^* \in GM(\bar{f}, E)$ is a solution of $GMVI(\bar{f}', E)$. We can prove that $\bar{f} \in IAR(E, x^*)$, to apply Theorem 3. By contradiction, let $x \in E$ be such that $\bar{f}(x(t_1)) > \bar{f}(x(t_2))$ for some $0 \leq t_1 < t_2$, $x(t) = (1-t)x^* + tx$. Let $c = \bar{f}(x(t_2))$. Since $x^* \in GM(\bar{f}, E)$, $\bar{f}(x^*) \leq \bar{f}(x(t_2)) = c$. Hence both $\bar{f}(x^*)$, $f(x(t_2)) \in \mathrm{lev}_{\leq c}\bar{f}$, while $(1 - \frac{t_1}{t_2})x^* + \frac{t_1}{t_2}x(t_2) = x(t_1) \notin \mathrm{lev}_c\bar{f}$. This contradicts the quasiconvexity assumption. □

We close this survey exploring the relations between the mvi we introduced in Section 3 and the one in Section 2. When f is differentiable over an open set containing K and K is a convex set, the two problems actually coincide. However, if K is just *st-sh*, but not convex, one can find solutions to $MVI(f', K)$ which are not solution to $GMVI(\bar{f}'_-, \mathbb{R}^n)$.

Example 3. *Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = x_2^2$, and $K = \{(x_1, x_2) : x_1 \geq 1$ or $(x_1 \geq -1$ and $|x_2| \leq 1)\}$. It is easy to check that the solution set of $MVI(f', K)$ is $\{x \in \mathbb{R}^2 : x_1 \geq -1, x_2 = 0\}$ and the solution set of the corresponding $GMVI(\bar{f}', \mathbb{R}^2)$ (with $\bar{f}(x) = +\infty$ for $x \notin K$) is $\{x \in \mathbb{R}^2 : x_1 \geq 1, x_2 = 0\}$.*

According to Theorem 4, solutions of $GMVI(\bar{f}'_-, \mathbb{R}^n)$ are also solutions of $P(f, K)$. However, Example 3 leave the possibility that solutions of $MVI(f', K)$ may not be solutions of $P(f, K)$.

Example 4. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x_1, x_2) = x_1x_2(x_1+x_2) - \frac{1}{3}(x_1+x_2-1)^2$ and let $K = \{[(-1, 0), (0, 1)] \cup [(0, 1), (0, 4)]\}$. The set K is st-sh with $\ker \mathrm{K} = \{(0, 1)\}$ and $MVI(f', K)$ has the unique solution $x^* = (-1, 0) \notin \ker \mathrm{K}$. This point does not belong to the set $GM(f, K)$ which is the singleton $\{(0, 4)\}$.*

The result of Example 4 is a very special case, indeed.

Proposition 8. *If $MVI(f', K)$ admits at least one solution $x^* \in \ker \mathrm{K}$, then each solution of $MVI(f', K)$ is a solution of the related minimization problem.*

Proof: Bu contradiction, assume $\bar{x} \neq x^*$ is a solution of $MVI(f', K)$. Since $x^* \in \ker \mathrm{K}$ solves $MVI(f', K)$ then also x^* solves $GMVI(f', K)$ and $f \in IAR(K, x^*)$. Hence f is increasing along the ray $R_{x^*,x} \cap K$. However, also $\bar{x}$ solves $MVI(f', K)$. Hence it can be easily seen that f is increasing along the ray $R_{\bar{x},x^*} \cap K$. This implies that all points along the segment wih enpoints x^* and $\bar{x}$ are minimizers of f over K. □

Remark 3. Example 4 proves that Minty Variational Principle, without the convexity of the feasible region may not hold. The last proposition can be seen as a weaker version of the principle.

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