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Some Considerations on Extensions of Cooperative Games

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1 Introduction

A cooperative game (transferable utility game) is specified by two factors, a set of players and a characteristic function which assigns a real number to each coalition (subset of the player set). A main problem in a TU-game is to fix an allocation rule (solution) such that the players may redistribute the utilities among themselves.

In order to deal with partial participation of players in a coalition, the concept of fuzzy coalitions was introduced by Aubin [1]. Some developments of cooperative fuzzy games can be seen in Nishizaki and Sakawa [7] and Brânzei [3]. If we regard an ordinary coalition as a vertex of the hypercube $[0,1]^n$, i.e., a point in $\{0,1\}^n$ (n is the number of players), a fuzzy coalition is a point in the hypercube $[0,1]^n$. Thus the domain of a characteristic function in a cooperative fuzzy game is $[0,1]^n$, while that in an ordinary cooperative game is $\{0,1\}^n$.

In this paper we consider special classes of cooperative fuzzy games which are obtained as extensions of ordinary cooperative games. These classes include the multilinear extensions and the Lovász extensions. Several properties such as superadditivity and convexity of these classes of games will be investigated. We also discuss solutions for fuzzy games obtained as extensions of ordinary games in terms of the concept of dividends, which are coefficients in representation of ordinary games as linear combinations of the unanimity games.

2 Cooperative Games

A transferable utility game (cooperative game) (N, v) or simply v consists of two factors: a set of players $N = \{1, 2, \ldots, n\}$ and a characteristic function $v : 2^N \to \mathbf{R}$ such that $v(\emptyset) = 0$. A subset $S \subseteq N$ is usually called a coalition and $v(S)$ is called the worth of S. We denote by Γ^N the set of all TU-games on N.

Definition 1 *A game $v \in \Gamma^N$ is said to be*

1. **monotonic** *if*

$$v(S) \le v(T), \ \ \forall S, T \subseteq N, \ s.t. \ S \subseteq T.$$

2. **superadditive** *if*

$$v(S \cup T) \ge v(S) + v(T), \ \ \forall S, T \subseteq N, \ s.t. \ S \cap T = \emptyset.$$

3. **convex** *if*

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T), \quad \forall S, T \subseteq N.$$

In other words, the game (N, v) is convex if and only if the set function v is supermodular (The terminology "supermodular" is used in discrete convex analysis).

The sum of two games $v, w \in \Gamma^N$, and the scalar multiplication of $v \in \Gamma^N$ by $\alpha \in \mathbf{R}$ are defined as follows:

$$\begin{array}{ll} (v + w)(S) = v(S) + w(S), & \forall S \subseteq N, \\ (\alpha v)(S) = \alpha v(S), & \forall S \subseteq N, \end{array}$$

Thus Γ^N is a $(2^n - 1)$-dimensional vector space.

Usually we consider unanimity games as the basis of this linear space. The unanimity game $u_T \in \Gamma^N$ for $T \subseteq N$, $T \neq \emptyset$, is defined by

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S \\ 0, & \text{otherwise} \end{cases}$$

A game $v \in \Gamma^N$ can be represented as a linear combination of the unanimity games as $v = \sum\limits_{T \subseteq N} d_T(v) u_T$. Hence $v(S) = \sum\limits_{T \subseteq S} d_T(v)$. Here the coefficient

$$d_T(v) = \sum_{S \subseteq T} (-1)^{|T| - |S|} v(S)$$

is called the Harsanyi dividend. It can be also given by the following recursive formula:

$$d_T(v) = \begin{cases} 0, & \text{if } T = \emptyset \\ v(T) - \sum_{S \subset T} d_S(v), & \text{if } T \neq \emptyset \end{cases}$$

3 Extension of Cooperative Games : Cooperative Fuzzy Games

An ordinary game on the player set N is specified by a characteristic function $v : 2^N \to \mathbf{R}$. Here each coalition $S \subseteq N$ is identified with a vector $e^S \in \{0, 1\}^n$ through the correspondence

$$e_i^S = \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases}$$

Thus v is a function defined on the set $\{0, 1\}^n$. In this section we consider an extension of v, that is a cooperative fuzzy game $\xi_v : [0, 1]^n \to \mathbf{R}$. Of course, each extension must satisfy the relations

$$\xi_v(e^S) = v(S), \quad \forall S \subseteq N.$$

Cooperative fuzzy games obtained as appropriate extensions from ordinary cooperative games form a important class of fuzzy games.

In this paper we use the following notations. First $e^i = e^{\{i\}}$. For $s, t \in [0,1]^n$,

$$(s \vee t)_i = \max\{s_i, t_i\},\ (s \wedge t)_i = \min\{s_i, t_i\},\ i = 1, 2, \ldots, n.$$

The support of a vector $s \in [0,1]^n$ is denoted by supp $s = \{i \in N \mid s_i > 0\}$. We denote by Δ^N the set of all cooperative fuzzy games on N.

Definition 2 *Cooperative fuzzy game $\xi \in \Delta^N$ is said to be*

1. **monotonic** *if*

$$\xi(s) \leq \xi(t),\ \ \forall s, t \in [0,1]^n,\ s.t.\ s \leq t.$$

2. **weakly superadditive** *if*

$$\xi(s) + \xi(t) \leq \xi(s \vee t),\ \ \forall s, t \in [0,1]^n,\ s.t.\ s \wedge t = 0.$$

3. **strongly superadditive** *if*

$$\xi(s) + \xi(t) \leq \xi(s + t),\ \ \forall s, t \in [0,1]^n,\ s.t.\ s + t \in [0,1]^n.$$

4. **convex** *if*

$$\xi(s) + \xi(t) \leq \xi(s \vee t) + \xi(s \wedge t),\ \ \forall s, t \in [0,1]^n.$$

It can be easily confirmed that a strongly superadditive game is weakly superadditive and that a convex game is weakly superadditive.

Lemma 1 *Let $\xi \in \Delta^N$ and assume that ξ is positively homogeneous, i.e., $\xi(\lambda s) = \lambda \xi(s)$ for any $\lambda > 0$ and $s \in [0,1]^n$ such that $\lambda s \in [0,1]^n$. Then ξ is strongly superadditive if and only if ξ is a concave function on $[0,1]^n$.*

Some extensions of cooperative games have been already considered, though they have not been necessarily regarded as cooperative fuzzy games. We consider the multilinear extension, the Butnariu's class and the Lovász extension.

Multilinear Extensions (Owen [8])

Definition 3 *The* **multilinear extension** *$m_v \in \Delta^N$ of $v \in \Gamma^N$ is defined by*

$$m_v(s) = \sum_{S \subseteq N} \Big(\prod_{i \in S} s_i \prod_{i \in N \setminus S} (1 - s_i)\Big) v(S),\ \ \forall s \in [0,1]^n.$$

Proposition 1 *Let $v, w \in \Gamma^N$ and $\alpha \in \mathbf{R}$. Then m_v is continuous w.r.t. v and linear in each component s_i. Moreover,*

$$m_{v+w} = m_v + m_w,\ \ m_{\alpha v} = \alpha m_v.$$

Proposition 2 *(Owen [8]) Let $v \in \Gamma^N$ and $m_v \in \Delta^N$ be its multilinear extension. Then the Shapley value of the game v for the player i is given by*

$$\phi_i(v) = \int_0^1 \frac{\partial}{\partial s_i} m_v(t, \dots, t)dt, \quad \forall i \in N$$

Butnariu's Class of Fuzzy Games

Butnariu [4] studied a class of cooperative fuzzy games, though it might not be regarded as an extension of ordinary cooperative games. For a vector $s \in [0,1]^n$, let

$$\langle s \rangle_h = \{i \in N \mid s_i = h\}, \quad \forall h \in [0,1].$$

Definition 4 *Given a cooperative game (N, v), the cooperative fuzzy game $b_v \in \Delta^N$ defined by*

$$b_v(s) = \sum_{h \in [0,1]} v(\langle s \rangle_h) \cdot h, \quad \forall s \in [0,1]^n$$

is called the **game with proportional values**.

This game is an extension of v, but not continuous with respect to v and therefore a little strange.

Proposition 3 *If a game $v \in \Gamma^N$ is superadditive, then the fuzzy game $b_v \in \Delta^N$ is weakly superadditive.*

Lovász Extensions

For $s \in [0,1]^n$, let $0 \le h_1 < \cdots < h_q$ be the components of s with different values, $h_0 = 0$ and

$$[s]_h = \{i \in N \mid s_i \ge h\}, \quad h \in [0,1].$$

Definition 5 *Let $v \in \Gamma^N$. Its* **Lovász extension** *$l_v \in \Delta^N$ is defined by*

$$l_v(s) = \sum_{p=1}^{q} v([s]_{h_p})(h_p - h_{p-1}), \quad \forall s \in [0,1]^n.$$

Remark 1 *1) Since $[s]_h = [s]_{h_p}$ for all $h_{p-1} < h \le h_p$,*

$$l_v(s) = \int_0^1 v([s]_h)dh, \quad \forall s \in [0,1]^n.$$

Thus the Lovász extension can be obtained from the original game by the Choquet integral. Tsurumi, Tanino and Inuiguchi [11, 12] introduced, independetly of Lovász, the class of cooperative fuzzy games with Choquet integral form and discussed solution concepts, the core function and the Shapley function.

2) If we put

$$S_p = \{i \in N : s_i \geq h_p\}$$

and

$$f_s(S) = \begin{cases} h_p - h_{p-1}, & \textit{if } S = S_p \\ 0, & \textit{otherwise}, \end{cases}$$

then,

$$l_v(s) = \sum_{S \subseteq N} f_s(S) v(S).$$

This formula is inspired by the equation

$$s = \sum_{S \subseteq N} f_s(S) e^S, \quad \forall s \in [0,1]^n,$$

and adopted as the definition of the Lovász extension in Bilbao [2].

3) The Lovász extension can be defined not only for $s \in [0,1]^n$, but also for any $s \in \mathbf{R}^n_+$. In this paper, however, we restrict its domain to $[0,1]^n$, because we regard it as a cooperative fuzzy game.

The Lovász extension (game with Choquet integral form) is continuous with respect to v.

Proposition 4 *Let $v, w \in \Gamma^N$ and $\alpha \in \mathbf{R}$. Then l_v is positively homogeneous, i.e., $l_v(\alpha s) = \alpha l_v(s)$ for all $\alpha \geq 0$ (when we consider the Lovász extension on $\mathbf{R}^n_+$) and*

$$l_{v+w} = l_v + l_w, \quad l_{\alpha v} = \alpha l_v.$$

Theorem 1 *If a game $v \in \Gamma^N$ is monotonic, then its Lovász extension $l_v \in \Delta^N$ is monotonic.*

Noting that, for any $h \in [0,1]$,

$$[s \vee t]_h = [s]_h \cup [t]_h, \quad [s \wedge t]_h = [s]_h \cap [t]_h.$$

we can prove the following results.

Theorem 2 *If a cooperative game $v \in \Gamma^N$ is superadditive, then its Lovász extension $l_v \in \Delta^N$ is weakly superadditive.*

Theorem 3 *If a cooperative game $v \in \Gamma^N$ is convex, then its Lovász extension $l_s \in \Delta^N$ is a convex cooperative fuzzy game.*

Proposition 5 *(Bilbao, Fujishige, Murota) Let $v \in \Gamma^N$ and $l_v \in \Delta^N$ be its Lovász extension. The game v is convex if and only if the function l_v is concave (on $\mathbf{R}^n_+$).*

Theorem 4 *A game $v \in \Gamma^N$ is convex if and only if its Lovász extension l_s is a strongly superadditive cooperative fuzzy game.*

4 Extensions Based on Dividends

In this section we consider a more general extension of an ordinary cooperative game $v \in \Gamma^N$. We recall that

$$v(S) = \sum_{T \subseteq N} d_T(v) u_T(S), \quad \forall S \subseteq N$$

with

$$d_T(v) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S), \text{ and } u_T(S) = \begin{cases} 1, & \text{if } S \supseteq T \\ 0, & \text{otherwise} \end{cases}$$

Now let us consider an extension of the unanimity game $u_T \in \Gamma^N$ to $a_T \in \Delta^N$. Namely

$$a_T(e^S) = \begin{cases} 1, & \text{if } S \supseteq T \text{ (i.e., if } e^S \geq e^T) \\ 0, & \text{otherwise} \end{cases}$$

Then we can define an extension $\xi_v^a \in \Delta^N$ of $v \in \Gamma^N$ depending on a by

$$\xi_v^a(s) = \sum_{T \subseteq N} d_T(v) a_T(s), \quad \forall s \in [0,1]^n$$

It is clear that

$$\xi_{u_S}^a(s) = a_S(s), \quad \forall s \in [0,1]^n$$

Proposition 6 *Let $v, w \in \Gamma^N$, and $\alpha \in \mathbf{R}$. Then*

$$\xi_{v+w}^a = \xi_v^a + \xi_w^a, \quad \xi_{\alpha v}^a = \alpha \xi_v^a.$$

Conversly, if the extension $v \mapsto \xi_v$ is linear w.r.t. v, it can be represented as

$$\xi_v = \sum_{T \subseteq N} d_T(v) a_T$$

for some $a_T \in \Delta^N$ which is the extension of u_T.

Simple examples of $a_T(s)$ are given as follows:

- $\prod_{i \in T} s_i$: multilinear extension (Owen [9])
- $\min_{i \in T} s_i$: Lovász extensdion (Bilbao [2])

Theorem 5 *Let $v \in \Gamma^N$ and $m_v \in \Delta^N$ be its multilinear extension. Then*

$$m_v(s) = \sum_{T \subseteq N} d_T(v) \prod_{i \in T} s_i.$$

Theorem 6 *Let $v \in \Gamma^N$ and $l_v \in \Delta^N$ be its Lovász extension. Then*

$$l_v(s) = \sum_{T \subseteq N} d_T(v) \min_{i \in T} s_i.$$

Other possibilities of a_T is a t-norms in fuzzy set theory, for example,

$$a_T(s) = \max\{0, \sum_{i \in T} s_i - |T| + 1\}.$$

and we may consider the extended fuzzy cooperative game

$$k_v(s) = \sum_{T \subseteq N} d_T(v) \max\{0, \sum_{i \in T} s_i - |T| + 1\}.$$

Since $\prod_{i \in T} s_i \leq \min_{i \in T} s_i$ and $\max\{0, \sum_{i \in T} s_i - |T| + 1\} \leq \min_{i \in T} s_i$, the following holds.

Proposition 7 *If $d_T(v) \geq 0$ for all $T \subseteq N$, then the following relations hold:*

$$m_v(s) \leq l_v(s), \text{ and } k_v(s) \leq l_v(s) \ \ \forall s \in [0,1]^n.$$

However, we should note these inequalities are not generally satisfied.

Theorem 7 *If $d_T(v) \geq 0$ and $a_T(s)$ is monotnically nondecreasing for all $T \subseteq N$ in a game v, then the extension ξ_v^a is a monotonic cooperative fuzzy game.*

All the $a_T(s)$ practically introduced in this paper satisfy the monotonicity, and therefore the extensions m_v, l_v and k_v are monotonic.

Lemma 2 *For any $s, t \in [0,1]^n$ and $T \subseteq N$,*

$$\prod_{i \in T}(s_i \vee t_i) + \prod_{i \in T}(s_i \wedge t_i) \geq \prod_{i \in T} s_i + \prod_{i \in T} t_i.$$

Theorem 8 *If $d_T(v) \geq 0$ for all $T \subseteq N$ in a game $v \in \Gamma^N$, then the multilinear extension m_v is a convex cooperative fuzzy game.*

5 Solutions for U-Extensions of Ordinary Cooperative Games

In this section we study solutions for cooperative fuzzy games obtained as extensions of ordinary cooperative games.

Definition 6 *The extension ξ_v of v is said to be a U-***extension** *of v if the way of extension is linear, i.e.,*

$$\xi_{v_1+v_2} = \xi_{v_1} + \xi_{v_2}, \ \forall v_1, v_2 \in \Gamma^N,$$
$$\xi_{\alpha w} = \alpha \xi_w, \ \forall w \in \Gamma^N, \forall \alpha \in \mathbf{R}.$$

*A U-extension is said to be a W-***extension** *if for any nonempty $T \subseteq N$,*

$$\xi_{u_T}(s) = \xi_{u_T}(s_{|T}), \ \forall s \in [0,1]^n,$$
$$\xi_{u_T}(s) \leq \xi_{u_T}(t), \ \forall s, t \in [0,1]^n \text{ such that } s \leq t.$$

Here $s_{|T} \in [0,1]^n$ is defined by $(s_{|T})_i = s_i$ if $i \in T$ and $(s_{|T})_i = 0$ otherwise.

Remark 2 *Let ξ_{u_T} be a W-extension of u_T. Note that $0 \leq s \leq e^{\mathrm{supp}\ s}$ for any $s \in [0,1]^n$. Therefore, if $T \not\subseteq \mathrm{supp}\ s$, then*

$$0 = \xi_{u_T}(0) \leq \xi_{u_T}(s) \leq \xi_{u_T}(e^{\mathrm{supp}\ s}) = u_T(\mathrm{supp}\ s) = 0.$$

Hence $\xi_{u_T}(s) = 0$ for $T \not\subseteq \mathrm{supp}\ s$.

Proposition 8 *(Moritani, Tanino and Tatsumi [6]) If $\xi_v \in \Delta^N$ is a W-extension of $v \in \Gamma^N$, then*

$$\xi_v(s) = \sum_{T \subseteq \mathrm{supp}\ s} d_T(v)\xi_{u_T}(s), \ \forall s \in [0,1]^n.$$

As a solution for a cooperative fuzzy game obtained as a U-extension from an ordinary cooperative game, we consider a point-valued solution, which is a function $f : [0,1]^n \times \Delta^N \to \mathbf{R}^n$. If we assume linearity of f with respect to ξ, it is written as

$$f_i(s,\xi_v) = f_i(s, \xi_{\sum_{T \subseteq N} d_T(v)u_T}) = f_i(s, \sum_{T \subseteq N} d_T(v)\xi_{u_T}) = \sum_{T \subseteq N} d_T(v) f_i(s, \xi_{u_T})$$

for each $v \in \Gamma^N$, $s \in [0,1]^n$ and $i \in N$. Hence this value is specified by values of $f_i(s,\xi_{u_T})$.

When a fuzzy coalition s is formed, the players participating in this coalition will share the worth $\xi_{u_T}(s)$ for the extension of the unanimity game ξ_{u_T}. We introduce a sharing system $p = (p_i^T)_{T \subseteq N, i \in T}$ satisfying $p_i^T \geq 0$ and $\sum_{i \in T} p_i^T = 1$. We put

$$f_i^p(s,\xi_{u_T}) = \begin{cases} p_i^T \xi_{u_T}(s), & \text{if } i \in T, \\ 0, & \text{if } i \notin T. \end{cases}$$

Hence the value corresponding to the sharing system p is given by

$$f_i^p(s,\xi_v) = \sum_{T \subseteq N, T \ni i} d_T(v) p_i^T \xi_{u_T}(s) \ \text{ for each } i \in N.$$

In particular, the case $p_i^T = \frac{1}{|T|}$ for all $i \in T$ corresponds to the Shapley value and the obtained value is

$$\phi_i(s,\xi_v) = \sum_{T \subseteq N, T \ni i} \frac{d_T(v)}{|T|} \xi_{u_T}(s) \ \text{ for each } i \in N.$$

When $\xi_v = l_v$ (Lovász extension),

$$\phi_i(s,l_v) = \sum_{T \subseteq N, T \ni i} \frac{d_T(v)}{|T|} \min_{j \in T} s_j \ \text{ for each } i \in N.$$

It is clear that all these values are zero for $i \notin \mathrm{supp}\ s$.

The above Shapley value for the Lovász extension was studied by Tsurumi et al. [12]. In Tsurumi et al. [12], though the explicit formula is slightly different, they proposed the following Shapley value for $v \in \Gamma^N$ and $s \in [0,1]^n$:

$$g_i(s,l_v) = \int_0^1 \varphi_i([s]_h, v)dh, \ \ i \in N,$$

$$[s]_h = \{j \in N \mid s_j \geq h\}, \quad h \in [0,1]$$

$\varphi_i([s]_h, v)$ is the Shapley value of $i \in N$ for the subgame $v|[s]_h$ of v on $[s]_h$, i.e., in terms of dividends

$$\varphi_i([s]_h, v) = \sum_{T \subseteq [s]_h, T \ni i} \frac{d_T(v)}{|T|}.$$

We should note that $T \subseteq [s]_h$ if and only if $s_j \geq h$ for all $j \in T$, i.e., if and only if $\min_{j \in T} s_j \geq h$. Therefore we can prove that $g_i(s, l_v) = \phi_i(s, l_v)$ for any $i \in N$.

The most fundamental set-valued solution for cooperative games is the core function (Tsurumi, Tanino and Inuiguchi [11]) defined by

$$C(s,\xi) = \{x \in \mathbf{R}^n | \sum_{i \in \text{supp } s} x_i = \xi(s), x_i = 0, \forall i \notin \text{supp } s, \sum_{i \in \text{supp } t} x_i \geq \xi(t), \ 0 \leq \forall t \leq s\}.$$

Finally we discuss some properties of the solution $f^p(s, \xi_v)$. The first result is the efficiency of this solution.

Proposition 9 *Let $v \in \Gamma^N$, $\xi_v \in \Delta^N$ be a U-extension of v and $s \in [0,1]^n$ be a fuzzy coalition. Then for any sharing system $p = (p_i^T)_{T \subseteq N, i \in T}$,*

$$\sum_{i \in N} f_i^p(s, \xi_v) = \xi_v(s).$$

Proposition 10 *Let $v \in \Gamma^N$, $\xi_v \in \Delta^N$ be a W-extension of v and $s \in [0,1]^n$ be a fuzzy coalition. Then for any sharing system $p = (p_i^T)_{T \subseteq N, i \in T}$,*

$$f_i^p(s, \xi_v) = \begin{cases} \displaystyle\sum_{T \subseteq \text{supp } s, \ T \ni i} d_T(v) p_i^T \xi_{u_T}(s) & i \in \text{supp } s, \\ 0 & i \notin \text{supp } s. \end{cases}$$

Proposition 11 *Let $v \in \Gamma^N$ be a positive cooperative game, i.e., $d_T(v) \geq 0$ for all $T \subseteq N$, $\xi_v \in \Delta^N$ be a W-extension of v and $s, t \in [0,1]^n$ be fuzzy coalitions such that $s \leq t$. Then for any sharing system $p = (p_i^T)_{T \subseteq N, i \in T}$,*

$$f_i^p(s, \xi_v) \leq f_i^p(t, \xi_v).$$

Theorem 9 *Let $v \in \Gamma^N$ be a positive cooperative game, $\xi_v \in \Delta^N$ be a W-extension of v, $s \in [0,1]^n$ be a fuzzy coalition and p be a sharing system. Then*

$$f^p(s, \xi_v) \in C(s, \xi_v).$$

6 Conclusion

We have dealt with classes of cooperative games which can be obtained as extensions of ordinary cooperative games. Some of them can be dealt with in a unified manner through the dividend representation. Typical classes are the multilinear extension and the Lovász extension. We studied several properties of those classes of games, for example, monotonicity, superadditivity and convexity. Finally we have discussed solutions of cooperative fuzzy games extended from ordinary cooperative games.

Because of page limitation, all the proofs of lemmas, propositions and theorems are omitted. Some of them can be found in Tanino [10].

References

[1] J.-P. Aubin: *Mathematical Methods of Game and Economic Theory*, revised ed., North-Holland (1982)

[2] J.M. Bilbao: *Cooperative Games on Combinatorial Structures*, Kluwer Academic Publishers (2000)

[3] R. Brânzei: Convex fuzzy games and partition monotonic allocation schemes, *Fuzzy Sets and Systems*, Vol. 139, pp. 267-281 (2003)

[4] D. Butnariu: Stability and Shapley value for an n-person fuzzy game, *Fuzzy Sets and Systems*, Vol. 4, pp. 63-72 (1980)

[5] L. Lovász: Submodulay functions and convexity, in A. Bachem et al. (eds.), *Mathematical Programming: The State of the Art*, Springer-Verlag, pp. 235-257 (1983)

[6] A. Moritani, T. Tanino and K. Tatsumi, Cooperative fuzzy games extended from ordinary cooperative games with restrictions on coalitions, *Kybernetika*, Vol. 42, pp. 461-474 (2006)

[7] I. Nishizaki and M. Sakawa: *Fuzzy and Multiobjective Games for Conflict Resolution*, Physica-Verlag (2001)

[8] G. Owen: Multilinear extensions of games, *Management Science*, Vol. 18, pp.64–79 (1972)

[9] G. Owen, *Game Theory*, 3rd ed., Academic Press (1995)

[10] T. Tanino: Cooperative fuzzy games as extensions of ordinary cooperative games, *Proceedings of 7th Czech-Japan Seminar on Data Analysis and Decision Making under Uncertainty*, pp. 26-31 (2004)

[11] M. Tsurumi, T. Tanino and M. Inuiguchi: The core and the related solution concepts in cooperative fuzzy games, *Journal of Japan Society for Fuzzy Theory and Systems*, Vol. 12, No. 1, pp. 193-202 (2000)

[12] M. Tsurumi, T. Tanino and M. Inuiguchi: A Shapley function of cooperative fuzzy games, *European Journal of Operational Research*, Vol. 129, pp. 596-618 (2001)