数理解析研究所講究録
第 1643 巻 2009 年 232-241

Existence and Multiplicity of Solutions for a Coupled Nonlinear Shrödinger System

Norimichi Hirano

Graduate School of Environment and Information Sciences, Yokohama National University

1. Introduction

In the present paper, we consider the multiple existence of non-radial positive solutions of coupled Schrödinger system

$$\text{(P)} \qquad \begin{cases} -\Delta u+\mu_1 u &= u^3+\beta uv^2 \qquad \text{in } \mathbb{R}^3 \\ -\Delta v+\mu_2 v &= v^3+\beta u^2 v \qquad \text{in } \mathbb{R}^3 \end{cases}$$

where $\mu_1, \mu_2 > 0$and $\beta \in R$.

Coupled nonlinear Schrödinger system (P) models many physical problems. In nonlinear optics, the phenomenon in Kerr-like photorefractive media is described by system (P) (cf. [**2**]). In this case, the solution $u$and $v$denote the components of the beam in Kerr-like photorefractive media, and the coupling constant $\beta$is the interaction between two components u and v.In case $\beta > 0$, the interaction is attractive, while the interaction is repulsive if $\beta < 0$.The bimodal pulse in optical fibers under birefringent effects is also governed by system (P) (cf. [**16**]). It is also known that system (P) is a model for a mixture of two Bose-Einstein condensates(cf. [**7**]).

Motivated by these physical interest, the existence of solutions of (P) has been investigated by several authors. In the case that $\beta > 0$,problem (P) was studied by Ambrosetti & Colorado[**3**], Maia, Montefusco & Pellacci[**15**] and Lin & Wei[**12**]. They proved the existence of least energy solutions $(u, v)$of (P) with $u, v > 0$.By the result of Troy(cf. [**22**]), we know that in this case, all positive solutions $(u, v)$of (P) satisfying

$$\text{(1.1)} \qquad u(x) \longrightarrow 0, v(x) \longrightarrow 0, \qquad \text{as } |x| \longrightarrow \infty$$

are radially symmetric functions. On the other hand, in case that $\beta < 0$,it is known that there is no least energy solution of (P)(cf. [**12**]).Moreover, positive solutions of (P) satisfying (1.1) is not always radial. In case that $\beta < 0$and $|\beta|$is small, there are positive solutions with one component concentrating on the origin and the other component concentrating around a regular polygon(cf. [**14**]). The existence of non radial positive solutions was also considered in [**24**] for the case that $\beta < 0$and $\mu_1 = \mu_2$.In this case, there are infinitely many nonradial solutions if $\beta < -1$.Recently, Sirakov[**21**] established the existence of ground state solutions of (P) in the case that coefficients of nonlinear terms u^3 and v^3 in (P) are different.

On the other hand, the existence of sign changing solutions of nonlinear scalar elliptic problem

$$-\Delta u + u = |u|^{p-1} u, \qquad u \in H_0^1(\Omega) \tag{1.2}$$

has been investigated by many authors in the last decade. Here Ω is a domain in $R^N (N \geq 3)$, and $p \in (1, (N+2)/(N-2)]$. We refer to **[5]**, **[6]** and **[17]** for related results of sign changing solutions of (1.2). The existence of sign changing solutions of (P) with $\beta > 0$ was considered in **[10]**. For the problem (P) with R^3 replaced by a bounded domain, we refer to **[13]** and **[19]**.

In the present paper, we first see the multiple existence of solutions of problem (P) in the case that $\beta > 0$. Next we consider the case that $\beta < 0$ and $|\beta|$ is small, i.e. the case that the interaction of two solutions are small and repulsive. We will show the multiple existence of nonradial solutions of (P) in this case with $\mu_1 \neq \mu_2$. Our results improve the results in **[14]**. (See Remark 2 and Remark 3).

To state our main results, we need some notations. We denote by $B_r(x)$ the open ball in R^3 centered at $x \in R^3$ with radius $r > 0$. The inner product in R^3 is denoted by $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$. We put $H = H^1(R^3)$ and $\mathcal{H} = H \times H$. We set $\mu_0 = 1$. We denote by $\|\cdot\|_{\mu_i}$ the norm of H defined by $\|u\|_{\mu_i}^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + \mu_i |u|^2) dx$ for $u \in H$ and $i \in \{0, 1, 2\}$. For simplicity of notations, we put $|u(x)|_{\mu_i}^2 = |\nabla u(x)|^2 + \mu_1 |u(x)|^2$ for $u \in H$ and $x \in R^3$. For each function $u \in H$, we set $u^+(x) = \max\{u(x), 0\}, u^-(x) = \max\{-u(x), 0\}$. For each $p \geq 1$, we denote by $|\cdot|_p$ the norm of the space $L^p(R^3)$. The Hilbert space $\mathcal{H}$ is equipped with the norm defined by $\|\mathcal{U}\|^2 = \|u\|_{\mu_1}^2 + \|v\|_{\mu_2}^2$ for $\mathcal{U} = (u, v) \in \mathcal{H}$. We recall that for each $i \in \{0, 1, 2\}$, problem

$$(P_i) \qquad \begin{cases} -\Delta u + \mu_i u & = & u^3 & \text{in } \mathbb{R}^3 \\ u(x) & > & 0 & \text{in } \mathbb{R}^3 \\ u(x) & \longrightarrow & 0 & \text{as } |x| \longrightarrow \infty \end{cases}$$

has a radial solution, denoted by U_i (cf. **[8]**, **[11]**). The function U_i is the unique smooth solution of (P_i) up to translation. Moreover we know that U_i satisfies $I_i(U_i) = c_i = \min\{I(v) : v \in S_i\}$, where I_i is the functional associated with problem (P_i) defined by

$$I_i(v) = \frac{1}{2} \|v\|_{\mu_i}^2 - \frac{1}{4} |v^+|_4^4 \qquad \text{for } v \in H \tag{1.3}$$

and S_i is the set defined by

$$S_i = \left\{ v \in H : \|v\|_{\mu_i}^2 = |v^+|_4^4 \right\} \qquad \text{for } i = 1, 2.$$

For each $x \in R^3$and $i \in \{0,1,2\}$,we put $U_{i,x}(\cdot) = U_i(\cdot - x)$.It is also known that

$$(1.4) \qquad |U_i(x)|_{\mu_i} |x| \exp(\sqrt{\mu_i}\,|x|) \longrightarrow c > 0, \text{ as } |x| \longrightarrow \infty \qquad \text{for } i \in \{0,1,2\}.$$

(cf. [**11**]). For each $u \in L^4(R^3)$,we put $\widehat{u}(x) = \int_{B_1(x)} |u(x)|^4\, dx$for $x \in R^3$.Then from (1.4),we can choose $R_0 > 0$such that

$$(1.5) \qquad \widehat{U}_i(z) < \frac{\left|\widehat{U}_i\right|_\infty}{3} \qquad \text{for all } z \in \mathbb{R}^3 \backslash B_{R_0}(0) \text{ and } i \in \{1,2\}.$$

Since we consider the case that $\mu_1 \neq \mu_2$,we may assume without any loss of generality that $\mu_2 < \mu_1$.We can now state our main results.

THEOREM 1. *Suppose that the following condition holds:*

$$0 < 2\sqrt{\mu_2} < \sqrt{\mu_1}.$$

Then there exists $\beta_0 > 0$such that for each $\beta \in (0,\beta_0)$,problem (P)possesses at least one ground state solution $U_0 \in H^1(R^3) \times H^1(R^3)$and one nonradial sign changing solution $U \in H^1(R^3) \times H^1(R^3)$.

THEOREM 2. *Suppose that $\sqrt{\mu_1/\mu_2}$is irrational. Then for each $i \in \{2,4,6,8,12,20\}$,there exists $\beta_i \in (-1,0)$such that for each $\beta \in (\beta_i,0)$,there exists a positive solution $U_i \in H$of (P)such that $ic_1 + c_2 < \Phi(U_i) < ic_1 + 2c_2$and $U_i$has the form*

$$(1.6) \qquad \mathcal{U}_i = (U,V) = (\sum_{j=1}^{i} U_{1,x_j} + u, U_2 + v)$$

where $\{x_1, x_2, \cdots, x_i\}$forms a regular $i-$polyhedra in R^3in case $i \neq 2$and $x_1 = -x_2$in case that $i = 2$,and $u,v \in H$such that $\|(u,v)\|$is so small that

$$(1.7) \qquad \widehat{U}(z) < \frac{1}{2}\left|\widehat{U}\right|_\infty \; \textit{for } z \in \mathbb{R}^3 \backslash (\cup_{j=1}^{i} B_{R_0}(x_j)) \textit{ and}$$

$$\widehat{V}(z) < \frac{1}{2}\left|\widehat{V}\right|_\infty \; \textit{for } z \in \mathbb{R}^3 \backslash B_{R_0}(0).$$

REMARK 1. *The expression (1.6)of $U_i = (U,V)$is unique when $\|(u,v)\|$is so small that condition (1.7)holds, i.e., for each U_i,$(x_1, x_2, \cdots, x_i)$is uniquely determined.*

REMARK 2. *The assertion of Theorem 1 implies that for $\beta < 0$with $|\beta|$sufficiently small, problem (P)possesses at least 6nonradial positive solutions. In [**14**], the existence of positive solutions of $(P)$of the form (1.6)was established in the case that $\{x_1, x_2, \cdots, x_i\}$forms regular cube or tetrahedra under the assumption*

$$\sqrt{\frac{\mu_1}{\mu_2}} < \begin{cases} \frac{\sqrt{3}}{3} & \textit{for the cube} \\ \frac{\sqrt{3}}{2} & \textit{for the tetrahedra.} \end{cases}$$

Our argument employed in this paper does not require the ratio of $\sqrt{\mu_1}$ *and* $\sqrt{\mu_2}$.

THEOREM 3. *Suppose that* $\sqrt{\mu_1/\mu_2}$ *is irrational. Then for each* $k \in N$, *there exists* $\tilde{\beta}_k \in (-1,0)$ *such that for each* $\beta \in (\tilde{\beta}_k, 0)$, *the problem* (P) *has a positive solution* $\tilde{\mathcal{U}}_k$ *such that* $kc_1 + c_2 < \Phi(\tilde{\mathcal{U}}_k) < kc_1 + 2c_2$ *and* $\tilde{\mathcal{U}}_k$ *has the form*

$$\tilde{\mathcal{U}}_k = (U, V) = (\sum_{j=1}^{k} U_{1,x_j} + u, U_2 + v) \tag{1.8}$$

where $\{x_1, x_2, \cdots, x_k\} \subset R^3$ *form a regular k-polygon in a two dimensional subspace of* R^3, *and* $u, v \in H$ *such that* $\|(u,v)\|$ *is so small that*

$$\widehat{U}(z) < \frac{1}{2}\left|\widehat{U}\right|_\infty \text{ for } z \in \mathbb{R}^3\backslash(\cup_{j=1}^k B_{R_0}(x_j)) \text{ and } \widehat{V}(z) < \frac{1}{2}\left|\widehat{V}\right|_\infty \text{ for } z \in \mathbb{R}^3\backslash B_{R_0}(0).$$

REMARK 3. *The existence of positive solutions of* (P) *of the form (1.8) was proved in* **[14]** *in the case that the spacial dimension is* 2 *and* μ_1, μ_2 *satisfy*

$$\sqrt{\frac{\mu_1}{\mu_2}} < \sin\frac{\pi}{k}.$$

We give a sketch of the proof of Theorem 2 for the case $i = 2$. The proofs of Theorem 2 for $i \neq 2$ and the proof of Theorem 3 are slight modifications of that of the case $i = 2$ of Theorem 2. The detail of the proofs can be found in **[9]**.

2. Preliminaries

Throughout the rest of this paper, we assume that $\sqrt{\mu_1/\mu_2}$ is irrational.

For each $u, v \in H = H^1(R^3)$, we put $\langle u,v\rangle = \int_{\mathbb{R}^3} uv$. We denote by $\langle\cdot,\cdot\rangle_{\mu_i}$ the inner product of H defined by $\langle u,v\rangle_{\mu_i} = \int_{\mathbb{R}^3}(\nabla u\cdot\nabla v + \mu_i uv)$ for $u, v \in H$ and $i \in \{0,1,2\}$. The inner product of $\mathbb{H}$ is defined by $\langle \mathcal{U}_1, \mathcal{U}_2\rangle_{\mathbb{H}} = \langle U_1, U_2\rangle_{\mu_1} + \langle V_1, V_2\rangle_{\mu_2}$ for $\mathcal{U}_1 = (U_1, V_1), \mathcal{U}_2 = (U_2, V_2) \in \mathbb{H}$. For each $u \in L^4(R^3)$, we put $\Omega(u) = \left\{x \in \mathbb{R}^3 : \widehat{u}(x) \geq \frac{|\widehat{u}|_\infty}{2}\right\}$ and

$$\mathcal{B}(u) = \frac{\int_{\Omega(u)} x(\widehat{u}(x) - \frac{|\widehat{u}|_\infty}{2})dx}{\int_{\Omega(u)}(\widehat{u}(x) - \frac{|\widehat{u}|_\infty}{2})dx}.$$

The mapping $\mathcal{B}$ is called generalized barycenter, which is introduced in **[18]**(cf. also **[4]**). By Sobolev's embedding theorem(**[1]**), for $p \in [2,6]$ there exists $m_p > 0$ such that

$$|z|_{L^p(B_r(0))} \leq m_p\left(|\nabla z|^2_{L^2(B_r(0))} + \mu_i |z|^2_{L^2(B_r(0))}\right)^{1/2}$$

for $r > 1, z \in H^1(B_r(0))$,and $i \in \{0,1,2\}$.For each $a \in R$,and a functional $F : H \to R$,we denote by $F^a$the level set defined by $F^a = \{v \in \mathbb{H} : F^a(v) \leq a\}$.The same notation is used for functionals defined on H.

It is easy to see that for each $u \in H\setminus\{0\}$with $u^+ \not\equiv 0$,there exists a unique positive number $t$such that $tu \in S_i$(cf. [**25**]). It follows from the definitions of $U_i$that $U_i(x) = \sqrt{\mu_i}U_0(\sqrt{\mu_i}x)$on R^3.Then one can see that

$$c_1 = \sqrt{\mu_1}c_0 > c_2 = \sqrt{\mu_2}c_0. \tag{2.1}$$

Let $i \in \{0,1,2\}$.It is known that $\{U_{i,x} : x \in \mathbb{R}^3\}$is a nondegenerate critical set of I_i(cf. [**23**]). More precisely, we have there exists $\lambda > 0$such that

$$\|u\|_{\mu_i}^2 - 3\left\langle U_i^2 u, u\right\rangle \geq \lambda \|u\|_{\mu_i}^2 \qquad \text{for all } u \in \left\{U_i, \frac{\partial U_i}{\partial x_1}, \frac{\partial U_i}{\partial x_2}, \frac{\partial U_i}{\partial x_3}\right\}^{\perp}. \tag{2.2}$$

We define a functional $\Phi : H \longrightarrow R$associated with problem (P) by

$$\begin{aligned}\Phi(\mathcal{U}) &= \frac{1}{2}(\|U\|_{\mu_1}^2 + \|V\|_{\mu_2}^2) - \frac{1}{4}(|U^+|_4^4 + |V^+|_4^4) - \frac{\beta}{2}\int_{\mathbb{R}^3}(U^+)^2(V^+)^2 \\ &= \Phi_1(\mathcal{U}) + \Phi_2(\mathcal{U}) \qquad \text{for } \mathcal{U} = (U,V) \in \mathbb{H},\end{aligned}$$

where

$$\Phi_1(\mathcal{U}) = \frac{1}{2}\|U\|_{\mu_1}^2 - \frac{1}{4}|U^+|_4^4 - \frac{\beta}{4}\int_{R^3}(U^+)^2(V^+)^2$$

and

$$\Phi_2(\mathcal{U}) = \frac{1}{2}\|V\|_{\mu_1}^2 - \frac{1}{4}|V^+|_4^4 - \frac{\beta}{4}\int_{R^3}(U^+)^2(V^+)^2.$$

Then a direct computation shows

$$\begin{aligned}\langle \nabla\Phi(\mathcal{U}), \mathcal{V}\rangle_{\mathbb{H}} &= \left\langle \begin{pmatrix} \nabla_u\Phi(\mathcal{U}) \\ \nabla_v\Phi(\mathcal{U}) \end{pmatrix}, \begin{pmatrix} W \\ Z \end{pmatrix} \right\rangle_{\mathbb{H}} \\ &= \langle -\Delta U + \mu_1 U - (U^+)^3 - \beta U^+(V^+)^2, W\rangle \\ &\quad + \langle -\Delta V + \mu_2 V - (V^+)^3 - \beta(U^+)^2V^+, Z\rangle\end{aligned}$$

for $U = (U,V), V = (W,Z) \in H$.We put

$$\begin{aligned}\mathcal{M}_+ = \{(U,V) \in \mathbb{H}\setminus\{0\} : \|U\|_{\mu_1}^2 &= |U^+|_4^4 + \beta\int_{\mathbb{R}^3}(U^+)^2(V^+)^2, \\ \|V\|_{\mu_2}^2 &= |V^+|_4^4 + \beta\int_{\mathbb{R}^3}(U^+)^2(V^+)^2\}.\end{aligned}$$

Then one can see that $U = (U,V) \in M_+$ is a critical point of Φif and only if $U$is a positive solution of problem (P). From the definition, we have

$$\Phi_1(\mathcal{U}) = \frac{1}{4}\|U\|_{\mu_1}^2, \Phi_2(\mathcal{U}) = \frac{1}{4}\|V\|_{\mu_2}^2 \text{ and } \Phi(\mathcal{U}) = \frac{1}{4}(\|U\|_{\mu_1}^2 + \|V\|_{\mu_2}^2) \tag{2.3}$$

for $U = (U,V) \in M_+$.We also have that for each $U = (U,V) \in M_+$,there exists $(s,t) \in R^+ \times R^+$such that $(sU,tV) \in M_+$.In fact, for each $U = (U,V)$,$U \not\equiv$

$0, V \not\equiv 0, (sU, tV) \in M_+$ if and only if

$$\begin{pmatrix} s^2 \\ t^2 \end{pmatrix} = A^{-1} \begin{pmatrix} \|U\|_{\mu_1}^2 \\ \|V\|_{\mu_2}^2 \end{pmatrix}$$

where

$$A = \begin{pmatrix} |U^+|_4^4 & \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 \\ \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 & |V^+|_4^4 \end{pmatrix}.$$

Since $\beta \in (-1, 0)$, we have by the Schwartz's inequality that A^{-1} exists and then there exists a unique solution $(s, t) \in R^+ \times R^+$. For given $U = (U, V) \in H$ with $U \not\equiv 0, V \not\equiv 0$, we put $NU = N(U, V) = (N_1 U, N_2 V) = (sU, tV) \in M_+$.

Now to prove Theorem 1 for the case that $i = 2$, we define $H_2 \subset H, H_2 \subset H$ and $M_2 \subset M_+$ by

$$H_2 = \{u \in H : u(x) = u(-x) \quad \text{for } x \in \mathbb{R}^3\}, \qquad \mathbb{H}_2 = H_2 \times H_2,$$

and

$$\mathcal{M}_2 = \mathcal{M}_+ \cap \mathbb{H}_2.$$

Since $\sqrt{\mu_1/\mu_2}$ is irrational, we can choose $\delta_2 \in (0, c_2)$ so small that

$$c_1 + kc_2 \notin [2c_1 + c_2 - \delta_2, 2c_1 + c_2 + \delta_2] \qquad \text{for all } k \in \mathbb{N}. \tag{2.4}$$

The following Lemmata are crucial for our argument.

LEMMA 1. (1) *There exists $\beta_1 \in (-1, 0)$ such that for each $\beta \in (\beta_1, 0)$ and each critical point $U \in M_2 \cap \Phi^{2c_1 + 2c_2}$ of Φ,*

$$\Phi(\mathcal{U}) \in \cup_{i \geq 1, j \geq 1} [ic_1 + jc_2 - \delta_2/2, ic_1 + jc_2 + \delta_2/2].$$

(2) *Let $\beta \in (\beta_1, 0)$ and $\{\mathcal{U}_n\} \subset M_2$ such that $\lim_{n \to \infty} \nabla \Phi(U_n) = 0$ and $\lim_{n \to \infty} \Phi(U_n) = 2c_1 + c_2 + \varepsilon$ with $\varepsilon \in (0, \delta_2/2)$. Then there exists a convergence subsequence $\{\mathcal{U}_{n_i}\} \subset \{\mathcal{U}_n\}$.*

LEMMA 2. *For given $\varepsilon > 0$, there exists $\beta_\varepsilon \in (\beta_1, 0)$ such that for each $\beta \in (\beta_\varepsilon, 0)$ and for each $x \in R^3 \setminus \{0\}$,*

$$\Phi(\mathcal{N}(U_{1,x} + U_{1,-x}, U_2)) < 2c_1 + c_2 + \varepsilon \tag{2.5}$$

a

3. Sketch of the Proof of Theorem 2 for $i = 2$.

Throughout this section we assume that $\beta \in (\beta_1, 0)$. We put

$$b_R(U) = \int_{\mathbb{R}^3 \setminus B_R(0)} |U|^2_{\mu_1} \qquad \text{for } U \in H \text{ and } R > 0$$

and

$$\Lambda_{2,\varepsilon}(R) = \left\{ \mathcal{U} = (U,V) \in \Phi^{2c_1+c_2+\varepsilon} \cap \mathcal{M}_2 : b_R(U) \geq 8c_1 - \min\left\{ \frac{1}{2m_4^4}, c_1 \right\} \right\}$$

for each $\varepsilon > 0$ and $R > 0$.

PROPOSITION 1. *For $\varepsilon > 0$ sufficiently small, there exists $(R_\varepsilon, \delta_\varepsilon, \alpha_\varepsilon, \gamma_\varepsilon) \in (\mathbb{R}^+)^4$ such that $\lim_{\varepsilon \to 0} \delta_\varepsilon = \lim_{\varepsilon \to 0} \alpha_\varepsilon = \lim_{\varepsilon \to 0} \gamma_\varepsilon = 0$ and each $\mathcal{U} = (U,V) \in \Lambda_{2,\varepsilon}(R_\varepsilon)$ has the form*

$$\mathcal{U} = (\alpha(U_{1,x} + U_{1,-x}) + u, \gamma U_2 + v) \tag{3.1}$$

where $\alpha \in (1-\alpha_\varepsilon, 1+\alpha_\varepsilon), \gamma \in (1-\gamma_\varepsilon, 1+\gamma_\varepsilon)$,

$$|x| \geq R_\varepsilon, \quad x = \mathcal{B}(U|_{B_{R_0}(x)}), \quad \widehat{U}(z) < \frac{1}{2}\left|\widehat{U}\right|_\infty \quad \text{for } z \in \mathbb{R}^3 \setminus \bigcup_{i=\pm 1} B_{R_0}(ix), \tag{3.2}$$

$$\widehat{V}(z) < \frac{1}{2}\left|\widehat{V}\right|_\infty \quad \text{for } z \in \mathbb{R}^3 \setminus B_{R_0}(0), \tag{3.3}$$

and

$$(u,v) \in \{U_{1,x}, U_{1,-x}\}^\perp \times \{U_2\}^\perp \text{ with } \|u\|^2_{\mu_1} + \|v\|^2_{\mu_2} \leq \delta_\varepsilon. \tag{3.4}$$

REMARK 4. *By (3.2) and the definition of $\mathcal{B}$, one can see that for each $\mathcal{U} \in \Lambda_{2,\varepsilon}(R_\varepsilon)$, $(x,-x) \in R^3 \times R^3$ in (3.1) is uniquely determined, and the mapping $\mathcal{U} \in \Lambda_{2,\varepsilon}(R_\varepsilon) \to (x,-x) \in R^3 \times R^3$ is continuous. We define a continuous mapping $\eta : \Lambda_{2,\varepsilon}(R_\varepsilon) \longrightarrow R^+$ by*

$$\eta(\mathcal{U}) = |x| \qquad \text{for } \mathcal{U} \in \Lambda_{2,\varepsilon}(R_\varepsilon). \tag{3.5}$$

We also need the following Proposition.

PROPOSITION 2. *There exists $M_0 > 0$ satisfying that for $\varepsilon > 0$ sufficiently small,*

$$\Phi(\mathcal{U}) \geq 2c_1 + c_2 - \beta M_0 e^{-2\sqrt{\mu_2}|x|} \qquad \text{for each } \mathcal{U} \in \Lambda_{2,\varepsilon}(R_\varepsilon),$$

where $x \in R^3$ such that $\mathcal{U}$ has the form (3.1).

Now for $x \in R^3 \setminus \{0\}$, we define a class $\Gamma_2(x) \subset C([0,1], M_2)$ by

$$\Gamma_2(x) = \{p \in C([0,1], \mathcal{M}_2) : p(0) = \mathcal{N}(U_1, U_2), p(1) = \mathcal{N}(U_{1,x} + U_{1,-x}, U_2)\}$$

and put

$$c_2(x) = \inf_{p \in \Gamma_2(x)} \sup_{t \in [0,1]} \Phi(p(t)).$$

We also note that from the definitions of N and Φ, we have that $N(U_{1,x}+ U_{1,-x}, U_2) - (U_{1,x} + U_{1,-x}, U_2) \longrightarrow 0$ in H as $|x| \longrightarrow \infty$ and then

$$\lim_{|x| \longrightarrow \infty} \Phi(\mathcal{N}(U_{1,x} + U_{1,-x}, U_2)) = 2I_1(U_1) + I_2(U_2) = 2c_1 + c_2. \tag{3.6}$$

Based on the preliminary results above, we can prove Theorem 2 for $i = 2$.

Proof of Theorem 2. Let $\varepsilon \in (0, \delta_2/2)$ sufficiently small. Let $\beta \in (\beta_\varepsilon, 0)$. To complete the proof, it is sufficient to show that there exists $\delta > 0$ and $R > 0$ such that

$$2c_1 + c_2 + \delta < c_2(x) < 2c_1 + c_2 + \delta_2/2 \qquad \text{for } |x| > R. \tag{3.7}$$

In fact, if the inequalities above hold, we have by (3.6) that we can choose $x \in R^3$ such that $|x| > R$ and

$$\Phi(\mathcal{N}(U_{1,x} + U_{1,-x}, U_2)) < 2c_1 + c_2 + \delta.$$

That is $\Phi(p(1)) < c_2(x)$ for all $p \in \Gamma_2(x)$. We also have $\Phi(p(0)) < \frac{7}{4}c_1 + c_2$. Then since the Palais-Smail condition holds by (2) of Lemma 1 on $\Phi^{(2c_1+c_2, 2c_1+c_2+\delta_2/2)}$, we have by a standard mountain pass argument that there exists a critical point U of Φ with $\Phi(U) = c_2(x)$.

From the definition of ε and Lemma 2, one can see the pass $p \in \Gamma_2(x)$ defined by

$$p(s) = \mathcal{N}(U_{1,sx} + U_{1,-sx}, U_2), \quad s \in [0,1]$$

satisfies $\max_{s \in [0,1]} \Phi(p(s)) \leq 2c_1 + c_2 + \varepsilon$. Then the second inequality of (3.7) holds. We now show that the first inequality of (3.7) holds. We first see that there exists $\overline{R} > 2R_\varepsilon$ such that ,

$$b_{R_\varepsilon}(U) \geq 8c_1 - \frac{1}{2} \min\left\{\frac{1}{2m_4^4}, c_1\right\} \quad \text{for } \mathcal{U} = (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon) \text{ with } \eta(U) \geq \overline{R}, \tag{3.8}$$

where η is the function defined by (3.5). By Proposition 1, each $U = (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon)$ has the form

$$\mathcal{U} = (\alpha(U_{1,x} + U_{1,-x}) + u, \gamma U_2 + v) \tag{3.9}$$

with $\alpha \in (1-\alpha_\varepsilon, 1+\alpha_\varepsilon), \gamma \in (1-\gamma_\varepsilon, 1+\gamma_\varepsilon)$and $(u,v) \in \{U_{1,x}, U_{1,-x}\}^\perp \times \{U_2\}^\perp$with $\|u\|^2_{\mu_1} + \|v\|^2_{\mu_2} \le \delta_\varepsilon$. Since $\lim_{\varepsilon \longrightarrow 0} \delta_\varepsilon = \lim_{\varepsilon \longrightarrow 0} \alpha_\varepsilon = 0$,we may assume that $\varepsilon > 0$is sufficiently small that

$$8\alpha_\varepsilon^2 c_1 - \delta_\varepsilon > 8c_1 - \frac{1}{2}\min\left\{\frac{1}{2m_4^4}, c_1\right\}. \tag{3.10}$$

Then noting that

$$b_{R_\varepsilon}(U) \ge \alpha^2 \|U_{1,x} + U_{1,-x}\|^2_{\mu_1} - \|u\|^2_{\mu_1} - 2\int_{B_{R_\varepsilon}(0)} |U_{1,x} + U_{1,-x}|^2_{\mu_1}$$

and

$$\|U_{1,x} + U_{1,-x}\|^2_{\mu_1} \longrightarrow 8c_1 \text{ and } \int_{B_{R_\varepsilon}(0)} |U_{1,x} + U_{1,-x}|^2_{\mu_1} \longrightarrow 0, \text{ as } |x| \longrightarrow \infty,$$

we find by (3.10) that there exists $\overline{R}$such that for each $U = (U,V) \in \Lambda_{2,\varepsilon}(R_\varepsilon)$with $\eta(U) \ge \overline{R}$,(3.8) holds. Now we choose $x \in R^3$so large that $|x| > \overline{R}$.Then

$$b_{R_\varepsilon}(\mathcal{N}_1(U_{1,x} + U_{1,-x})) \ge 8c_1 - \frac{1}{2}\min\left\{\frac{1}{2m_4^4}, c_1\right\}.$$

Let $p = (p_1, p_2) \in \Gamma_2(x)$such that $\sup_{t\in[0,1]} \Phi(p(t)) \le 2c_1 + c_2 + \varepsilon$.From the definition,

$$\eta(p_1(1)) = \eta(\mathcal{N}_1(U_{1,x} + U_{1,-x})) > \overline{R} \text{ and } b_{R_\varepsilon}(p_1(1)) \ge 8c_1 - \frac{1}{2}\min\left\{\frac{1}{2m_4^4}, c_1\right\}.$$

On the other hand, recalling that $\Phi_2(U) \ge c_2$,we have that $\Phi_1(U) \le \frac{7}{4}c_1$.Then by the definition of ε,$b_{R_\varepsilon}(p_1(0)) < 7c_1 \le 8c_1 - \min\left\{\frac{1}{2m_4^4}, c_1\right\}$, there exists $t \in (0,1)$such that $b_{R_\varepsilon}(p_1(t)) = 8c_1 - \min\left\{\frac{1}{2m_4^4}, c_1\right\}$.Then by (3.8), $\eta(p_1(t)) < \overline{R}$. Therefore by the continuity of η,we have that there exists $t_0 \in (0,t)$such that $\eta(p_1(t_0)) = \overline{R}$.By Proposition 2, we have

$$\Phi(p(t_0)) \ge 2c_1 + c_2 + \beta M_0 e^{-2\sqrt{\mu_2}\overline{R}}.$$

Therefore we obtain that $\sup_{t\in[0,1]} \Phi(p(t)) > 2c_1 + c_2 + \beta M_0 e^{-2\sqrt{\mu_2}\overline{R}}$.Thus by the mountain pass theorem(cf. [**20**]), we find that there exists a critical point $U$of $\Phi$with $\Phi(U) = c_2(x)$. □

References

1. R. A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.
2. N. Akhmediev and A. Ankiewicz, *Partially coherent solitons on a finite background*, 'jus. Rev. Lett **82** (1999), 2661–2664.
3. A. Ambrosetti and E. Colorado, *Bounded and ground states of coupled nonlinear schrodinger equations*, preprint.
4. T. Bartsch and Tobias Weth, *Three nodal solutions of singularly perturbed elliptic equatuions on domanis without topology*, Ann. I. H. Poincare **22** (2005), 259–281.
5. A. Castro, J. Cossio, and J. M. Neuberger, *A sign-changing solution for a superlinear dirichlet problem*, Rocky Mountain J. Math. **27** (1997), 1041–1053.
6. M. Clapp and T. Weth, *Minimal nodal solutions of the pure critical exponent problem on a symmetric domain*, Cal.Var. and Partial Differential Equations **21** (2004), 1–14.

7. B. D. Esry, C. H. Greene, J. P. Burke, and J. L. Bohn, *Jartree-fock theory for double condensates*, Phys. Rev. Lett **78** (1997), 3494–3497.
8. B. Gidas, W. M. Ni, and L. Nirenberg, *Symmetry of positive solutions fo nonlinear elliptic equation in r^n*, Adv. Math. Suppl. Stud. **7** (1981), 369–402.
9. N. Hirano, *Multiple existence of nonradial positive solutions for a coupled nonlinear schrodinger system*, to appear.
10. N. Hirano and N. Shioji, *Multiple existence of solutions for a coupled nonlinear schrodinger equations*, Nonlinear Analysis TMA, to appear.
11. M. K. Kwong, *Uniqueness of positive solutions of $-\delta u - u + u^p = 0$ in r^n*, Arch. Rat. Mech. Anal. **105** (1989), 243–266.
12. T. C. Lin and J. Wei, *Ground state of n coupled nonlinear schrodinger equations in r^n*, Communications in Mathematical Physics **255** (2005), 629–653.
13. ———, *Spikes in two-component systems of nonlinear schrodinger equations*, Annales de l'Institut H. Poincare **22** (2005), 403–439.
14. ———, *Solitary and self-similar solutions of two-component system of nonlinear schrodinger equations*, Phys. D **2** (2006), 99–115.
15. L.A. Maia, E. Montefusco, and B. Pellacci, *Positive solutions for a weakly coupled nonlinear schrodinger system*, preprint.
16. C. R. Menyuk, *Nonlinear pulse propagation in birefrignent optical fibers*, IEEE J. Quantum Electron **23** (1987), 174–176.
17. E. S. Noussair and J. Wei, *On the effect of domain geometry on the existence of nodal solutions in singular perturbations problems*, Indiana Univ. Math. J. **46** (1996), 1255–1271.
18. G. Cerami & D. Passaseo, *The effect of concentrating potentials in some singulerly perturbed problems*, Carc. Var. PDE. **17** (2003), 257–281.
19. A. Pomponio, *Coupled nonlinear schrodinger systems with potentials*, J. Diff. Equations **227** (2006), 258–281.
20. P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, C. B. M. S. Regional Conf. Math. Series 65. A.M.S. Providence, 1986.
21. B. Sirakov, *Leat energy solitary waves for a system of nonlinear schrodinger equations in r^n*, Communications in Mathematical Physics **271** (2007), 199–221.
22. W. C. Troy, *Symmetry properties in systems of semilinear elliptic equations*, J. Diff. Equations **42** (1981), 400–413.
23. J. Wei, *On the construction of single-peaked solutions to a singularly perturbed semilinear dirichlet problem*, J. Diff. Equations **129** (1996), 315–333.
24. J. Wei and T. Weth, *Existence of nonraidal symmetric bound states for a system of coupled schrodinger equations*, preprint.
25. X. Zhu, *A perturbation result on positive entire solutions of a semilinear elliptic equation*, J. Diff. Equations **92** (1991), 163–178.