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Enumeration of Graph Sandwiches

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Abstract

This paper is concerned with problems to list, sample, and count graphs with edge constraints. The objects we look at are *graph sandwiches*; for a prescribed graph property Π, given a pair of graphs $\overline{G}$ and $\underline{G}$ satisfying $\underline{G} \subseteq \overline{G}$, a graph H is called a graph sandwich of $\overline{G}$ and $\underline{G}$ with Π if H satisfies Π and $\underline{G} \subseteq H \subseteq \overline{G}$. This paper mainly focuses on the classes of chordal graphs and interval graphs as the property Π. It is known that both problems finding a chordal sandwich and finding an interval sandwich are NP-complete for a general input, and we assume a restriction on the input that $\overline{G}$ or $\underline{G}$ satisfy the same property Π as objects. This assumption provides a recursive structure on the set of graph sandwiches with Π and may allow to construct some effective algorithms for our problems. Note that, our objects are a natural generalization of chordal/interval completions/deletions.

1 Introduction

For a pair of graphs G and H on a common vertex set V, we write $G \subseteq H$ (and $G \subsetneq H$) when their edge sets satisfies $E(G) \subseteq E(H)$ (and $E(G) \subsetneq E(H)$, respectively). For a prescribed graph property Π, the *graph sandwich* problem for Π is, given a pair of graphs $\overline{G}$ and $\underline{G}$ satisfying $\underline{G} \subseteq \overline{G}$, to find a graph H satisfying Π and $\underline{G} \subseteq H \subseteq \overline{G}$. Golumbic, Kaplan, and Shamir [6] proposed the graph sandwich problem, and showed NP-hardness of the problem for many properties Π, such as *chordal*, *interval*, *proper interval*, and so on.

Let $\Omega_\Pi(\overline{G}, \underline{G})$ denote the set of graphs defined by

$$\Omega_\Pi(\overline{G}, \underline{G}) \stackrel{\text{def.}}{=} \{G \mid G \text{ satisfies a property } \Pi,\ \underline{G} \subseteq G \subseteq \overline{G}\}. \tag{1}$$

This paper is concerned with the following three types of problems: given a pair of graphs $\overline{G}$ and $\underline{G}$ with $\underline{G} \subseteq \overline{G}$

- output all graphs in $\Omega_\Pi(\overline{G}, \underline{G})$ (listing);
- output the number $|\Omega_\Pi(\overline{G}, \underline{G})|$ (counting);
- output a graph in $\Omega_\Pi(\overline{G}, \underline{G})$ at random according to a prescribed distribution (sampling).

Note that the graphs in $\Omega_\Pi(\overline{G}, \underline{G})$ are assumed to be "labeled," meaning that we distinguish $G \in \Omega_\Pi(\overline{G}, \underline{G})$ from $G' \in \Omega_\Pi(\overline{G}, \underline{G})$ when their edge sets are different even if they are isomorphic graphs. This paper mainly investigates chordal sandwiches and interval sandwiches, with some restricted inputs such as at least one of the input graphs satisfies the prescribed property Π, i.e., chordal/interval, respectively. This assumption is a generalization of chordal/interval completion/deletion problems, and may give an recursive structure on the objective set.

2 Enumeration of Chordal Sandwiches

A graph is *chordal* if it has no induced cycle of length more than three. Given a pair of graphs $\overline{G}$ and $\underline{G}$ satisfying $\underline{G} \subseteq \overline{G}$, let $\Omega_{\mathrm{C}}(\overline{G}, \underline{G})$ be a set of graphs defined by

$$\Omega_{\mathrm{C}}(\overline{G}, \underline{G}) \stackrel{\text{def.}}{=} \{G \mid G \text{ is chordal}, \ \underline{G} \subseteq G \subseteq \overline{G}\}. \tag{2}$$

A graph in $\Omega_{\mathrm{C}}(\overline{G}, \underline{G})$ is called a *chordal sandwich* for the pair $\overline{G}$ and $\underline{G}$, while $\overline{G}$ and $\underline{G}$ are called the *ceiling graph* and the *floor graph* of $\Omega_{\mathrm{C}}(\overline{G}, \underline{G})$, respectively. If $\overline{G}$ is a complete graph, then a chordal sandwich is called a *chordal completion* of $\underline{G}$. If $\underline{G}$ is an empty graph (i.e. has no edge), then a chordal sandwich is called a *chordal deletion* of $\overline{G}$.

2.1 Background

The class of chordal graphs often appears as a tractable case of a lot of problems arising from various areas such as statistics, optimization, numerical computation, etc. In those areas, we often approximate a given graph by a chordal graph and then apply efficient algorithms for chordal graphs to the obtained graph. Evaluation criteria for chordal approximations depend on applications. For example, in the context of graphical modeling in statistics, a chordal approximation is desired to minimize AIC (Akaike's Information Criterion), BIC (Bayesian Information Criterion), MDL (Minimum Description Length), etc. [20, 28, 32]; in the context of numerical computation, a chordal approximation is desired to minimize the number of added edges (a.k.a. the minimum fill-in problem) [22, 23, 30, 3]; in the context of discrete optimization, a chordal approximation is desired to minimize the size of a largest clique (a.k.a. the treewidth problem) [21, 15, 16, 2].

Since we are concerned with various sorts of criteria and often these computational problems are NP-hard, listing algorithms and random-sampling algorithms can be useful universal decision-support schemes. An exhaustive list found by an algorithm may provide an exact solution, whereas random samples may provide an approximative solution. Our goal is to provide efficient algorithms for listing problems and random-sampling problems of graphs, or to show the intractability of the problems.

2.2 Graded poset of chordal graphs

As stated previously, the chordal graph sandwich problem, if $\Omega_{\mathrm{C}}(\overline{G}, \underline{G}) \setminus \{\overline{G}, \underline{G}\}$ has an element, is NP-hard for general inputs $\overline{G}$ and $\underline{G}$, whereas it is polynomial time solvable if at least one of $\overline{G}$ and $\underline{G}$ is chordal. It is from the following by Rose, Tarjan, and Lueker [23].

Theorem 2.1 [23] *For a graph $G = (V, E)$ and a chordal graph $G' = (V, E \cup F)$ with $E \cap F = \emptyset$, the graph G' is a minimal chordal completion of G (i.e., $\Omega_{\mathrm{C}}(G', G) = \{G'\}$) if and only if $G' - f$ is not chordal for each $f \in F$.*

From the result by Rose, Tarjan, and Lueker [23], we have the following fact.

Proposition 2.2 [12] *Suppose a pair of chordal graphs $\overline{G} = (V, \overline{E})$ and $\underline{G} = (V, \underline{E})$ satisfies $\underline{G} \subseteq \overline{G}$, and let $k = |\overline{E} \setminus \underline{E}|$. Then there exists a sequence of chordal graphs $G_0, G_1, \ldots, G_k$ that satisfies $G_0 = \underline{G}$, $G_k = \overline{G}$, and $G_{i+1} = G_i + e_i$ with an appropriate edge $e_i \in \overline{E} \setminus \underline{E}$ for each $i \in \{0, \ldots, k-1\}$.*

Table 1: The time delay of listing chordal sandwiches w.r.t. input pair.

floor graph $\underline{G}$ \ ceiling graph $\overline{G}$	general	chordal	complete graph (cf. chordal completion)
general	(cf. NP-hard)	$O(k(n+m))$	$O(n^3)$
chordal	$O(k(n+m))$	$O(k(n+m))$	$O(n^3)$
empty graph (cf. chordal deletion)	$O(1)$	$O(1)$	$O(1)$

Proof. The proof is done by induction on k. If $k = 0$, then $\underline{G} = \overline{G}$ and we are done. Now assume that $k \geq 1$, and the proposition holds for all $k' < k$. In this case, $\overline{G}$ is not a minimal chordal completion of $\underline{G}$ since $\underline{G} \neq \overline{G}$ and $\underline{G}$ is actually a minimal chordal completion of itself. By the result of Rose, Tarjan, and Leuker above, there must exist an edge $f \in \overline{E} \setminus \underline{E}$ such that $\overline{G} - f$ is chordal. Then, letting $G_{k-1} = \overline{G} - f$ and $e_{k-1} = f$, we have $\overline{G} = G_k = G_{k-1} + e_{k-1}$. Further, by the induction hypothesis, there exists a sequence of chordal graphs $\underline{G} = G_0, G_1, \ldots, G_{k-1}$ such that $G_{i+1} = G_i + e_i$ for some $e_i \in (\overline{E} \setminus \{e_{k-1}\}) \setminus \underline{E}$. □

Note that Proposition 2.2 implies that the set of chordal sandwiches forms a graded poset with respect to the inclusion relation of edge sets. For later reference, we write this assumption as a condition.

Condition 1 *A pair of graphs $\overline{G}$ and $\underline{G}$ satisfies $\underline{G} \subsetneq \overline{G}$, and at least one of $\overline{G}$ and $\underline{G}$ is chordal.*

2.3 Listing

On the listing problem of $\Omega_C(\overline{G}, \underline{G})$, Kiyomi and Uno [13] gave a constant time delay algorithm for chordal deletions, which corresponds the case that $\underline{G}$ is empty graph. Kiyomi, Kijima, and Uno [14] gave an $O(n^3)$ time delay algorithm for chordal deletions, where n is the number of vertices, and this is the case corresponding to the case that $\overline{G}$ is complete graph. Those two algorithms are based on the reverse search technique devised by Avis and Fukuda [1].

On Condition 1, Kijima, Kiyomi, Okamoto, and Uno [12] gave an algorithm based on the binary partition. Their algorithm runs $O(k{\cdot}\tau)$ time delay, where $k = |E(\overline{G}) \setminus E(\underline{G})|$ and τ denotes the time complexity for checking the chordality of a graph, hence their algorithm runs $O(k(n+m))$ time delay with an $O(n+m)$ time recognition algorithm by Rose, Tarjan, and Lueker [23]. The running time can be improved by using Ibarra's dynamic data structure [10] to $O(kn + n \log n)$ when $\overline{G}$ is chordal, and to $O(k \log^2 n + n)$ when $\underline{G}$ is chordal. See also Table 1 for the time complexity of listing chordal sandwiches.

2.4 Counting

On the counting problem of $\Omega_C(\overline{G}, \underline{G})$, Wormald [33] gave a generating function for the number of labeled chordal graphs with n vertices, that corresponds to counting $\Omega_C(K_n, I_n)$ where K_n denotes the complete graph with n vertices and I_n denotes the empty graph with n vertices. Wormald [33] also said that the number of labelled chordal graphs with n vertices is asymptotically $\sum_r \binom{n}{r} 2^{r(n-r)} = 2^{\Theta(n^2)}$.

Kijima, Kiyomi, Okamoto, and Uno [12] showed that counting chordal sandwiches is #P-hard even when $\underline{G}$ is connected chordal. The reduction is from counting forests in parsimonious way,

Table 2: The complexity of counting chordal sandwiches w.r.t. input pair.

floor graph $\underline{G}$ \ ceiling graph $\overline{G}$	general	chordal	complete graph (cf. chordal completion)
general	#P-complete	open	open
chordal	#P-complete	open	open
empty graph (cf. chordal deletion)	#P-complete	open	(cf. asymptotic analysis by Wormald [33])

meaning that the reduction preserves the error ratio of approximate counting. It is open if there is a *fully polynomial time randomized approximation scheme* (*FPRAS*) for counting forests. Kijima, Kiyomi, Okamoto, and Uno [12] also showed the #P-hardness of counting chordal deletions, i.e., computation of $|\Omega_{\mathrm{C}}(\overline{G}, I_n)|$, by a Cook reduction from counting forests. For other cases, the complexity of counting is open (see Table 2).

2.5 Sampling

Here, we consider a uniform sampling on $\Omega_{\mathrm{C}}(\overline{G}, \underline{G})$ satisfying Condition 1. It is well-known that approximate counting and uniform sampling is deeply related (see e.g., [11]). From Proposition 2.2, we have a simple and natural Markov chain for uniform sampling on $\Omega_{\mathrm{C}}(\overline{G}, \underline{G})$ on Condition 1.

Let $\mathcal{M}$ be a Markov chain with a state space $\Omega_{\mathrm{C}}(\overline{G}, \underline{G})$ with Condition 1. A transition of $\mathcal{M}$ from a current state $G \in \Omega_{\mathrm{C}}(\overline{G}, \underline{G})$ to a next state G' is defined as follows; Choose an edge $e \in (\overline{E} \setminus \underline{E})$ uniformly at random. We consider the following three cases.

1. If $e \notin E(G)$ and $G + e$ is chordal, then set $H = G + e$.
2. If $e \in E(G)$ and $G - e$ is chordal, then set $H = G - e$.
3. Otherwise set $H = G$.

Let $G' = H$ with the probability 1/2, otherwise let $G' = G$. Clearly $G' \in \Omega_{\mathrm{C}}(\overline{G}, \underline{G})$. Note that this $\mathcal{M}$ can be easily modified into ones for non-uniform distributions by a Metropolis-Hastings method.

The Markov chain $\mathcal{M}$ is irreducible from Proposition 2.2. The chain $\mathcal{M}$ is clearly aperiodic, and hence $\mathcal{M}$ is ergodic. The unique stationary distribution of $\mathcal{M}$ is the uniform distribution on $\Omega_{\mathrm{C}}(\overline{G}, \underline{G})$, since the detailed balanced equation holds for any pair of $G \in \Omega_{\mathrm{C}}(\overline{G}, \underline{G})$ and $G' \in \Omega_{\mathrm{C}}(\overline{G}, \underline{G})$. From Proposition 2.2, the diameter of $\mathcal{M}$ is at most $2k$, where $k = |\overline{E} \setminus \underline{E}|$.

Now, we discuss the mixing time of the Markov chain. Let μ and ν be a pair of distributions on a common finite set Ξ. The *total variation distance* $d_{\mathrm{TV}}(\mu, \nu)$ between μ and ν is defined by $d_{\mathrm{TV}}(\mu, \nu) \stackrel{\text{def.}}{=} \frac{1}{2} \sum_{x \in \Xi} |\mu(x) - \nu(x)|$. For an arbitrary positive ε, the *mixing time* $\tau(\varepsilon)$ of an ergodic Markov chain MC with a state space Ξ is defined by $\tau(\varepsilon) \stackrel{\text{def.}}{=} \max_{x \in \Xi} \min\{t \mid \forall s \geq t,\ d_{\mathrm{TV}}(P_x^s, \pi) \leq \varepsilon\}$ where π is the unique stationary distribution of MC, and P_x^s denotes a distribution of MC at time s starting from a state x.

Unfortunately, the Markov chain $\mathcal{M}$ for uniform sampling on $\Omega_{\mathrm{C}}(\overline{G}, \underline{G})$ is not rapidly mixing for some inputs. Kijima, Kiyomi, Okamoto, and Uno [12] gave the following.

Proposition 2.3 [12] *There exist infinitely many pairs of chordal graphs $\overline{G}$ and $\underline{G}$ satisfying $\underline{G} \subseteq \overline{G}$ for which the mixing time of $\mathcal{M}$ on $\Omega_{\mathrm{C}}(\overline{G}, \underline{G})$ is exponential in n, where n is the number of vertices of $\overline{G}$ (and $\underline{G}$).*

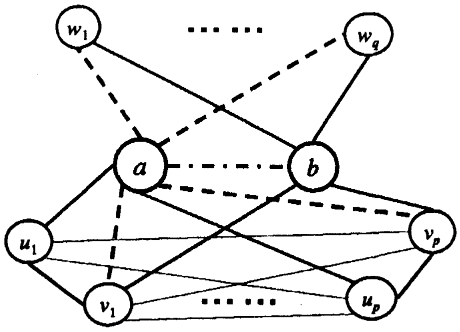

Figure 1: Example of an input pair on which the simple Markov chain $\mathcal{M}$ mixes slowly.

Figure 1 shows an example. Let V be a set of vertices $\{a, b, v_1, \ldots, v_p, u_1, \ldots, u_p, w_1 \ldots, w_q\}$. Let $\underline{G} = (V, E(\underline{G}))$ be a graph defined by

$$\begin{aligned} E(\underline{G}) \ &\stackrel{\text{def.}}{=}\ \{\{a, u_i\} \mid i \in \{1, \ldots, p\}\} \cup \{\{b, v_i\} \mid i \in \{1, \ldots, p\}\} \\ &\cup \{\{b, w_i\} \mid i \in \{1, \ldots, q\}\} \cup \{\{u_i, v_j\} \mid (i,j) \in \{1, \ldots, p\}^2\}. \end{aligned}$$

Let $\overline{G} = (V, E(\overline{G}))$ be a graph defined by

$$E(\overline{G}) \ \stackrel{\text{def.}}{=}\ E(\underline{G}) \cup \{\{a, v_i\} \mid i \in \{1, \ldots, p\}\} \cup \{\{a, w_i\} \mid i \in \{1, \ldots, q\}\} \cup \{\{a, b\}\}.$$

In Figure 1, $\underline{G}$ is described by solid lines, and $\overline{G}$ is described by solid lines and dashed lines. Note that both $\underline{G}$ and $\overline{G}$ are chordal.

3 Enumeration of Interval Graph Sandwiches

A graph G is *interval* if there is a one-to-one correspondence between its vertices and a set of intervals on the real line, such that two vertices are adjacent iff the corresponding intervals have an intersection. The set of intervals is called an interval representation of G. It is known that the class of interval graphs is a subclass of chordal graphs. Precisely, a graph is interval iff chordal and *asteroidal triple free* (*AT-free*), where three vertices of a graph form an *asteroidal triple* (*AT*) if for every pair of them are connected by a path avoiding the neighborhood of the remaining vertex.

Given a pair of graphs $\overline{G}$ and $\underline{G}$ satisfying $\underline{G} \subseteq \overline{G}$, let $\Omega_{\mathrm{I}}(\overline{G}, \underline{G})$ be a set of graphs defined by

$$\Omega_{\mathrm{I}}(\overline{G}, \underline{G}) \stackrel{\text{def.}}{=} \{G \mid G \text{ is interval}, \ \underline{G} \subseteq G \subseteq \overline{G}\}.$$

The interval graph sandwich problem is known to be NP-hard due to Golumbic, Kaplan, and Shamir [6]. The minimum interval completion is also NP-hard (see e.g., [4]). The minimality check of interval completion, that is the problem if $\Omega_{\mathrm{I}}(\overline{G}, \underline{G}) \setminus \{\overline{G}, \underline{G}\}$ has an element where $\overline{G}$ is interval, is polynomial time solvable due to Heggernes, Suchan, Todinca, and Villanger [9]. On the other hand, the complexity of the minimality check of interval deletion i.e. if $\Omega_{\mathrm{I}}(\overline{G}, \underline{G}) \setminus \{\overline{G}, \underline{G}\}$ has an element in case that $\underline{G}$ is interval, is open.

3.1 Listing

Kiyomi, Kijima, and Uno [14] gave a listing algorithm for interval completions and deletions. Their algorithm is based on the reverse search technique on the interval representation.

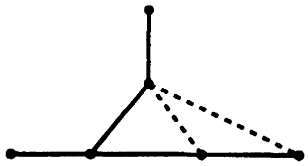

Figure 2: An example of $\Omega_{\mathrm{I}}(\overline{G}, \underline{G}) = \emptyset$ even though both $\overline{G}$ and $\underline{G}$ are interval.

Table 3: The time delay of listing interval sandwiches w.r.t. input pair.

floor graph $\underline{G}$ \ ceiling graph $\overline{G}$	general	interval	complete graph (cf. interval completion)
general	(cf. NP-hard)	open	$O(n^3)$
interval	open	open	$O(n^3)$
empty graph (cf. interval deletion)	$O((n+m)^2)$	$O((n+m)^2)$	$O(n^3)$

Unfortunately, the set of interval sandwiches does not form a graded poset in general, while chordal graphs does. Figure 2 is an example both $\overline{G}$ and $\underline{G}$ are interval but no interval graphs between them. The complexity of listing interval sandwiches is open when $\overline{G}$ or $\underline{G}$ is interval (see Table 3).

3.2 Counting and sampling

On counting interval graphs, Hanlon [7] gave a generating function for counting the number of labeld interval graphs with n vertices. Whereas Kijima, Kiyomi, Okamoto, and Uno [12] showed that counting interval sandwiches is #P-hard even when $\underline{G}$ is a connected interval graph. For other cases, the complexity is open (see Table 4).

On sampling interval sandwiches, it is open if we have an irreducible Markov chain on $\Omega_{\mathrm{I}}(\overline{G}, \underline{G})$ in which any transition can be handled efficiently. Note that for the instance in Fig 1, which is an example for slow mixing of a Markov chain on chordal sandwiches, $\overline{G}$ and $\underline{G}$ are interval and $\Omega_{\mathrm{C}}(\overline{G}, \underline{G}) = \Omega_{\mathrm{I}}(\overline{G}, \underline{G})$. This implies that a "nearest neighbor" chain does not mix rapidly, even if we have such a Markov chain.

4 Other Classes

4.1 Proper interval graph

An graph is *proper interval graph* if it is an interval graph that has an interval representation in which no interval properly contains another. The proper interval graph is equivalent to *unit interval graph* that is an interval graph which has an interval representation in which every interval has a unit length. The unit interval graph sandwich problem is known to be NP-hard due to Golumbic, Kaplan, and Shamir [6]. The minimum proper interval graph completion is also NP-hard (see e.g., [8]).

The set of proper interval graphs forms the Catalan structure, hence a proper interval graph

Table 4: The hardness of counting interval sandwiches w.r.t. input pair.

floor graph $\underline{G}$ \ ceiling graph $\overline{G}$	general	interval	complete graph (cf. interval completion)
general	#P-complete	open	open
interval	#P-complete	open	open
empty graph (cf. interval deletion)	open	open	(cf. generating function by Hanlon [7])

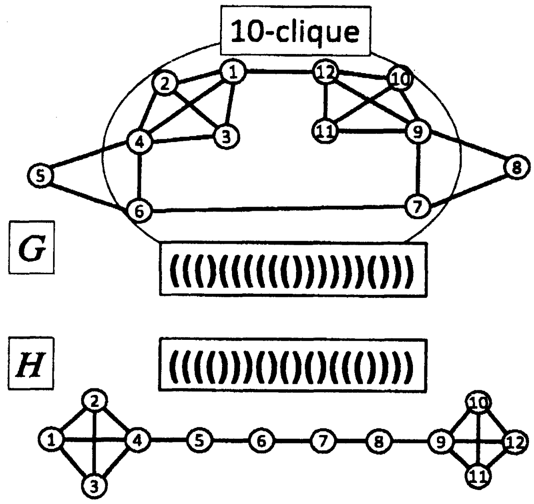

Figure 3: Example of a pair of proper interval graphs G and H such that $H \subseteq G$.

has a representation by n pairs of parentheses (see e.g., [24]). Saitoh, Yamanaka, Kiyomi, and Uehara [24] gave algorithms for listing and sampling for the set of perfect interval graphs with n vertices, where the graph is not labeled hence every pair in the set are not graph isomorphic. We can introduce a natural partial order $\preceq$ for a representation by n pairs of parentheses. If a pair of proper interval graphs G and H satisfies $H \preceq G$, then $H \subseteq G$ holds. However, the converse is not true; there is a pair of proper interval graph G and H satisfying $H \subseteq G$ but $H \not\preceq G$. Figure 3 shows an example, in which $H \subseteq G$ but $H \not\preceq G$. The time complexity of subgraph isomorphism of a pair of proper interval graphs is open.

4.2 Perfect graph

A graph is *perfect* if the chromatic number is equal to the size of maximum clique for any induced subgraphs. The class of perfect graphs is known to be a superclass of chordal graphs. The time complexity of perfect graph sandwich problem is open [6]. Perfect graphs does not form a graded poset regarding to edge sets. Fig 4 shows an example that the graph G is perfect though $G - e$ is not perfect for any edge e, and $G + \{u, v\}$ is not perfect for any pair $\{u, v\} \notin E(H)$. The time

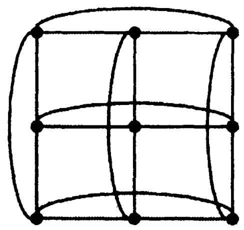

Figure 4: Example of a perfect graph G for which no graph of $G - e$ nor $G + \{u, v\}$ are interval.

complexity of listing, counting, and sampling of perfect graph sandwiches are open.

5 Concluding Remarks

This paper investigates listing, counting and sampling graph sandwiches for chordal sandwiches and interval sandwiches. There still exists several open problems for enumeration of graph sandwiches and related topics.

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