A Two-level Algorithm for Generating Multiset Permutations

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Abstract: We present an algorithm that generates multiset permutations in O(1) time for each permutation, that is, by a loop-less algorithm with O(n) extra memory requirement. There already exist several such algorithms that generate multiset permutations in various orders. For multiset permutations, we combine two loop-less algorithms that are designed in the same principle of tree traversal. Our order of generation is different from any existing order, and the algorithm is simpler and faster than the previous ones.

1. Introduction.

O(1) time generation for combinatorial objects, such as permutations and combinations, is known also as loop-less algorithms. That is, given the current object, we generate the next object loop-lessly, e.g., with a finite number of statements. This is also called combinatorial Gray code, since the idea is a generalization of binary Gray code to more general combinatorial objects. Ehrich [5] was the first to investigate this topic. Since then many loop-less algorithms were invented for various objects, such as permutations, combinations, parenthesis strings, etc. To list up just a few, see Bitner, et al. [1], Ehrlich [5], Lehmer [11], Eades and McKay [4], Chase [3] for combinations, Johnson [7] and Heap [6] for permutations, Mikawa and Takaoka [12], Vajnovszki [18], and Walsh [20] for parenthesis strings. We focus on the recent topic of multiset permutations in this paper.

There have been several algorithms that generate multiset permutations in O(1) time per permutation. Canfield and Williamson (1995) [2] and Korsh and Lipschutz (1997) [9] were the first to investigate this problem. Their algorithms achieves O(1) time, but uses a linked list as a container of permutations. Thus we can go from (1, 1, 1, 2, 2, 2) to (2, 2, 2, 1, 1, 1) in O(1) time by pointer manipulations. Since these papers, there was a question whether we can do the work with only arrays. There were two solutions by Takaoka [16] and Vajnovzski [19]. Both require O(kn) space apart from the main container of permutations, where k is the number of distinct items and n is the size of permutation. Thus we have a question whether we can do the work with only O(n) extra space. Again we have two solutions by Korsh and LaFollette (2004) [10] and Takaoka and Violich (2005) [17]. Those algorithms are based on the two-level approach, which we use in this paper also. In [9], the upper structure is based on Johnson-Trotter algorithm and the lower structure is based on Eades and McKay [4] for combination generation. The upper structure for the latter method [17] is the same, Johnson-Trotter, and the lower is based on Chase's algorithm for combination generation.

Suppose we have a multi-set (1,...,1,2,...,2,......k,...,k), where there are n[i] items of i, for i=1, ..., k. The upper algorithm controls which item i to move, and the lower algorithm controls the movement of those n[i] identical items by regarding those as ones and others as zeros.

In this paper, we use Johnson-Trotter for the upper structure and Takaoka's algorithm [15] for combination generation for the lower structure, which is less restrictive than Eades-McKay and Chase. Both the upper structure and lower structure are based on the idea of tree traversal. Thus our algorithm has a more transparent and consistent design methodology and the program code is shorter than existing ones. To make the problem more visible we start from an example. The following is the list of permutations of (1, 2, 3, 4) by Johnson-Trotter, which should be read column-wise.

1 2 3 4	3 2 4 1	1 3 4 2	4 3 2 1	1 4 2 3	2 4 3 1
2 1 3 4	3 2 1 4	3 1 4 2	4 3 1 2	4 1 2 3	2 4 1 3
2 3 1 4	3 1 2 4	3 4 1 2	4 1 3 2	4 2 1 3	2 1 4 3
2 3 4 1	1 3 2 4	3 4 2 1	1 4 3 2	4 2 3 1	1 2 4 3

Table 1. Permutations of (1, 2, 3, 4) by Johnson-Trotter

In this list 1 moves right over (2, 3, 4). Then 2 moves right. Then 1 moves left over (3, 2, 4), etc., alternately. If we remove 1 at the top of each column, we have permutations (2, 3, 4), (3, 2, 4), (3, 4, 2), (4, 3, 2), (4, 2, 3), (2, 4, 3). This list of permutations has a similar pattern of the movement of 2 going back and forth alternately. Later we show an $O(1)$ time algorithm for Johnson-Trotter.

The next list is the list of combinations of four items, called 4-combinations out of 6 items {1, 2, 3, 4, 5, 6} in in-place expression and binary vector form. The binary vector is for illustration purposes. In later sections, we generate only in-place forms. In combination generation, we generate combinations one by one, which we call the forward generation. When we use combination generation repeatedly as the lower structure of multiset permutation generation, we use forward generation and backward generation alternately. Backward generation is to generate combinations in reverse order of forward generation. This forward generation is from [15].

Forward Generation			Backward Generation	
In-place	binary vector		in-place	binary vector
1 2 3 4	1 1 1 1 0 0		4 3 6 5	0 0 1 1 1 1
1 2 3 5	1 1 1 0 1 0		2 3 6 5	0 1 1 0 1 1
1 2 3 6	1 1 1 0 0 1		2 3 4 5	0 1 1 1 1 0
1 2 4 6	1 1 0 1 0 1		2 3 4 6	0 1 1 1 0 1
1 2 4 5	1 1 0 1 1 0		2 5 4 6	0 1 0 1 1 1
1 2 6 5	1 1 0 0 1 1		1 5 4 6	1 0 0 1 1 1
1 3 6 5	1 0 1 0 1 1		1 3 4 6	1 0 1 1 0 1
1 3 4 5	1 0 1 1 1 0		1 3 4 5	1 0 1 1 1 0
1 3 4 6	1 0 1 1 0 1		1 3 6 5	1 0 1 0 1 1
1 5 4 6	1 0 0 1 1 1		1 2 6 5	1 1 0 0 1 1
2 5 4 6	0 1 0 1 1 1		1 2 4 5	1 1 0 1 1 0
2 3 4 6	0 1 1 1 0 1		1 2 4 6	1 1 0 1 0 1
2 3 4 5	0 1 1 1 1 0		1 2 3 6	1 1 1 0 0 1
2 3 6 5	0 1 1 0 1 1		1 2 3 5	1 1 1 0 1 0
4 3 6 5	0 0 1 1 1 1		1 2 3 4	1 1 1 1 0 0

Table 2. List of 4-combinations out of six items

We note that there is only one change from combination to combination in the in-place list, and in the binary vector list, a one moves over a block of ones, and does not change the relative order of zeroes. From 011110 to 011011, for example, the 4th element and 6th element are swapped, and the 4th element, 1, goes over the 5th element, 1. In general, the 1 may go over several consecutive 1's in a larger example. Note also that four ones at the left end finally come to the right end of the binary vector, and vice versa for backward generation. Chase does not have this property. We can modify the algorithm so that forward generation and backward generation alternate to fit Johnson-Trotter. The source code is given in Appendix.

Suppose we have a list of multiset, such as (1, 1, 2, 2, 2, 3, 3, 4), we move 1's to the right using Takaoka's algorithm by regarding them as ones and other items as zeroes. When all 1's

arrive at the right end, that is, (2, 2, 2, 3, 3, 4, 1, 1), we perform one step of the movement of 2's to the right, and 1's start to move to the left, resulting in (1, 1, 2, 2, 3, 2, 3, 4). That is, the movement of each item in Johnson-Trotter is generalized by the movements of several identical items by a combination generation algorithm. In Korsh and LaFollette [10], the combination algorithm is that of Eades and McKay, and in Takaoka and Violich [17], it was Chase's algorithm.

The following is the complete list of permutations of the multiset (1, 1, 2, 2, 3) in this paper.

1 1 2 2 3	2 3 2 1 1	1 1 3 2 2
1 2 1 2 3	2 3 1 2 1	1 3 1 2 2
1 2 2 1 3	2 3 1 1 2	1 3 2 1 2
1 2 2 3 1	2 1 1 3 2	1 3 2 2 1
2 1 2 3 1	2 1 3 1 2	3 1 2 2 1
2 1 2 1 3	2 1 3 2 1	3 1 2 1 2
2 1 1 2 3	1 2 3 2 1	3 1 1 2 2
2 2 1 1 3	1 2 3 1 2	3 2 1 1 2
2 2 1 3 1	1 2 1 3 2	3 2 1 2 1
2 2 3 1 1	1 1 2 3 2	3 2 2 1 1

Table 3. List of permutations of (1, 1, 2, 2, 3)

2. General framework for 2-level loop-less algorithms
We have two kinds of 2-level algorithms for combinatorial generation. Let array "a" be the main container for the combinatorial objects, which are obtained by combining upper and lower objects.

Type 1
1. Initialize array "a" , the upper and lower objects and other data structures
2. **repeat**
3.   **repeat**
4.   compute the next lower object;
5.   make an effect on the object in array "a"
6.   **until** last lower object
7.   re-initialization for lower objects
8.   compute the next upper object;
9. **until** last upper object

Example [17] Mixed parenthesis strings are described briefly. Upper objects are well-formed parenthesis strings. Lower objects are the set of binary strings in Gray code. The upper algorithm takes care of the parenthesis strings, while the lower algorithm carries out the changes of parentheses and brackets driven by the binary strings in Gray code. The central algorithmic issue here is the computation of positions of changes between parentheses and brackets. At line 7, we cannot afford to spend more than O(1) time, meaning we need to generate binary strings repeatedly or reversely.

Lower	Upper	
0 0	( ) ( )	(( ))
0 1	( ) [ ]	[( )]
1 1	[ ] [ ]	[[ ]]
1 0	[ ] ( )	([ ])

For each parenthesis string, the lower objects carry out necessary changes.

Type 2.
1. Initialize array "a" and other data structures. Let i be current upper object
2. **repeat**
3.   compute the next lower object for upper object i
4.   make an effect on the object in array "a"
5.   **if** last lower object **then begin**
6.   re-initialization for lower objects for i
7.   compute the next upper object and let it be the current i
8.   **end**
9. **until** last upper object has been processed

We use type 2 for multiset permutations. Upper objects are permutations (specifically, moving items) by Johnson-Trotter and the lower objects are combinations of items i.

3. General Form of Loop-less Algorithms by Tree Traversal

We borrow some materials from [15] to describe the concept of tree traversal used in this paper. Let $\Sigma = \{\sigma_0, \dots , \sigma_{r-1}\}$ be an alphabet for combinatorial objects. A combinatorial object is a string $a_1 \dots a_n$ of length n such that each a_i is taken from Σ and satisfies some property. An order is defined on Σ with $\sigma_i < \sigma_{i+1}$. Let Σ^n be the set of all possible strings on Σ of length n with the lexicographic order $<$. The order $<$ on a set of combinatorial objects, $S \subseteq \Sigma^n$, is defined by projecting the lexicographic order on Σ^n onto S.

The lexicographic tree, or lexico-tree for short, of S is defined in the following way. Each $a \in S$ corresponds to a path from the root to a leaf. The root is at level 0. If $a = a_1 \dots a_n$, a_i corresponds to a node at level i. We refer to a_i as the label for the node. We sometimes do not distinguish between node and label. If a and b share the same prefix of length k, they share the path of length k in the tree. The children of each node are ordered according to the labels of the children. A path from the root to a leaf corresponds to a leaf itself, so a corresponds to a leaf. The combinatorial objects at the leaves are thus ordered in lexicographic order on S.

The twisted lexico tree of a set S of combinatorial objects is defined as follows together with the parity function. We proceed to twist a given lexico-tree from the root to leaves in a breadth-first search manner. Let the parity of the root be even. Suppose we processed up to the i-th level. If the parity of a node v at level i is even, we do not twist the branches from v to its children. If the parity of v is odd, we arrange the children of v in reverse order. If we process all nodes at level i, we give parity to all the nodes at level i+1 from first to last alternately starting from even. We denote the parity of node v by parity(v). When we process nodes at level i in the following algorithms, which are children of a node v such that parity(v)=p, we say the current parity of level i is p. Note that (labels of) nodes at level i are in increasing order if the parity of the parent if even, or equivalently if the current parity of level i is even. If the parity is odd, they are in decreasing order. We draw trees lying horizontally for notational convenience. We refer to the top child of a node as the first child and the bottom as the last child.

If the labels on the paths from the root to two adjacent leaves in the twisted lexico-tree for S are different at a fixed number of nodes, we can generate S from object to object with the fixed number of changes. We design an efficient algorithm that traverses the twisted lexico-tree and generate combinatorial objects in O(1) time per object. Thus the fixed number of changes is a necessary condition for our algorithm, but not sufficient.

The current parity at level i is given by parity[i], and the procedure "output" is only to see the result; for O(1) time this will be eliminated. We allow O(n) time for initialization, but for the repeated use of generation cannot afford to re-initialize arrays in O(n) time. This will be addressed in later sections. The parity 0 is for even and 1 for odd. Let $\Sigma = \{0, \dots ,r-1\}$. We associate an object with a leaf of a tree. The path from the root to the leaf identifies the object.

Traversal from a node to the next sibling node corresponds to generating the next object, making some changes on the object. We maintain the parity information for each level in array "parity". The level of the tree corresponds to various aspects of the objects as we will see in later sections. A generic form of the algorithm follows. All algorithms are given in Pascal style pseudo code.

Algorithm 1. Iterative tree traversal in pseudo code. i keeps track of the current level of the tree.

```
1.  initialize array a to be the first object in S;
2.  initialize v₁, ... , vₙ to be nodes on the path to the first object (top path);
3.  for i:=0 to n do up[i]:=i;
4.  for i:=0 to n do parity[i]:=0;
5.  repeat
6.    output(a);
7.    i:=up[n]; up[n]:=n;
8.    perform changes on a at vᵢ and related positions;
9.    let vᵢ go to the next node at level i based on the current parity;
10.   if vᵢ is the last child of its parent then begin
11.     let vᵢ further go to the next node at level i;
12.     up[i]:=up[i-1]; up[i-1]:=i-1;
13.     parity[i]:=1-parity[i]
14.   end
15. until i=0.
```
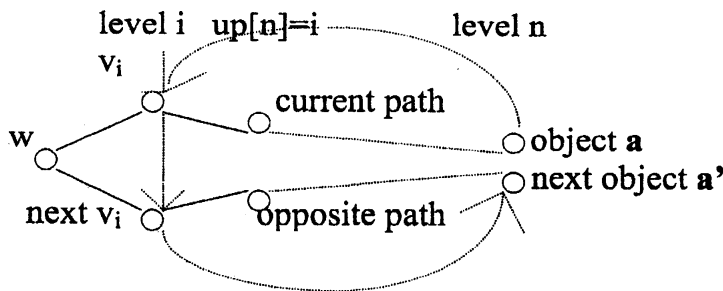

When we come to the last child of a parent (w in the above figure), we have to update up[i] to up[i-1] so that when we visit the last leaf of the sub-tree rooted at w, we can come back directly from the leaf to w or its ancestor if w itself is a last child. We refer to the paths from v_i to a and from next v_i to a' as the current path and the opposite path. A current path and opposite path consist of last children and first children respectively except for the left ends. If next v_i in the above figure is a last child, we further set v_i to the next node of next v_i, say u, so we can avoid O(n) time to set up the environment for such u's later. This is illustrated in the above figure by the path from "next v_i" to "next object a' ". That is, when we traverse down the current path, we prepare for the opposite path so that we can jump over the opposite path from level i to the leaf. In this situation a and a' share the same prefix from position 1 to i-1. We call i the difference point. The two strings also share the same suffix (possibly empty). Let the longest such is from position j+1 to n. Then we call j the solution point. Intuitively this is the point where the difference caused upstream is solved. Algorithms for permutations and for combinations in this paper are designed along the paradigm in this section.

4. Review of Johnson-Trotter

In the iterative algorithm for Johnson-Trotter, the level of the tree corresponds to the item we are trying to move. Parity corresponds to the direction of movement, left or right, represented by -1

(odd parity) or +1 (even parity). Array "p" is to hold the position of item x. Array element "c[i]" is to count the number of nodes at level i and check the last child condition. Procedure "move(x)" is to move item x to the direction given by d[x], and update the positions of affected items. The comment /*output */ shows the point where a new permutation is generated. Array "d" plays the role of parity in Algorithm 1. In the program, level variable i corresponds to item i. If we regard the position of each item as the label of the node, we have a close correspondence with the twisted lexico-tree. In the following figure, level numbers are reversed to allow item moves first.

level 4   level 3   level 2   level 1

```
                                            move(1)     1 2 3 4
                                            move(1)     2 1 3 4
                              move(2)        move(1)     2 3 1 4
                                                         2 3 4 1
                                            move(1)     3 2 4 1
                                            move(1)     3 2 1 4
                              move(2)        move(1)     3 1 2 4
                                            move(1)     1 3 2 4
                                            move(1)     1 3 4 2
    move(3)                                              3 1 4 2
                                            move(1)     3 4 1 2
                                                         3 4 2 1
                                            move(1)     4 3 2 1
                                            move(1)     4 3 1 2
                              move(2)        move(1)     4 1 3 2
                                                         1 4 3 2
                                            move(1)     1 4 2 3
                                            move(1)     4 1 2 3
                              move(2)        move(1)     4 2 1 3
                                                         4 2 3 1
                                            move(1)     2 4 3 1
                                            move(1)     2 4 1 3
                                            move(1)     2 1 4 3
                                                         1 2 4 3
```
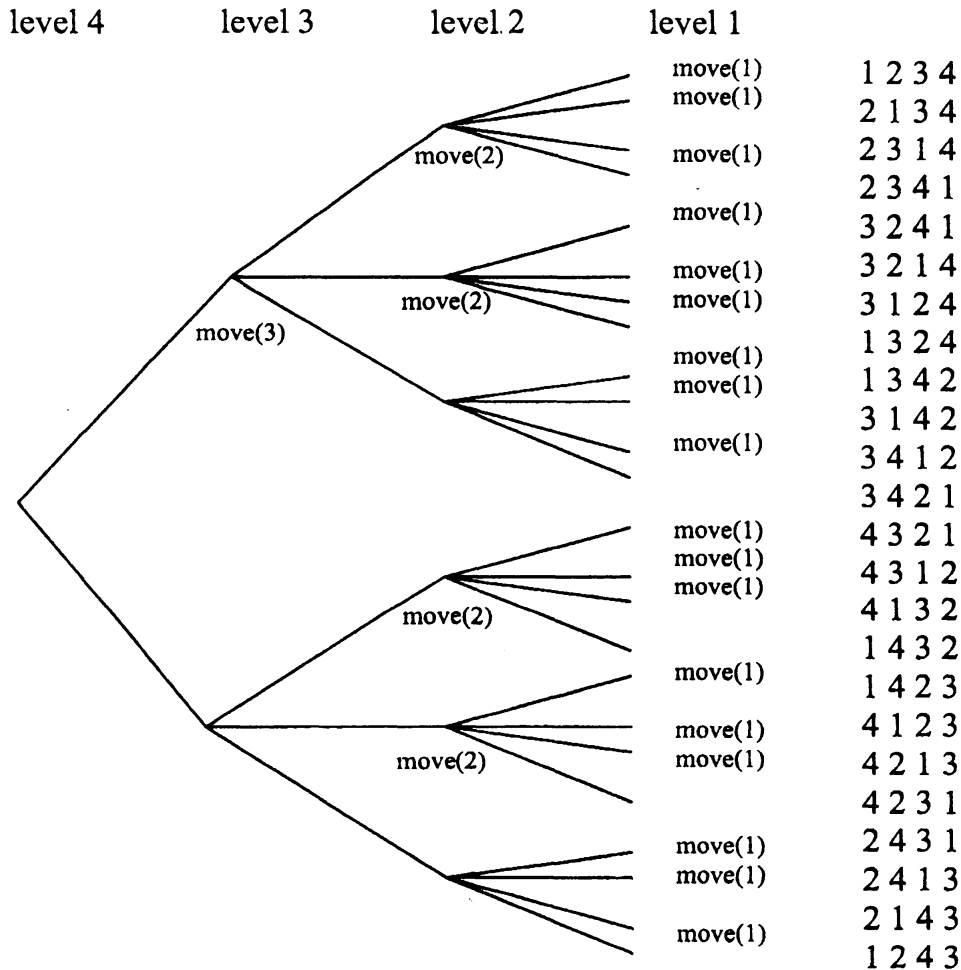
Figure 1 Tree for Johnson-Trotter

In this figure levels increase from leaves to the root. Variable i keeps track of the current level of tree traversal. The action move(x) is attached to each node of the tree. Variable c[i] is called the child counter, which can tell if we are hitting a last child at the current level. In the following sections, we say x moves actively in the procedure call "move(x)", and the item neighboring x moves passively by swapping.

Algorithm 2. Iterative algorithm for Johnson-Trotter
procedure move(x);
begin var w;
  w:=a[p[x]+d[x]]; a[p[x]+d[x]]:=x; a[p[x]]:=w;
  p[w]:=p[x]; p[x]:=p[x]+d[x]; c[i]:=c[i]+1; /* output */
end;
begin {main program}

```
for i:=1 to n do begin
    a[i]:=i, up[i]:=i; p[i]:=i; c[i]:=1; d[i]:=1;
end;
repeat
    i:=up[1]; up[1]:=1;
    move(i);
    if c[i] = n-i+1 then begin   /* currently hitting a last child at level i */
        up[i]:=up[i+1]; up[i]:=i+1;   /* inherit the up-value of the parent */
        c[i]:=1;
        d[i]:=-d[i];                  /* parity (direction) reversed */
    end
until i=n;
end.
```

5. The Two Level Approach

Now we combine the loop-less algorithm in Section 4 for permutations and the combination generation algorithm described later in the next section in detail. Let us generate permutations of multiset $(1,1,...,1,2,2,...,2,...,k,k,...,k)$. Let $n[i]$ be the number of items i in the multiset, that is, the multiplicity of i. Let $n=n[1]+ ... + n[k]$, that is, the total size of the multiset.

When items 1 move in the container array $a[1 .. n]$, the range of movement is $[1 .. n]$, the whole span of array "a". The range of items 2 is limited to $[1 .. n-n[1]]$ when all 1s are at the right end, or range $[n[1]+1 .. n]$ if all 1s are at the left end. If we define the capsule of 2 as range $[1 .. n-n[1]]$, the latter range of the capsule can be obtained by adding the base, $base[2]=n[1]$. In general, we move item j in the capsule $[1 .. size[i]]$ by combination generation where $size[i]$ is the size of the capsule. The capsule is activated by performing one step of combination generation, when all lower items are to the left or to the right, that is, outside of it. Note that when we talk about movement by swapping two items, we mean active movement of the lower item, not passive movement of the higher item. Thus items k never move actively. The absolute range of a capsule is obtained by adding $base[i]$ to it. The values of $size[i]$ and $base[i]$ for item i are defined as follows:

$$size[i] = n[i] + ... + n[k], i=1, ..., k-1$$
$$base[i] = \text{the number of all lower items to the left of any occurrence of item i}$$

The computation of $base[i]$ will be explained later. We maintain positions of items i by the combination generation algorithm. Suppose the positions of items i changed from $(q[1], ..., q[j], ..., q[n[i]])$ to $(q[1], ..., q'[j], ..., q[n[i]])$ in the capsule by the combination generation algorithm. Then we see that an item i moves from position $q[j]$ to position $q'[j]$ in the capsule of item i in the main container array "a". In Algorithm 4, those values of $q[j]$ and $q'[j]$ are returned by the call to "combination-server" in global variables "from" and "to". The values of "from" and "to" are adjusted by adding $base[i]$ to the positions in the range. Let $C(n[i], n)$ be the binomial coefficient of $n[i]$-combinations out of n items.

Example. Suppose the first item of 2 moved right from $(1,1,1,3,2,2,3)$ to $(1,1,1,3,3,2,2)$. Since 2s are moving right globally, the positions of 2s before and after the move are $(5, 6) = 3+ (2, 3)$ and $(7, 6)=3+(4, 3)$, modified by adding the base 3 to 2-combinations, $(2,3)$ and $(4,3)$ out of 4 items $\{1, ... , 4\}$. Thus 2 moves from position 5 to 7.

In the following, the work "maintain combinations" includes "return the next combination". For readability, computation of $base[i]$ is omitted in the program, and explained in the text.

Algorithm 3. Multiset Permutation Generation
Lower Level
Combination-server   /* This server performs the following operations on demand */
begin
    Initialize $n[1]$-combinations out of size[1] items
    Initialize $n[2]$-combinations out of size[2] items
    ...
    Initialize $n[k-1]$-combinations out of size[k-1] items
    Maintain $n[1]$-combinations out of size[1] items
    Maintain $n[2]$-combinations out of size[2] items
    ...
    Maintain $n[k-1]$-combinations out of size[k-1] items
end

Upper Level
procedure move(x);
begin var w;
    c[i]:=c[i]+1;
    Let the change of combination be from "from" to "to";
    from:=from + base[i]; to:=to + base[i];
    w:=a[from]; a[from]:=a[to]; a[to]:=w;   /* output */
end;
begin {main program}
    for i:=1 **to** k-1 **do call** combination-server to
                    initialize $n[i]$-combinations out of size[i] items;
    for i:=1 **to** k **do begin**
        a[i]:=i, up[i]:=i; p[i]:=i; c[i]:=1; d[i]:=-1;
    end;
    repeat
        i:=up[1]; up[1]:=1;
        call combination-server for next combination for item i;
        if i<k **then** move(i);
        if c[i] = C(n[i], n) **then begin** /* checking a last child */
            Re-initialize combinations of item i for reverse direction;
            up[i]:=up[i+1]; up[i]:=i+1;
            c[i]:=1;
            d[i]:=-d[i];
        end
    until i=n;
end.

Re-initialization of combinations of item i must be done in $O(1)$ time. This will be mentioned in the next section. The value of $C(n[i], n)$ is potentially large, and may not be contained in a single precision variable. We can use the termination condition in the Algorithm 4 (i=0) in the appendix, and bring the effect to the calling site. As initialization for combinations of item i takes $O(n[i])$ time, Algorithm 3 takes $O(n)$ time at the beginning. As the size of an object of $n[i]$-combinations is $O(n[i])$, the total space requirement is $O(n)$. In the appendix we declare fixed-sized arrays for readability. It is straightforward to allocate necessary space dynamically using the "calloc" function in C.

    Now we explain how to compute base[i]. Let $l=n[i]$ occurrences of item i be $i_1, ..., i_l$. Let the number of items lower than i occurring to the left of i_m ($1 \leq m \leq l$) be $N(i_m)$. We define LEFT[i]
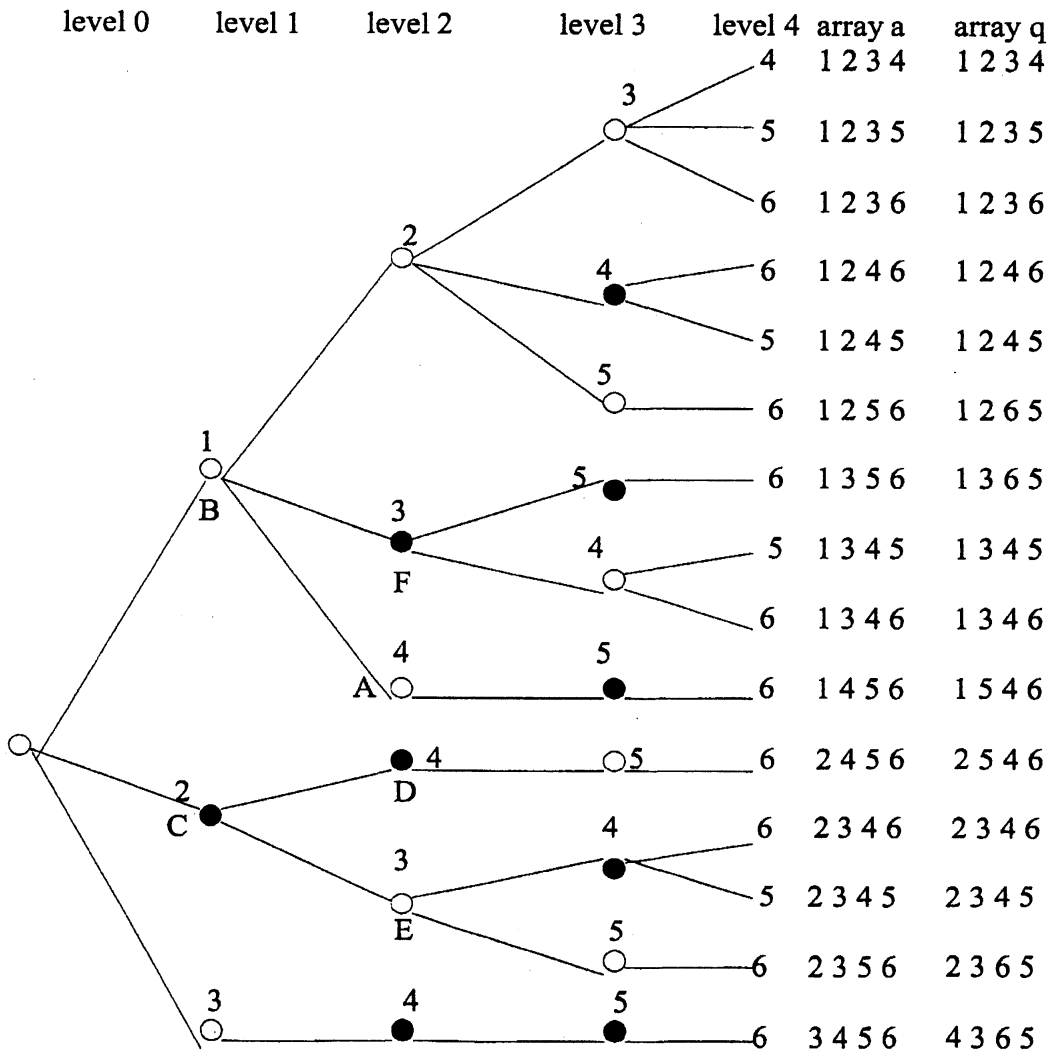
as follows: LEFT[i] = N(i₁) + ... + N(iⱼ). Then base[i] can be computed as base[i] = LEFT[i]/n[i]. LEFT[i] is initialized to n[i]*(n[1]+...+n[i-1]). When item i and item j are swapped in "move" where i=a[from] and j=a[to], LEFT[j] is updated by LEFT[j]=LEFT[j]+(from - to).

Example. (1,1,3,2,2,3,4) changes to (1,1,3,3,2,2,4). Since from=4 and to=6, LEFT[3] changes from 6 to 4. This is because item 3 goes left over two of items 2. Note that LEFT[2] does not change as LEFT is concerned with lower items.

6. Review of Takaoka's algorithm for in-place combination generation.
We describe an O(1) time algorithm for generating the set S of combinations of n elements out of the set {1, ..., r}. In the following array "q" is the main container of combinations and "a" is an auxiliary array for book-keeping; it keeps track of tree traversal with some delay. We state a few lemmas describing necessary properties for our multi-set permutation generation.

Example The twisted lexico tree for combinations of 4 elements out of 6 elements is given with the contents of array "a" and "q" at the leaves. A white circle is for even parity and black for odd.

level 0	level 1	level 2	level 3	level 4	array a	array q
				4	1 2 3 4	1 2 3 4
			3	5	1 2 3 5	1 2 3 5
				6	1 2 3 6	1 2 3 6
		2	4	6	1 2 4 6	1 2 4 6
				5	1 2 4 5	1 2 4 5
			5	6	1 2 5 6	1 2 6 5
	1 B		5	6	1 3 5 6	1 3 6 5
		3 F	4	5	1 3 4 5	1 3 4 5
				6	1 3 4 6	1 3 4 6
		4 A	5	6	1 4 5 6	1 5 4 6
		4 D	5	6	2 4 5 6	2 5 4 6
	2 C		4	6	2 3 4 6	2 3 4 6
		3 E		5	2 3 4 5	2 3 4 5
			5	6	2 3 5 6	2 3 6 5
	3	4	5	6	3 4 5 6	4 3 6 5

Lemma 1. The set S generated by the twisted lexico-tree generates combinations with one change per combination. See [14] for a proof.

Now we describe implementation details. There are many nodes which have only one child down to the leaf, causing straight lines. Traversal of these lines downwards will cause $O(n)$ time. Having one child is caused by a node with the maximum possible label at the level. To control the position to which we come down, we keep an array "down". In the above example, we use "up" to go from A to B, and use "down" to go from C to D. A snapshot of the movement is like (..., F,A,B,C,D,E, ...). As we see below, we cannot generate combinations in labels attached to the nodes of the twisted lexico-tree with a fixed number of changes. Let array a contain those labels. Then we have a situation with

$$\mathbf{a} = (a_1, \dots ,a_{i-1}, a_i, \dots ,a_j, \dots ,a_n) \text{ and } \mathbf{a'} = (a_1, \dots ,a_{i-1}, a_i+1, \dots ,a_j+1 \dots ,a_n),$$

where $a_{k+1}=a_k+1$ for $i \le k < j$, or a symmetric case where $\mathbf{a'} = (a_1, \dots ,a_{i-1}, a_i-1, \dots ,a_j-1, \dots ,a_n)$, where $a_{k+1}=a_k-1$. Note that "i" is the difference point and "j" is the solution point. In the first case, "a_i" goes out of the combination and "a_j+1" comes into the combination like a revolving door system. We keep the combinations in array "q" and use array "a" for book-keeping. As we cannot change $j-i+1$ places in "a" in $O(1)$ time, we just change a_i and a_j, and change the contents of q correspondingly. Thus we do not implement line 11 of Algorithm 2 to prepare for the opposite path to the next object. To keep track of the positions of these elements in q, we use array "pos". Solution points are maintained in array "solve". Since some parts of array "a" are not maintained to the corresponding labels in the tree, we use Boolean array "mark" to show no maintenance. When mark[i]=true, the proper value of a[i-1] is given by its child a[i]. The main part of the algorithm follows in which up[i], down[i], solve[i], and pos[i] are initialized to i for all i. The values of d[i] are initialized to 1 and those of "mark" to false. Line-by-line explanation follows the algorithm. If we change the termination condition at line 23 to **false** we can keep generating combinations in forward order and reverse order alternately for ever. The proof of the next lemma is omitted.

Lemma 2. The generation of combinations in the binary vector form by the above twisted lexico-tree does not change the relative order of 0's. Items 1 move from the left end and finish at the right end for one run in the binary vector. From the second run on, the movement alternates in direction.

To continue to generate combinations in reverse order, we can simply perform i:=down[i], and repeat from line 3. In the source list in the Appendix, we use Algorithm 4 as "combination-server", where all variables are indexed by item. Thus simple variables become one-dimensional arrays and one-dimensional arrays become two-dimensional arrays. Re-initialization in Algorithm 3 can be implemented by performing i[I]=down[I][i[I]] for item I. Also "checking a last child" can be implemented by testing "i[I]=0". In the program, Algorithm 3 and Algorithm 4 communicate through global variables, which make the algorithm structure less transparent, but efficiency is gained. When Algorithm 4 is used as "combination-server", line 2 and 23 will be removed.

Algorithm 4. In-place algorithm for combinations {arrays a, q, d, up, pos, sol, solve, mark used}
1. d[n+1]:= -1; up[0]:=0; i:=n; initialize d[i] to 1, and other arrays to i for all i; d[0]:=0;
2. **repeat**
3.   output(q);   /* This is to output q */
4.   **if** mark[i]=**true then begin** a[i-1]:=a[i]-1; mark[i]:=**false end;**
5.   **if** d[i+1]<0 **then begin**   /* Moving from an even node to an odd node*/
6.   q[pos[a[i]]]:=a[i]+d[i]; pos[a[i]+d[i]]:=pos[a[i]]
7.   **end else begin**
8.   **if** d[i]>0 **then**   /* Moving from odd to even, with "a" value increasing */

```
9.      q[pos[a[i]]]:=a[solve[i]]+d[i]; pos[a[solve[i]]+d[i]]:=pos[a[i]]
10.     end else if d[i]<0 then begin   /* Moving from odd to even, with "a" value decreasing */
11.     q[pos[a[solve[i]]]]:=a[i]+d[i]; pos[a[i]+d[i]]:=p[a[solve[i]]]
12.     end;
13.     a[i]:=a[i]+d[i];
14.     if d[i+1]>0 then a[solve[i]]:=a[solve[i]]+d[i];
15.     up[i]:=i;
16.     if (d[i]>0) and (a[i]=r-n+i) or (d[i]<0) and (a[i]=a[i-1]+1) then begin
17.        up[i]:=up[i-1]; up[i-1]:=i-1;
18.        down[up[i]]:=i;
19.        if d[i]<0 then begin solve[[up[i]]:=i; mark[i]:=true end;
20.        d[i]:= -d[i];
21.        if (d[i]<0) or (i=n) then i:=up[i] else i:=down[i]
22.     end else i:=down[i]
23. until i=0;   { This termination condition is replaced by false for an infinite loop}
24. output(q).
```

Line 4: Update the parent when maintenance is not done.

Lines 5-12: Update the contents of q.

Lines 5-7: Note that d[i+1] was altered in a previous step. Thus we regard parity(a$_i$) as even despite d[i+1]<0. Move from B to C with i=1 for example in the figure.

Lines 8-10: Similarly move from F to A with i=2 for example.

Lines 10-12: Similarly move from D to E with i=2 for example.

Line 13: Take the next child.

Line 14: Change the value of a at the solution point.

Line 15: Originate the destination for descendants to come up to. This value will possibly propagate down in line 17.

Line 16: Check if the node is a last child.

Line 17: Compute the correct value of "up".

Line 18: Let the ancestor know where to come down next.

Line 19: If parity is odd, update the solution point for ancestor by i. This may further be updated by larger i. Also signal that the parent a[i-1] is not updated for the next object. If up[i]=i-1, a[i-1] has been given a proper value of a[i]-1, but signaling "mark[i]=true" will not cause any harm.

Line 20: Change the parity at i.

Line 21: Go up if you hit a last child with even parity or at a leaf. Otherwise go down.

Line 22: Go down if you do not hit a last child.

Line 23: Go out of iteration if you come to the root, or go back to line 2 for repeated generation.

7. Concluding Remarks

We showed how to design a loop-less algorithm for multiset permutation generation based on a two-level approach. To avoid complications, we used an abstract algorithm, called "combination server", which delivers combinations of various sizes one by one at the request of the upper algorithm of Johnson-Trotter. If we implement our algorithm completely in a procedural language, we can use two-dimensional arrays, another dimension corresponding to which kind of items are moving. The source code in Appendix is based on this approach by two dimensional arrays. Note that with this approach, memory requirement can be O(n), where n is the total size of each permutation. Precisely speaking, we would need to maintain eight arrays of size n[i] for each item i. Those arrays need to be maintained by pointers for dynamic memory allocation, although the main container of permutations remains to be a fixed array of size n. In the upper structure of the algorithm we need eight arrays of size k where k is the number of distinct items. In [10], 20

arrays of various sizes are used and in [17] an array of size n and nine arrays of size k are used. It remains to be seen whether we can further simplify the algorithm or reduce memory requirement.

It is hoped that the multi-level approach can be extended to the O(1) time generation of other complicated combinatorial objects.

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Appendix . Program Source List in C

For the readability this program is based on two-dimensional arrays for local variables for combination generation. Available at http://www.cosc.canterbury.ac.nz/tad.takaoka/multiperm.c

```c
// up to 10 items. up to multiplicity of 20
// array M is the main container of multiset permutations
int tt;
int I, K, N[10], A[10], P[10], D[10], UP[10];
int R, count; int countk, init, COUNT, from, to;
int M[40], base[40], limit[40]; /* array "limit" is "size" in the text */
int a[10][20], q[10][20], up[10][20], solve[10][20],
    d[10][20], pos[10][20], mark[10][20], down[10][20],
    i[10];
int true=1, false=0;
init_base() {int i, k;
  base[1]=0;                              // base[i] is LEFT[i] in the text
  for(k=2;k<=K;k++)base[k]=base[k-1]+N[k-1];
  for(k=1;k<=K;k++)limit[k]=R-base[k];    // limit[k] size of the world for k
  for(k=1;k<=K;k++)base[k]=base[k]*N[k];  // base[k] is multiplied by N[k]
}
OUT() {int i;
  for(i=1;i<=R;i++)printf("%d ",M[i]);
  printf(" base ");
  for(i=1;i<=K;i++)printf("%d ",base[i]);
  printf("COUNT=%d¥n", COUNT);
}
MOVE() {int w;
  from=from+base[I]/N[I]; to=to+base[I]/N[I];    // no. of i < 2
  w=M[from]; M[from]=M[to]; M[to]=w;             // 3 3 1 1 2 2 2
  base[M[from]]=base[M[from]]+from-to;           //       *
  COUNT++;  OUT();                               //       base[2] =2*3=6
}                                                // 3 3 2 1 1 2 2
main() {int k, j;                                //       base[2]= 6-2 = 4
printf("Input K :number of kinds of items ");
scanf("%d", &K); getchar(); // no of distinct items
printf("Input multiplicities of each kind of tems ");
for(I=1;I<=K;I++)scanf("%d",&N[I]);
getchar();
R=0; for(I=1;I<=K;I++) R=R+N[I]; // Total size
// M is the main container of permutations. M is initialized to (1,1,1,2,2,3,3...) etc.
tt=clock();
init=1;
for(countk=1;countk<=K;countk++) {
  for(I=init;I<=init+N[countk]-1;I++) {
    M[I]=countk;
  }
  init=init+N[countk];
}
printf("M= ");
```

```
for(I=1;I<=R;I++)printf("%d ", M[I]); printf("\n");
init_base();
for(I=1;I<=K-1;I++){combi_init(I, N[I], limit[I]);}
/*** Johnson-Trotter upper structure ***/
for(I=1;I<=K;I++){A[I]=I; D[I]=1; P[I]=I; UP[I]=I;}
COUNT=1; OUT();
do{
  I=UP[1]; UP[1]=1;
  combi(I, N[I], limit[I]);
  if(I<K)MOVE(I);
  if(i[I]==0){
    i[I]=down[I][i[I]];
    UP[I]=UP[I+1]; UP[I+1]=I+1;
    D[I]=-D[I];
  }
}while(I!=K);
printf("time %d\n", clock()-tt);
}
/*** Initialization for combi ***/
combi_init(int t, int n, int r){
  { int i;
  for(i=1; i<=n; i++){a[t][i]=i; q[t][i]=i; d[t][i]=1;}
  for(i=0; i<=n; i++){up[t][i]=i; down[t][i]=i;
            pos[t][i]=i; solve[t][i]=i;}
  }
  d[t][n+1]= -1; up[t][0]=0; i[t]=n;
}
/*** Combination generation server ***/
combi(int t, int n, int r){ // t for item
    int ii;
    ii=i[t];
    if(mark[t][ii]==true)
      {a[t][ii-1]=a[t][ii]-1;
      mark[t][ii]=false ;}
    if(d[t][ii+1]<0){
      from=q[t][pos[t][a[t][ii]]];
      q[t][pos[t][a[t][ii]]]
        =a[t][ii]+d[t][ii];
      to=q[t][pos[t][a[t][ii]]];
      pos[t][a[t][ii]+d[t][ii]]=pos[t][a[t][ii]];
    } else
    if(d[t][ii]>0){
      from=q[t][pos[t][a[t][ii]]];
      q[t][pos[t][a[t][ii]]]=a[t][solve[t][ii]]+d[t][ii];
      to=q[t][pos[t][a[t][ii]]];
      pos[t][a[t][solve[t][ii]]+d[t][ii]]=
        pos[t][a[t][ii]];
    } else if(ii>0){
      from=q[t][pos[t][a[t][solve[t][ii]]]];
```

```
      q[t][pos[t][a[t][solve[t][ii]]]]=a[t][ii]+d[t][ii];
      to=q[t][pos[t][a[t][solve[t][ii]]]];
      pos[t][a[t][ii]+d[t][ii]]=pos[t][a[t][solve[t][ii]]];
    }
  a[t][ii]=a[t][ii]+d[t][ii];
  if(d[t][ii+1]>0) a[t][solve[t][ii]]=a[t][solve[t][ii]]+d[t][ii];
  up[t][ii]=ii;
  if(((d[t][ii]>0)&&(a[t][ii]==r-n+ii))||
      ((d[t][ii]<0)&&(a[t][ii]==a[t][ii-1]+1))){
    up[t][ii]=
      up[t][ii-1]; up[t][ii-1]=ii-1;
    down[t][up[t][ii]]=ii;
    if(d[t][ii]<0){solve[t][up[t][ii]]=ii;
      mark[t][ii]=true;}
    d[t][ii]= -d[t][ii];
    if((d[t][ii]<0) || (ii==n))
      i[t]=up[t][ii]; else i[t]=down[t][ii];
  } else i[t]=down[t][ii];
}
```