数理解析研究所講究録
第 1645 巻 2009 年 100-107


# Low-Frequency 3D Electromagnetic Scattering on Dielectric Structures

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## Introduction

In this article we give the survey of some results which have been received in [1]-[2]. We consider the singular volume integral equations describing the electromagnetic wave scattering in three-dimensional bounded inhomogeneous media. We analyze the problem of finding the spectrum of these non-selfadjoint integral operators. A closed-form expression describing the continuous part of the spectrum on the complex plane is presented. For low-frequency scattering problems, by using differential formulation of the problems, we find a domain on the complex plane where the discrete spectrum of our integral equations is located. We describe a generalized simple iteration method and, using the information of the spectrum, show that it can be very effective tool in low-frequency scattering problems.

## 1. Formulation of the problem

We will consider the following class of electromagnetic problems. The medium in a finite 3D domain $Q$ is characterized by a dielectric permittivity function $\varepsilon$ that is a Holder continuous everywhere and constant $\varepsilon = \varepsilon_0 = \text{const}$ outside $Q$; the permeability is constant everywhere, $\mu = \mu_0 = \text{const}$. The problem is to find the electromagnetic field excited in the medium by an external field with time dependence given by the factor $\exp(-i\omega t)$. The corresponding mathematical problem is stated as follows: find unknown vector functions $\vec{E}$ and $\vec{H}$ satisfying Maxwell equations

$$\operatorname{rot}\vec{H} = -i\omega\varepsilon\vec{E} + \vec{J}^0, \quad \operatorname{rot}\vec{E} = i\omega\mu_0\vec{H} \tag{1}$$

and the radiation condition at infinity

$$\lim_{r\to\infty}\left[r\left(\frac{\partial u}{\partial r} - ik_0 u\right)\right] = 0, \tag{2}$$

where $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$. In (1) $\vec{J}^0$ is the external current generating the external field $\vec{E}^0$, $\vec{H}^0$; and $\operatorname{Im}\varepsilon_0 \ge 0$, $\operatorname{Im}\mu_0 \ge 0$, and $\operatorname{Im}k_0 \ge 0$.

Using the polarization current and known formulas for the vector potentials, one can obtain the integral-differential equation with respect to electric field $\vec{E}$

$$\vec{E}(x) - k_0^2 \int_Q (\varepsilon_r(y) - 1)\vec{E}(y)G(R)\,dy - \operatorname{grad}\operatorname{div}\int_Q (\varepsilon_r(y) - 1)\vec{E}(y)G(R)\,dy = \vec{E}^0(x). \tag{3}$$

In (3) $G$ is the Green function of the Helmholtz equation

$$G(R)=\frac{\exp(ik_0 R)}{4\pi R},\quad R=|x-y|, x=(x_1,x_2,x_3), y=(y_1,y_2,y_3),\quad \varepsilon_r=\varepsilon/\varepsilon_0. \quad (4)$$

In (3) $\vec{E}^0(x)$ is the electric field generated by known current $\vec{J}^0$ in the homogeneous space with parameters $\varepsilon_0$ and $\mu_0$. If we know the electric field we can calculate magnetic field by using second equation (1).
Note that we cannot apply *grad div* under the integral sign in (3) because in this case one must differentiate function $G$ twice with respect to coordinates which yields the term $\sim 1/R^3$ in the kernel of the integral equation and the corresponding improper integrals diverge. Equation (3) can be reduced to the singular volume integral equation

$$\vec{E}(x)+\frac{1}{3}(\varepsilon_r(x)-1)\vec{E}(x)-p.v.\int_Q((\varepsilon_r(y)-1)\vec{E}(y),\text{grad})\,\text{grad}\,G(R)\,dy-$$

$$k_0^2\int_Q(\varepsilon_r(y)-1)\vec{E}(y)G(R)\,dy=\vec{E}^0(x),\quad x\in Q. \quad (5)$$

Here $p.v.\int$ denotes a singular integral, for which an infinitely small ball occupying the vicinity of the point $x=y$ is extracted from the domain of integration; and $(*,*)$ denotes the inner product of three-dimensional vectors.
We will consider integral equation (5) with respect to vector function $\vec{E}$ in a domain $Q$. The electric field outside $Q$ is represented through the value of $\vec{E}$ in this domain by formula (5), where, obviously, singular integral should be considered as proper ones.

## 2. Spectrum of integral operator

The *spectrum* of the operator $\hat{A}$ on the complex plane $Z$ is the set of points $\lambda$ such that the operator $(\hat{A}-\lambda\hat{I})$ does not have an inverse defined everywhere in the Hilbert space $H$. The points $\lambda$ such that the operator $(\hat{A}-\lambda\hat{I})$ is not Fredholm belong to the continuous part of the spectrum of $\hat{A}$. The points $\lambda$ such that $(\hat{A}-\lambda\hat{I})$ is a Fredholm operator of index zero and there exists a nontrivial solution $u, \hat{A}u-\lambda u=0$ belong to the discrete part of the spectrum of $\hat{A}$.
First, we have to specify appropriate functional space. The integrals of squared field characteristics stand in the conservation law for electromagnetic scattering problems. Therefore, one may assume that the Hilbert space of square-integrable vector-functions $\vec{L}_2(Q)$ with the inner product

$$\left(\vec{U},\vec{V}\right)=\int_Q \vec{U}(x)\vec{V}^*(x)\,dx \quad (6)$$

is the most appropriate from the physical viewpoint as applied to the analysis of the integral equation (5). In (6) symbol * means the complex conjugation.
The following statement is valid [1].

**Theorem 1.** *The operator of the singular integral equation (5) is a Fredholm operator in the Hilbert space* $\vec{L}_2(Q)$ *if and only if the following condition is satisfied*

$$\varepsilon(x),\ x \in Q. \tag{7}$$

Rewrite integral equation (5) in the symbolic form

$$\hat{A}u \equiv u - \hat{S}((\varepsilon_r - 1)u) = f. \tag{8}$$

Obviously

$$\hat{A} - \lambda\hat{I} = (1-\lambda)\left[\hat{I} - \hat{S}\left(\frac{\varepsilon_r - \lambda}{1-\lambda} - 1\right)\right]. \tag{9}$$

By comparing (8) and (9) from Theorem 1 we find that the continuous part of the spectrum of the operator in equation (5) contains the set $\sigma_1$ of points on the complex plane given by the formula

$$\lambda = \varepsilon_r(x),\ x \in Q. \tag{10}$$

It follows from (10) that the point $\lambda = 1$ belongs to $\sigma_1$ since $\varepsilon_r = 1$ on the boundary of the domain $Q$. Note that, by virtue of the Holder continuity of the permittivity function $\varepsilon_r(x)$, the set $\sigma_1$ is a connected subset of the complex plane.
Denote the boundary of $\sigma_1$ by $\gamma_1$ and the set of all points of the complex plane $Z$ lying on and inside the boundary $\gamma_1$ by $\sigma^+$. If, in particular, if $\sigma_1$ is a non closed curve then $\sigma^+ = \sigma_1 = \gamma_1$. It is obvious that the set $Z \setminus \sigma^+$ is a connected subset of the complex plane and for any $\lambda \in Z \setminus \sigma^+$ operator $(\hat{A} - \lambda\hat{I})$ is Fredholm. Therefore index of the operator $(\hat{A} - \lambda\hat{I})$ is the same for any $\lambda \in Z \setminus \sigma^+$. Operator $(\hat{A} - \lambda\hat{I})$ has an inverse if $|\lambda| > \|\hat{A}\|$ and therefore his index is zero. Thus we arrive to the following assertion.

**Theorem 2.** *The continuous spectrum of the operator of the integral equation (5) contains the set* $\sigma_1$ *on the complex plane given by formula (10). Moreover, the operator* $(\hat{A} - \lambda\hat{I})$ *is Fredholm of index zero in the Hilbert space* $\vec{L}_2(Q)$ *if* $\lambda \in Z \setminus \sigma^+$.

It follows from Theorem 2 that each point $\lambda \in Z \setminus \sigma^+$ on the complex plane belongs to either the resolvent set or the discrete spectrum of the operator $\hat{A}$.

### 3. Spectrum for low-frequency case

In the general case it is impossible to describe the localization domain of the discrete spectrum of the operator accurately. However, this can be done in a special case which is important in practice. Consider low-frequency electromagnetic wave scattering problems such that the diameter of $Q$ is much less than the wavelength, $D << \lambda$, where $\lambda = 2\pi / k_0$.

Equation (5) can be applied when the wave number $k_0 = 0$, i.e., for the static case. Obviously, all preceding assertions remain valid in the static case. It follows from (5) that

$$\left(\hat{A}(k_0) - \hat{A}(0)\right)\vec{V} = -k_0^2 \int\limits_Q (\varepsilon_r - 1)\vec{E}(y)G(R)dy - \int\limits_Q ((\varepsilon_r - 1)\vec{V}(y), \text{grad})\,\text{grad}\, G_0(R)dy\,,$$

$$G_0(R) = \frac{\exp(ik_0R) - 1}{4\pi R}, \tag{11}$$

where $\hat{A}(k_0)$ and $\hat{A}(0)$ are the operators in the integral equations for the stationary and static cases, respectively. The second integral operator in (11) does not contain a singular integral since the kernel of this operator has no singularity at $x=y$ and is a smooth function of the coordinates. Therefore, from (11) we obtain

$$\lim_{k_0 \to 0} \left\| \hat{A}(k_0) - \hat{A}(0) \right\| = 0\,. \tag{12}$$

From (12) we have the following assertion.

**Lemma 1.** *The spectrum of the low-frequency integral operator* $\hat{A}(k_0)$ *tends to the spectrum of the static operator* $\hat{A}(0)$ *as* $k_0 \to 0$.

The integral equations (3) and (5) are equivalent among themselves. In the static case, the integral equation (3) can be represented in the form

$$\vec{E}(x) - \text{grad div} \int\limits_Q (\varepsilon_r(y) - 1)\,\vec{E}(y)(1/4\pi R))\,dy = \vec{E}^0(x)\,. \tag{13}$$

The solution of the homogeneous equation (13) satisfies the differential equations

$$\text{rot}\,\vec{E} = 0, \quad \text{div}(\varepsilon_r\vec{E}) = 0\,. \tag{14}$$

The first equation (14) follows from the identity $\text{rot grad} = 0$, and the second equation follows from the identities $\text{grad div} = \text{rot rot} + \Delta$ and $\text{div rot} = 0$ and the differential equation $\Delta\vec{A} = -\vec{J}$ which is valid for the volume potential $\vec{A}(x) = \int \vec{J}(y)\left(1/4\pi R\right)dy$.

From the first equation in (14), we have $\vec{E} = \text{grad}\,\varphi$. Then equations (14) can be reduced to a second-order differential equation for the function $\varphi$

$$\text{div}(\varepsilon_r\ \text{grad}\,\varphi) = 0\,. \tag{15}$$

Let $\psi$ be an everywhere-defined differentiable function. We have an obvious identity

$$\text{div}(\psi\varepsilon_r\ \text{grad}\,\varphi) = \psi\,\text{div}(\varepsilon_r\ \text{grad}\,\varphi) + (\text{grad}\,\psi, \varepsilon_r\ \text{grad}\,\varphi)\,. \tag{16}$$

Let $\psi = \varphi^*$. Then, by integrating relation (16) over the space and by taking into account (15) and the divergence theorem, we obtain the integral relation

$$\int \varepsilon_r |\mathrm{grad}\,\varphi|^2 \, d\upsilon = \lim_{R\to\infty} \int_{S_R} \varphi^* \frac{\partial\varphi}{\partial n} dS, \tag{17}$$

where $S_R$ is the sphere of radius $R$ centered at the origin and $n$ is the normal to the sphere. Since $\varphi$ is a harmonic function outside $Q$, it follows that $\varphi^* \partial\varphi/\partial n$ decreases as $R^{-3}$ at infinity. Therefore, the limit on the right-hand side in (17) is zero, and each solution of the homogeneous equation (13) satisfies the integral relation

$$\int \varepsilon_r |\mathrm{grad}\,\varphi|^2 \, d\upsilon = \int \varepsilon_r |\vec{\mathrm{E}}|^2 \, d\upsilon = 0. \tag{18}$$

Denote

$$\varepsilon_r^+(\lambda, x) = (\varepsilon_r(x) - \lambda)/(1-\lambda), \quad \lambda \notin \sigma_1. \tag{19}$$

It follows from (9) and (19) that $\lambda$ is a point of the discrete spectrum of the operator (13) if there exists a nonzero solution $\vec{E}$ of homogeneous integral equation (13) with permittivity $\varepsilon_r^+(\lambda, x)$. Moreover, it follows from (18) and (19) that the corresponding value of $\lambda$ is given by the formula

$$\lambda = \frac{\int \varepsilon_r |\vec{E}(\lambda)|^2 \, d\upsilon}{\int |\vec{E}(\lambda)|^2 \, d\upsilon}. \tag{20}$$

It is impossible to find the corresponding functions $\vec{E}(\lambda)$. However, using formula (20), one can find the localization domain of points of the discrete spectrum on the complex plane: the points of the discrete spectrum of the integral operator (13) can lie only inside the convex envelope of the set $\sigma_1$ given by formula (10). Set $\sigma^+$ lies inside convex envelope of the set $\sigma_1$. Therefore we arrive to the statement.

**Theorem 3.** *Spectrum of the integral operator* (13) *can lie only inside the convex envelope of the set* $\sigma_1$ *given by the formula* (10).

Theorem 3 and Lemma 1 provide approximate information about a convex envelope of the spectrum of the integral operator for the low-frequency case.

Let's set a simple example.
Let domain $Q$ in the integral equation (5) be a ball and suppose that function of dielectric permittivity has the following form in the spherical system of the coordinates

$$\frac{\varepsilon(r)}{\varepsilon_0}=\begin{cases}\varepsilon_2, \ d_2 \ge r \ge 0 \\ \varepsilon_2+(\varepsilon_1-\varepsilon_2)\dfrac{r-d_2}{d_1-d_2}, \ d_1 \ge r \ge d_2 \\ \varepsilon_1+(1-\varepsilon_1)\dfrac{r-d_1}{R-d_1}, \ R \ge r \ge d_1\end{cases}. \tag{21}$$

In (21) $R$ is a radius of a ball, and $R > d_1 > d_2 > 0$.

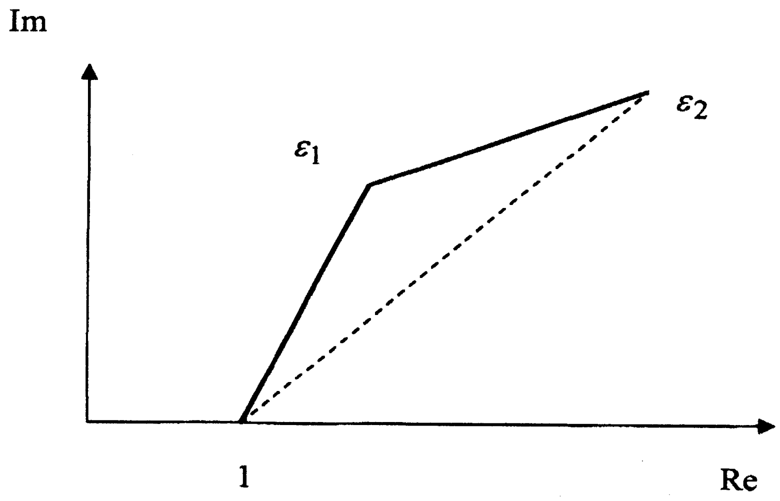


*Figure 1: Spectrum of integral operator.*

On Fig. 1 fat solid line schematically outline the continuous part of spectrum for the case (21). The spectrum of the integral operator for the low-frequency case lies inside the triangle.

## 4. Generalized simple iteration method

In the Banach space $B$, we consider the linear operator equation

$$\hat{A}u = f, \tag{22}$$

where $\hat{A}$ is a bounded operator.
Rewrite Eq. (22) in the equivalent form

$$u - \hat{B}_\mu = f / \mu. \tag{23}$$

Here $\hat{B}_\mu$ is the linear operator given by the formula $\hat{B}_\mu = \left(\mu \hat{I} - \hat{A}\right)/\mu$ and $\mu \neq 0$ is an arbitrary complex number.
The successive approximations

$$u_{n+1} = \hat{B}_\mu u_n + f / \mu, \quad n = 0,1,... \tag{24}$$

converge to the solution of Eq. (23) and hence of Eq. (22) for any $u_0, f \in B$ provided that

$$\rho_0(\mu) = \sup|\eta(\mu)| < 1, \quad \eta(\mu) \in \sigma(\hat{B}_\mu). \tag{25}$$

One can readily show that there is a one-to-one correspondence between points of the spectrum $\sigma(\hat{A})$ of the operator $\hat{A}$ and points of the spectrum $\sigma(\hat{B}_\mu)$ of the operator $\hat{B}_\mu$; this correspondence is given by the formula

$$\eta = (\mu - \lambda)/\mu, \quad \lambda \in \sigma(\hat{A}), \quad \eta \in \sigma(\hat{B}_\mu). \tag{26}$$

The iterations (24) can be represented in the simpler form

$$u_{n+1} = u_n - \frac{1}{\mu}\left(\hat{A}u_n - f\right), \quad n = 0,1,\ldots \tag{27}$$

One can prove the following statement [1].

**Theorem 4.** *A necessary and sufficient condition for the existence of complex number $\mu$ such that the iterations (27) converge to the solution of Eq. (22) for arbitrary $u_0, f \in B$ is that the origin of the complex plane lies outside a convex envelope of the spectrum of $\hat{A}$.*

The convex envelope is illustrated in Fig. 2.

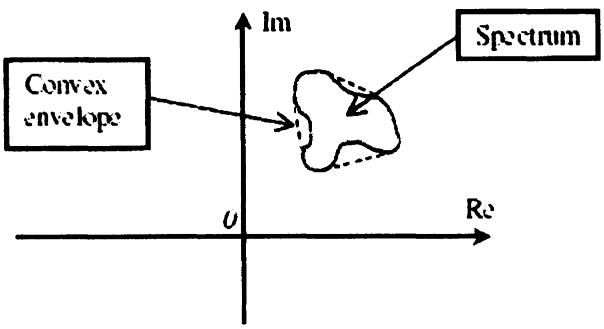


*Figure 2: Spectrum and its convex envelope.*

The iteration converges to the solution at the rate of a geometric progression; i.e.

$$\|u_n - u\| \le C[\rho_0(\mu)]^n, \quad C = \text{const} , \tag{28}$$

where, by (25) and (26), $\rho_0(\mu)$ is given by the formula

$$\rho_0(\mu) = \frac{\sup|\mu - \lambda|}{|\mu|}, \quad \lambda \in \sigma(\hat{A}). \tag{29}$$

Obviously, the best convergence of the iterations is attached at the value of $\mu$ for which the function $\rho_0(\mu)$ takes the minimum value. By $S_\mu$ we denote the disk on the complex plane with center $\mu$ and the least radius $R$ which contains all points of the spectrum of $\hat{A}$. Obviously, $R = \sup|\mu - \lambda|$, $\lambda \in \sigma(\hat{A})$. From the origin, we draw the tangents to the disk $S_\mu$ and denote the angle between them by $\alpha$. Then it follows from (29) that $\rho_0(\mu) = \sin(\alpha/2)$. Thus we have proved the following statement.

**Theorem 5.** *Let the origin of the complex plane lies outside the convex envelope of the spectrum of* $\hat{A}$ *. Let* $S_0$ *be the disk which contains all points of the spectrum of* $\hat{A}$ *and is "seen" from the origin at minimal angle* $\alpha_0$*. Then the best convergence of the iterations (27) to the solution of Eq. (22) is attained at the complex value* $\mu_0$ *which is the center of the disk* $S_0$*. The iterations converge to the solution at the rate of a geometric progression with the denominator* $\rho_0 = \sin(\alpha_0/2)$.

## Conclusion

The convex envelope of the spectrum of an integral operator on a complex plane depending on the form of the dielectric permittivity function has been defined above for the integral equation (3) in the case of low-frequency electromagnetic scattering problems. It follows from Theorems 4 and 5 and relation (10) that the generalized simple iteration method can be used for Eq. (3) for arbitrary real media; moreover, we can readily evaluate the optimal iterative parameter $\mu_0$. Numerical experiments have shown that this method is a very effective tool for the numerical solution of low-frequency scattering problems.

## References

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