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短パルスモデル方程式の周期解

Periodic solutions of the short pulse model equation

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Abstract

The short pulse model equation describes the propagation of ultra-short optical pulses in nonlinear media. We develop a systematic method for solving the short pulse equation and address the construction of the two-phase periodic solutions and their properties. The detail of the content of this paper is described in Ref. [11].

1.1 Maxwell equation

We start from the following Maxwell equation

$$div\ \mathbf{D} = \rho, \quad div\ \mathbf{B} = 0, \quad rot\ \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad rot\ \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \tag{1.1a}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{B} = \mu_0 \mathbf{H} \tag{1.1b}$$

where $\mathbf{E}$ and $\mathbf{H}$ are electric and magnetic field vectors, respectively, and $\mathbf{D}$ and $\mathbf{B}$ are corresponding electric and magnetic flux density.

We assume that $\rho = 0$ and $\mathbf{j} = 0$ and consider the one-dimensional propagation. Then Eq. (1.1) reduces to

$$\mathbf{E} = E_3(x,t)\mathbf{e}_3, \quad \mathbf{H} = H_2(x,t)\mathbf{e}_2 \tag{1.2}$$

$$\frac{\partial H_2}{\partial x} = \frac{\partial D_3}{\partial t}, \quad \frac{\partial E_3}{\partial x} = \mu_0 \frac{\partial H_2}{\partial t}. \tag{1.3}$$

Using (1.3) and the relation $D_3 = \epsilon_0 E_3 + P_3$, we eliminate H_2 from (1.3) to obtain

$$E_{xx} - \frac{1}{c^2} E_{tt} = P_{tt} \tag{1.4}$$

where we have put $E = E_3, P = P_3/(\epsilon_0 c^2), c^2 = (\epsilon_0 \mu_0)^{-1}$. We further assume the relation

$$P = P_{lin} + P_{nl} = \int_{-\infty}^{\infty} \chi(t-\tau) E(x,\tau) d\tau + \chi_3 E^3 \tag{1.5a}$$

$$\chi_{tt} = \chi_0 \delta(t). \tag{1.5b}$$

Substituting (1.5) into (1.4), we obatin the nonlinear wave equation

$$E_{xx} - \frac{1}{c^2}E_{tt} = \chi_0 E + \chi_3 (E^3)_{tt}. \tag{1.6}$$

1.2 Singular perturbation

In accordance with Schäfer and Wayne (2004), we apply the singular perturbation method to Eq. (1.6) to derive the short pulse (SP) equation. We expand E with respect to the small parameter ϵ

$$E(x,t) = \epsilon u_0(\phi, X) + \epsilon^2 u_1(\phi, X) + \cdots \tag{1.7a}$$

where the new independent variables ϕ and X are defined by

$$\phi = \frac{t - \frac{x}{c}}{\epsilon}, \quad X = \epsilon x. \tag{1.7b}$$

If we introduce (1.7) into (1.6), we obtain, at the lowest order $O(\epsilon)$, the following PDE

$$-\frac{2}{c}\frac{\partial^2 u_0}{\partial\phi\partial X} = \chi_0 u_0 + \chi_3 \frac{\partial^2 u_0^3}{\partial\phi^2}. \tag{1.8}$$

After an appropriate change of the variables, we arrive at the normalized form of the SP equation:

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}. \tag{1.9}$$

1.3 Remarks

• The SP equation is a model equation describing the propagation of ultra-short optical pulses in nonlinear media.

• The SP equation has been derived in a mathematical context concerning the integrable PDE (Robelo (1989)).

• The following solutions are known for the SP equation:

Soliton and breather solutions: Sakovich et al (2006), Kuetche et al (2006), Matsuno (2007)

Periodic solutions of traveling type (one-phase solutions): Parkes (2008)

• Analogous integrable equations (Matsuno (2006))

$$u_{xt} = \alpha u + \frac{1}{2}(1-\beta)u_x^2 - uu_{xx}$$

$\beta = 2$: Short-pulse model for Camassa-Holm equation

$\beta = 3$: Short-pulse model for the Degasperis-Procesi equation, Vakhnenko equation

$\alpha = 0, \beta = 2$: Hunter-Saxton equation

All the above equations have the solutions expressed by the parametric reprsentation.

2. Exact method of solution

2.1 Transformation to the sine-Gordon equation

Introduce the new variable r:

$$r^2 = 1 + u_x^2. \tag{2.1}$$

We rewrite the SP equation (1.9) into the form

$$r_t = \left(\frac{1}{2}u^2 r\right)_x. \tag{2.2}$$

By means of the hodograph transformation $(x, t) \to (y, \tau)$

$$dy = r dx + \frac{1}{2}u^2 r dt, \quad d\tau = dt \tag{2.3a}$$

or equivalently

$$\frac{\partial}{\partial x} = r\frac{\partial}{\partial y}, \ \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \frac{1}{2}u^2 r \frac{\partial}{\partial y} \tag{2.3b}$$

(2.1) and (2.2) are transformed to

$$r^2 = 1 + r^2 u_y^2, \quad r_\tau = r^2 u u_y. \tag{2.4}$$

Using the transformation

$$u_y = \sin\phi, \quad \phi = \phi(y, \tau) \tag{2.5}$$

(2.4) can be put into the form

$$\frac{1}{r} = \cos\phi. \tag{2.6}$$

It follows from (2.4)-(2.6) that $u = \phi_\tau$. Substituting this into (2.5), we obtain the sine-Gordon(sG) equation :

$$\phi_{y\tau} = \sin\phi. \tag{2.7}$$

We see from (2.3) that $x = x(y, \tau)$ satisfies the following linear PDE

$$x_y = \frac{1}{r}, \quad x_\tau = -\frac{1}{2}u^2. \tag{2.8}$$

2.2 Parametric representation of the solution

Since the integrability of Eq. (2.8), i.e. $x_{y\tau} = x_{\tau y}$ is assured by (2.4), we can integrate (2.8) to obtain

$$x(y, \tau) = \int^y \cos\phi \, dy + d \tag{2.9}$$

where d is an integration constant. The expression of u in terms ϕ is given by

$$u(y, \tau) = \phi_\tau. \tag{2.10}$$

2.3 A criterion for the single-valued functions

To derive a criterion for single-valued functions, we may simply require that $u_x = \tan\phi$ exhibits no singularities. Thus, if

$$-\frac{\pi}{2} < \phi < \frac{\pi}{2}, \quad (\mathrm{mod}\ \pi), \ (-\sqrt{2}+1 < \tan\frac{\phi}{4} < \sqrt{2}-1). \tag{2.11}$$

then the parametric solutions (2.9) and (2.10) will become single-valued functions for all values of x and t.

3. Periodic solutions

Here, we are concerned with the construction of the periodic solutions of the SP equation, particularly focusing on the two-phase solutions.

3.1 Method of solution

We first introduce the two independent phase variables ξ and η according to

$$\xi = ay + \frac{t}{a} + \xi_0, \ \eta = ay - \frac{t}{a} + \eta_0 \tag{3.1}$$

where $a(\neq 0), \xi_0$ and η_0 are arbitrary constants. Then, the sG equation is transformed to

$$\phi_{\xi\xi} - \phi_{\eta\eta} = \sin\phi, \ \phi = \phi(\xi, \eta). \tag{3.2}$$

We seek solutions of the sG equation of the form

$$\phi = 4\tan^{-1}\left[\frac{f(\xi)}{g(\eta)}\right]. \tag{3.3}$$

This ϕ satisfies the sG equation provided that

$$f'^2 = -\kappa f^4 + \mu f^2 + \nu \tag{3.4a}$$

$$g'^2 = \kappa g^4 + (\mu - 1)g^2 - \nu. \tag{3.4b}$$

Now, the parametric representation of u follows from (2.10) and (3.3)

$$u = \frac{4}{a}\frac{f'g + fg'}{f^2 + g^2}. \tag{3.5}$$

To obtain the parametric form of x, we note the relation

$$\cos\ \phi = 1 - \frac{8f^2g^2}{(f^2 + g^2)^2}. \tag{3.6}$$

We modify the right-hand side of (3.6) by introducing the function $Y = Y(\xi, \eta)$

$$Y = \frac{c_1(f^2)' + c_2(g^2)'}{f^2 + g^2}. \tag{3.7}$$

We calculate Y_y. Using (3.4), we can modify this in the form

$$Y_y = \frac{a}{(f^2+g^2)^2}\Big[-2\kappa(c_1 f^6 + 3c_1 f^4 g^2 - 3c_2 f^2 g^4 - c_2 g^6) - 4c_2 f^2 g^2 \\ +2(c_1+c_2)\left\{-2fgf'g' + 2\mu f^2 g^2 - \nu(f^2-g^2)\right\}\Big]. \tag{3.8}$$

If we put $c_1 + c_2 = 0$ and $c_1 = -2/a$, then (3.8) simplifies to

$$Y_y = 4\kappa(f^2+g^2) - \frac{8f^2g^2}{(f^2+g^2)^2}. \tag{3.9}$$

If we compare (3.6) and (3.9), we obtain

$$\cos\ \phi = 1 + Y_y - 4\kappa(f^2+g^2). \tag{3.10}$$

Finally, substituting (3.10) into (2.9) and integrating, we obtain the parametric representation of x:

$$x = y - \frac{4}{a}\frac{ff' - gg'}{f^2+g^2} - \frac{4\kappa}{a}\left(\int f^2(\xi)d\xi + \int g^2(\eta)d\eta\right) + d. \tag{3.11}$$

3.2 Examples

Here, we present the three examples of the periodic solutions:

a. Example 1

$$f(\xi) = A\ \mathrm{cn}(\beta\xi, k_f),\ \ g(\eta) = \frac{1}{\mathrm{cn}(\Omega\eta, k_g)} \tag{3.12}$$

$$k_f^2 = \frac{A^2}{1+A^2}\left(1 + \frac{1}{\beta^2(1+A^2)}\right) \tag{3.13a}$$

$$k_g^2 = \frac{A^2}{1+A^2}\left(1 - \frac{1}{\Omega^2(1+A^2)}\right) \tag{3.13b}$$

$$\Omega^2 = \beta^2 + \frac{1-A^2}{1+A^2}. \tag{3.13c}$$

The inequality $0 \le k_f \le 1$ implies that the parameter β must be restricted by the condition

$$\frac{A}{\sqrt{1+A^2}} \le \beta. \tag{3.14}$$

The parametric solution takes the form

$$u = \frac{4A}{a}\frac{-\beta\ \mathrm{sn}(\beta\xi,k_f)\mathrm{dn}(\beta\xi,k_f)\mathrm{cn}(\Omega\eta,k_g) + \Omega\ \mathrm{cn}(\beta\xi,k_f)\mathrm{sn}(\Omega\eta,k_g)\mathrm{dn}(\Omega\eta,k_g)}{A^2\mathrm{cn}^2(\beta\xi,k_f)\mathrm{cn}^2(\Omega\eta,k_g)+1} \tag{3.15a}$$

$$x = y + \frac{4\beta}{a}\frac{\mathrm{cn}(\beta\xi,k_f)\mathrm{cn}(\Omega\eta,k_g)}{A^2\mathrm{cn}^2(\beta\xi,k_f)\mathrm{cn}^2(\Omega\eta,k_g)+1}\Big\{A^2\mathrm{sn}(\beta\xi,k_f)\mathrm{dn}(\beta\xi,k_f)\mathrm{cn}(\Omega\eta,k_g)$$

$$-\frac{\beta k_f^2}{\Omega k_g'^2}\,\mathrm{cn}(\beta\xi,k_f)\mathrm{sn}(\Omega\eta,k_g)\mathrm{dn}(\Omega\eta,k_g)\Big\}$$

$$-\frac{4\beta}{a}\left[E(\beta\xi,k_f)-k_f'^2\beta\xi-\frac{\beta k_f^2}{A^2\Omega k_g'^2}\left\{E(\Omega\eta,k_g)-k_g'^2\Omega\eta\right\}\right]+d. \tag{3.15b}$$

Properties of the solution

• The solution is a multiply peridic function. It becomes a single-valued function if $0 < A < \sqrt{2}-1$.
• Under the condition $L = m_\xi L_\xi/a = m_\eta L_\eta/a$, $(m_\xi, m_\eta) = 1$ where $L_\xi \equiv 4K(k_f)/\beta$ and $L_\eta \equiv 4K(k_g)/\Omega$, the solution has a period Λ

$$\Lambda = L\left[1-4\beta^2\left\{\frac{E(k_f)}{K(k_f)}-\frac{k_f^2}{A^2(1-k_g^2)}\frac{E(k_g)}{K(k_g)}+\frac{1}{\beta^2(1+A^2)}\right\}\right] \tag{3.16}$$

where $K(k_f)$ and $E(k_f)$ are the complete elliptic integral of the first and second kinds, respectively. Figure 1 shows a profile of u at $t = 0$ for Example 1.

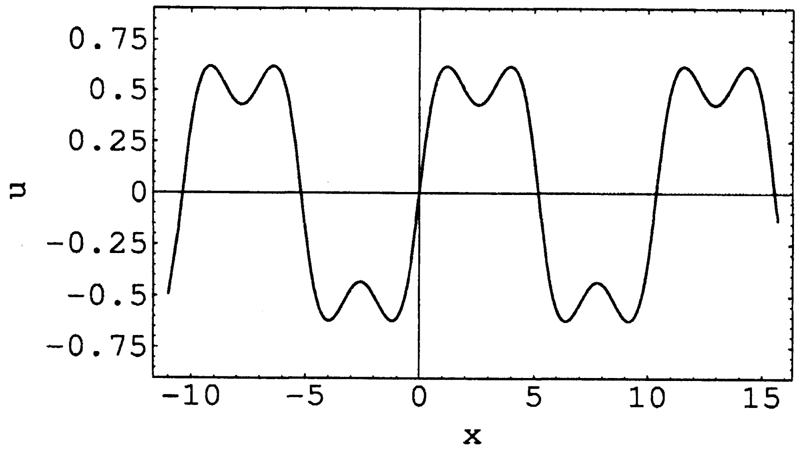

Figure 1: $A = 0.2, m_\xi = 1, m_\eta = 2, a = 1.0, \beta = 0.5832, \Omega = 1.124, k_f = 0.3837, k_g = 0.0958, \Lambda = 10.37$.

Long-wave limit $\Lambda \to \infty$

In the long-wave limit, the parametric solution reduces to

$$u \sim \frac{4A\Omega}{a}\frac{-A\ \sinh\beta\xi\cos\Omega\eta+\cosh\beta\xi\sin\Omega\eta}{\cosh^2\beta\xi+A^2\cos^2\Omega\eta} \tag{3.17a}$$

$$x \sim y-\frac{2\Omega}{a}\frac{\sinh 2\beta\xi+A\ \sin 2\Omega\eta}{\cosh^2\beta\xi+A^2\cos^2\Omega\eta}+d. \tag{3.17b}$$

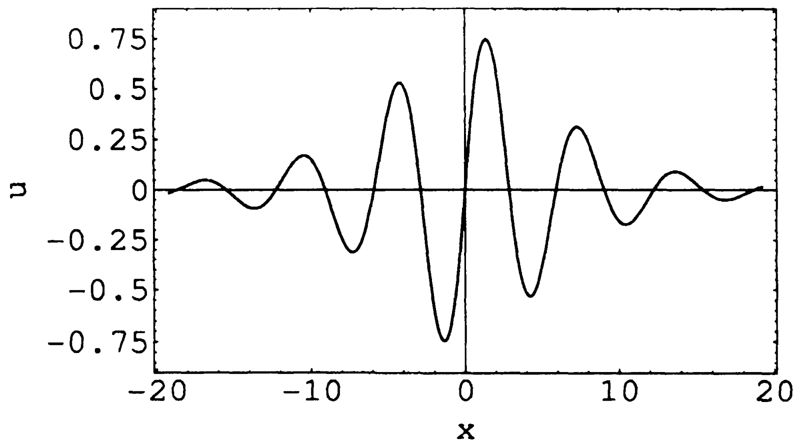

Figure 2: Long-wave limit of the solution depicted in Figure 1.

b. Example 2

$$f(\xi) = A\,\frac{\mathrm{sn}(\beta\xi, k_f)}{\mathrm{cn}(\beta\xi, k_f)},\quad g(\eta) = \frac{1}{\mathrm{dn}(\Omega\eta, k_g)} \tag{3.18}$$

$$k_f^2 = 1 - A^2 + \frac{A^2}{\beta^2(1 - A^2)} \tag{3.19a}$$

$$k_g^2 = 1 - \frac{1}{A^2} + \frac{1}{\Omega^2(1 - A^2)} \tag{3.19b}$$

$$\Omega = \beta A \tag{3.19c}$$

$$\frac{1}{\sqrt{1 - A^2}} \le \beta \le \frac{1}{1 - A^2} \tag{3.20}$$

$$u = \frac{4A}{a}\,\frac{\beta\ \mathrm{dn}(\beta\xi, k_f)\mathrm{dn}(\Omega\eta, k_g) + k_g^2\Omega\ \mathrm{sn}(\beta\xi, k_f)\mathrm{cn}(\beta\xi, k_f)\mathrm{sn}(\Omega\eta, k_g)\mathrm{cn}(\Omega\eta, k_g)}{A^2\mathrm{sn}^2(\beta\xi, k_f)\mathrm{dn}^2(\Omega\eta, k_g) + \mathrm{cn}^2(\beta\xi, k_f)} \tag{3.21a}$$

$$x = y - \frac{4\beta}{a}\,\frac{1}{A^2\mathrm{sn}^2(\beta\xi, k_f)\mathrm{dn}^2(\Omega\eta, k_g) + \mathrm{cn}^2(\beta\xi, k_f)}\times$$
$$\times\Big[(A^2\mathrm{dn}^2(\Omega\eta, k_g) - 1)\mathrm{sn}(\beta\xi, k_f)\mathrm{cn}(\beta\xi, k_f)\mathrm{dn}(\beta\xi, k_f)$$
$$+k_g^2 A^3\mathrm{sn}^2(\beta\xi, k_f)\mathrm{sn}(\Omega\eta, k_g)\mathrm{cn}(\Omega\eta, k_g)\mathrm{dn}(\Omega\eta, k_g)\Big] + \frac{4\beta}{a}\left(-E(\beta\xi, k_f) + AE(\Omega\eta, k_g)\right) + d \tag{3.21b}$$

$$\Lambda = L\left[1 - 4\beta^2\left\{\frac{E(k_f)}{K(k_f)} - A^2\frac{E(k_g)}{K(k_g)}\right\}\right]. \tag{3.22}$$

Figure 3 shows a profile of u at $t = 5$ for Example 2.

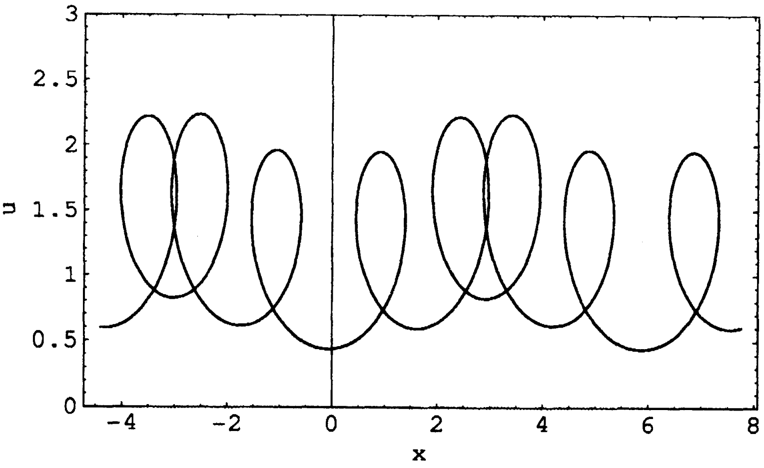

Figure 3: $A = 0.2, m_\xi = 2, m_\eta = 1, a = 1.0, \beta = 1.027, \Omega = 0.2053, k_f = 0.9998, k_g = 0.8421, \Lambda = 5.938$.

Long-wave limit $\Lambda \to \infty$

$$u \sim \frac{4\beta A}{a} \frac{\cosh\beta\xi\cosh\Omega\eta + A\sinh\beta\xi\sinh\Omega\eta}{A^2\sinh^2\beta\xi + \cosh^2\Omega\eta} \tag{3.23a}$$

$$x \sim y - \frac{2\beta}{a}\frac{A^2\sinh 2\beta\xi - A\sinh 2\Omega\eta}{A^2\sinh^2\beta\xi + \cosh^2\Omega\eta} + d. \tag{3.23b}$$

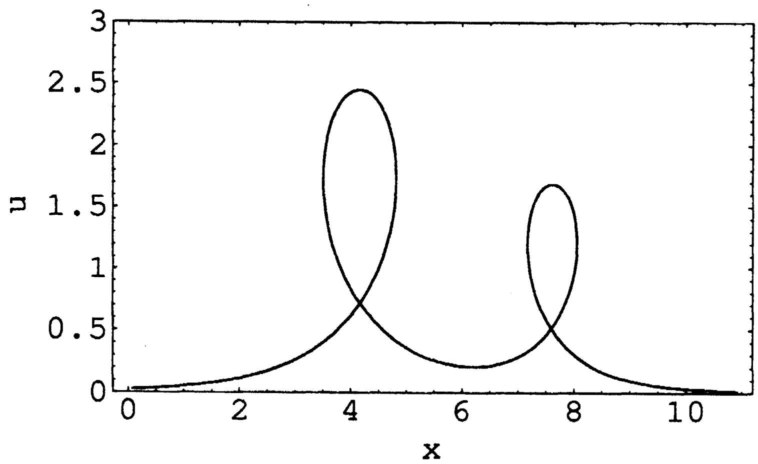

Figure 4: Long-wave limit of the solution depicted in Figure 3.

c. Example 3

$$f(\xi) = A\ \mathrm{dn}(\beta\xi, k_f),\ \ g(\eta) = \frac{\mathrm{cn}(\Omega\eta, k_g)}{\mathrm{sn}(\Omega\eta, k_g)} \tag{3.24}$$

$$k_f^2 = 1 - \frac{1}{A^2} + \frac{1}{\beta^2(A^2-1)} \tag{3.25a}$$

$$k_g^2 = 1 - A^2 + \frac{A^2}{\Omega^2(A^2-1)} \tag{3.25b}$$

$$\Omega = \frac{\beta}{A} \tag{3.25c}$$

$$\frac{A}{\sqrt{A^2-1}} \leq \beta \leq \frac{A^2}{A^2-1}, \quad A > 1 \tag{3.26}$$

$$u = -\frac{4A}{a}\frac{\Omega\ \mathrm{dn}(\beta\xi, k_f)\mathrm{dn}(\Omega\eta, k_g) + \beta k_f^2\ \mathrm{sn}(\beta\xi, k_f)\mathrm{cn}(\beta\xi, k_f)\mathrm{sn}(\Omega\eta, k_g)\mathrm{cn}(\Omega\eta, k_g)}{A^2\mathrm{dn}^2(\beta\xi, k_f)\mathrm{sn}^2(\Omega\eta, k_g) + \mathrm{cn}^2(\Omega\eta, k_g)} \tag{3.27a}$$

$$x = y - \frac{4\beta}{a}\frac{1}{A^2\mathrm{dn}^2(\beta\xi, k_f)\mathrm{sn}^2(\Omega\eta, k_g) + \mathrm{cn}^2(\Omega\eta, k_g)}\times$$
$$\times\Big[\frac{1}{A}(1 - A^2\mathrm{dn}^2(\beta\xi, k_f))\mathrm{sn}\ (\Omega\eta, k_g)\mathrm{cn}\ (\Omega\eta, k_g)\mathrm{dn}(\Omega\eta, k_g)$$
$$-k_f^2A^2\mathrm{sn}(\beta\xi, k_f)\mathrm{cn}(\beta\xi, k_f)\mathrm{dn}(\beta\xi, k_f)\mathrm{sn}^2(\Omega\eta, k_g)\Big] - \frac{4\beta}{a}\left(E(\beta\xi, k_f) - \frac{1}{A}E(\Omega\eta, k_g)\right) + d \tag{3.27b}$$

$$\Lambda = L\left[1 - 4\beta^2\left\{\frac{E(k_f)}{K(k_f)} - \frac{1}{A^2}\frac{E(k_g)}{K(k_g)}\right\}\right]. \tag{3.28}$$

Figure 5 shows a profile of u at $t = 5$ for Example 3.

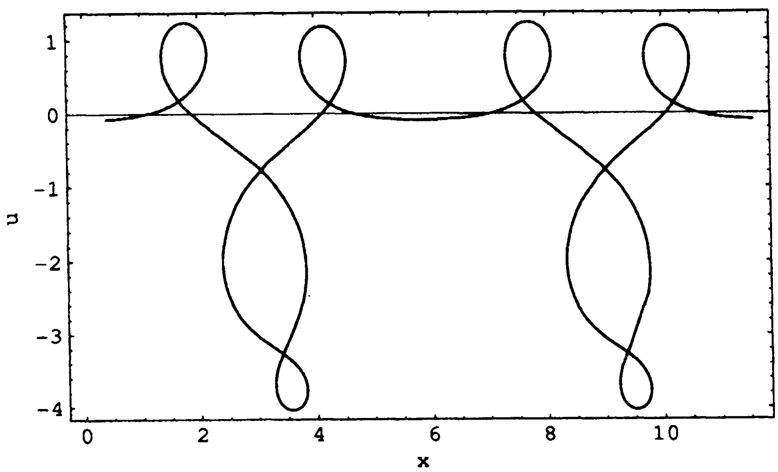

Figure 5: $A = 5, m_\xi = 2, m_\eta = 1, a = 1.0, \beta = 1.027, \Omega = 0.2053, k_f = 0.9998, k_g = 0.8421, \Lambda = 5.938$.

Long-wave limit $\Lambda \to \infty$

$$u \sim -\frac{4\beta}{a}\frac{\cosh\beta\xi\cosh\Omega\eta + A\sinh\beta\xi\sinh\Omega\eta}{\cosh^2\beta\xi + A^2\sinh^2\Omega\eta} \tag{3.29a}$$

$$x \sim y - \frac{2\beta}{a}\frac{\sinh 2\beta\xi - A\sinh 2\Omega\eta}{\cosh^2\beta\xi + A^2\sinh^2\Omega\eta} + d. \tag{3.29b}$$

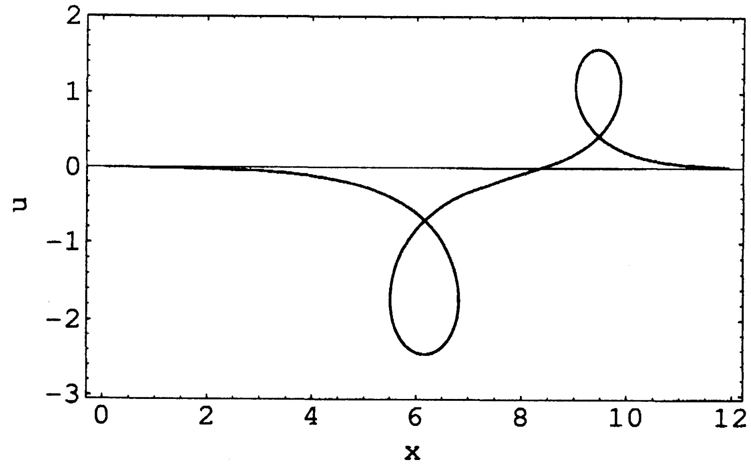

Figure 6: Long-wave limit of the solution depicted in Figure 5.

4. Conclusion

• By means of a novel method of exact solution, we obtained periodic solutions of the SP equation and investigated their properties.

• Of particular interest is the nonsingular periodic solution which reduces to the breather solution in the long-wave limit.

• The construction of a more general class of periodic solutions is under study. It is produced by the multiphase solutions of the sG equation expressed by Riemann's theta functions.

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