数理解析研究所講究録
第 1646 巻 2009 年 37-40

On weak forking

前園 久智 (Hisatomo MAESONO)
早稲田大学メディアネットワークセンター
(Media Network Center, Waseda University)

Abstract

Weak dividing was defined in [1] and has been characterized in simple theory ([2], [3]). We consider some generalized notion of it.

1. Weak dividing and weak forking

We recall some definitions.

Definition 1 Let $\varphi(x_0, x_1, \cdots\cdots, x_{n-1})$ be a formula and $p(x)$ be a type. We denote the type $\{\varphi(x_0, x_1, \cdots\cdots, x_{n-1})\} \cup p(x_0) \cup p(x_1) \cup \cdots\cdots \cup p(x_{n-1})$ by $[p]^{\varphi}$.

Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ divides over$A$ if there are a formula $\varphi(x, b) \in p(x)$ and an infinite sequence $\{b_i : i < \omega\}$ with $b \equiv b_i\,(A)$ such that $\{\varphi(x, b_i) : i < \omega\}$ is $k-$inconsistent for some $k < \omega$.

$p(x)$ weakly divides over A if there is a formula $\varphi(\bar{x}) \in L_n(A)$ such that $[p\lceil A]^{\varphi}$ is consistent, while $[p]^{\varphi}$ is inconsistent.

A collection of types $\{\mathrm{tp}(a_\alpha/Ab\{a_\beta \mid \beta < \alpha\}) \mid \alpha < |T|^+\}$ is a *weak dividing left − chain* if for each $\alpha < |T|^+$, $\mathrm{tp}(a_\alpha/Ab\{a_\beta|\beta < \alpha\})$ weakly divides over $A \cup \{a_\beta \mid \beta < \alpha\}$.

A collection of types $\{\mathrm{tp}(b/A\{a_\beta \mid \beta < \alpha\}) \mid \alpha < |T|^+\}$ is a *dividing right− chain* if for each $\alpha < |T|^+$, $\mathrm{tp}(b/A\{a_\beta \mid \beta \leqq \alpha\})$ divides over $A \cup \{a_\beta \mid \beta < \alpha\}$.

In this note, we call such formula "$\varphi(\bar{x})$" in the definition the *witness formula* of weak dividing for the sake of convenience.

I show examples from [3].

Example 2 Let T be the theory of an equivalence relation with two infinite classes of the language $L = \{$a binary relation $E(x, y)\}$. And let $\models \neg E(a, b)$. Then the type $\mathrm{tp}(a/b)$ does not divide over $\emptyset$, while $\mathrm{tp}(a/b)$ weakly divides over $\emptyset$ by the formula $\neg E(x, y)$.

Example 3 Let $(V, <, >)$ be a vector space V over a finite field equipped with an inner product giving orthogonality between two independent vectors. Let a, b, c be independent vectors in V such that $a \perp b$, while $b \not\perp c$ and $a \not\perp c$. Then $\mathrm{tp}(a/bc)$ does not weakly divide over $\emptyset$. But $\mathrm{tp}(a/bc)$ weakly divides over c by the formula $\varphi(x, y) :=$ "x is a linear combination of y and c".

In various characterizations, one of the most important results is the next theorem.

Theorem 4 (Kim [3])
The following are equivalent ;
(1) *T is stable.*
(2) *Weak dividing is symmetric in T.*
(3) *There is no weak dividing left-chain in T.*

I consider some generalization of weak dividing.

Definition 5 Let $A \subset B$ and $p(x) \in S(B)$.
$p(x)$ weakly forks over A if there is a complete type $q(x, y) \in S(A)$ such that $p(x) \cup q(x, y)$ is consistent, and any completion $r(x, y) \in S(B)$ of $p(x) \cup q(x, y)$ weakly divides over A.

We can easily prove the next fact.

Fact 6 *Let T be any theory.*
If $\mathrm{tp}(a/bA)$ *forks over A, then* $\mathrm{tp}(b/aA)$ *weakly forks over A.*

And I consider the definition of weak dividing by the use of formulas. But it can be defined in relation to complete types.

Definition 7 Let $A, B \subset C$. And $\varphi(x)$ is a $L_m(B)-$formula.
$\varphi(x)$ weakly divides over A if there are a $L_{mn}(A)-$formula $\phi(\bar{x})$ and a complete type $p(x) \in S_m(A)$ such that $p(x) \cup \{\varphi(x)\}$ is consistent, and $[p]^{\phi}$ is consistent, while $[\varphi]^{\phi}$ is inconsistent.
$\varphi(x)$ weakly forks over A if there are $L(B)-$formulas $\psi_i(x, y)$ $(i < n)$ for some $n < \omega$, a complete type $p(x, y) \in S(A)$ and $L(A)-$formulas $\phi_i(\bar{x_i}, \bar{y_i})$ $(i < n)$ such that $\varphi(x) \vdash \bigvee_{i<n} \exists y \psi_i(x, y)$, $p(x, y) \cup \{\varphi(x)\}$ is consistent, and $\psi_i(x, y)$ weakly divides ove A with respect to $p(x, y)$ and $\phi_i(\bar{x_i}, \bar{y_i})$ for $i < n$.

2. Restricted notions of weak dividing

I considered that we can divide witness formulas into some classes according to properties of stability theory. I told about characterizations of the next restricted weak dividing before.

Definition 8 Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *$\mathcal{M}$−weakly divides over* A if there are a formula $\varphi(\bar{x}) \in L_n(A)$ and a Morley sequence $I = \{a_i : i < n+1\}$ of $p\lceil A$ such that $\models \varphi(a_0, a_1, \cdots\cdots, a_{n-1})$, while the type $[p]^{\varphi}$ is inconsistent.

$p(x)$ *M−weakly divides over* A if there are a formula $\varphi(\bar{x}) \in L_n(A)$ and a Morley sequence $I = \{a_i : i < n+1\}$ of $p\lceil A$ such that $\models \varphi(a_0, a_1, \cdots\cdots, a_{n-1})$, while there is no Morley sequence $J = \{b_i : i < n+1\}$ of p over A such that $\models \varphi(b_0, b_1, \cdots\cdots, b_{n-1})$.

If we set the sequence I indiscernible over A in the definition above, we can define $\mathcal{I}$−weak dividing and I−weak dividing in the same way.

Another variant of dividing, "*thorn*"−dividing has been characterized in rosy theory of late years. (see e.g. [6]) I define weak notion of $\mathfrak{p}$−dividing (thorn-dividing). We recall some definitions.

Definition 9 Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *strongly divides over* A if there is a formula $\varphi(x, b) \in p(x)$ such that $b \notin acl(A)$ and $\{\varphi(x, b_i) : b_i \models \mathrm{tp}(b/A)\}$ is k−inconsistent for some $k < \omega$.

$p(x)$ *$\mathfrak{p}$−divides over* A if $p(x)$ strongly divides over Ac for some parameter c.

Weak notions of $\mathfrak{p}$−dividing could be defined in many ways. As $\mathfrak{p}$−dividing implies dividing, we expect that weak $\mathfrak{p}$−dividing implies weak dividing.

Definition 10 Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *weakly $\mathfrak{p}$−divides over* A if there is a formula $\varphi(\bar{x}) = \exists y \bigwedge_{i<n} \psi(x_i, y) \in L_n(A)$ such that $[p\lceil A]^{\varphi}$ is consistent, while $[p]^{\varphi}$ is inconsistent.

$p(x)$ *weakly $\mathfrak{p}$ − forks over* A if there is a complete type $q(x, y) \in S(A)$ such that $p(x) \cup q(x, y)$ is consistent, and any completion $r(x, y) \in S(B)$ of $p(x) \cup q(x, y)$ weakly $\mathfrak{p}$−divides over A.

Lastly I raise a question.

Problem

Characterize theories T in which there is no weak $\mathfrak{p}$− forking left-chain. Are such theories included in rosy theories properly?

References

[1] S.Shelah, Simple unstable theories, Annals of Pure and Applied Logic 19 (1980) 177-203

[2] A.Dolich, Weak dividing, chain conditions, and simplicity, Archive for

Mathematical Logic 43 (2004) 265-283
[3] B.Kim and N.Shi, A note on weak dividing, preprint
[4] A.Kolesnikov, n-simple theories, Annals of Pure and Applied Logic 131 (2005) 227-261
[5] B.Kim, A.Kolesnikov and A.Tsuboi, Generalized amalgamation and n-simplicity, preprint
[6] A.Onshuus, Properties and consequences of Thorn-independence, Journal of Symbolic Logic 71 (2006) 1-21
[7] H.Adler, Introduction to theories without the independence property, preprint
[8] E.Hrushovski and A.Pillay, On NIP and invariant measure, preprint
[9] F.O.Wagner, Simple theories, Kluwer Academic Publishers (2000)