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Maximum principle for fully nonlinear equations with linear and superlinear terms in Du

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Abstract. The maximum principle for L^p-viscosity solutions of fully nonlinear second order elliptic partial differential equations containing linear and superlinear growth in the first derivatives with unbounded coefficients is established.

1 Introduction

We are concerned with fully nonlinear second order elliptic partial differential equations (PDEs for short) in a bounded domain $\Omega \subset \mathbb{R}^n$:

$$F(x, u(x), Du(x), D^2u(x)) = f(x) \quad \text{in } \Omega, \tag{1.1}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ are given measurable functions. Here S^n denotes the set of $n \times n$ symmetric matrices with the standard ordering.

Since our PDEs have possibly discontinuous coefficients and inhomogeneous terms, we adapt the notion of L^p-viscosity solutions introduced in [3] (see also [1] and [2]).

Throughout this paper, for the sake of simplicity, we assume

$$\Omega \subset B_1 \quad (i.e.\ \mathrm{diam}(\Omega)/2 \leq 1).$$

It is easy to extend the results below to general bounded domains Ω by scaling and translation.

To obtain the maximum principle for L^p-viscosity solutions, as in [9] and [10] (see also [7]), it is essential to consider the associated extremal PDEs: for instance,

$$\mathcal{P}^{\pm}(D^2u) + H(x, Du) = f(x) \quad \text{in } \Omega. \tag{1.2}$$

Here, $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is given, and the Pucci operators $\mathcal{P}^{\pm} : S^n \to \mathbb{R}$ are defined by

$$\mathcal{P}^+(X) = \max\{-\mathrm{trace}(AX) \mid A \in S^n_{\lambda,\Lambda}\} \quad \text{and} \quad \mathcal{P}^-(X) = \min\{-\mathrm{trace}(AX) \mid A \in S^n_{\lambda,\Lambda}\},$$

where for fixed uniformly ellipticity constants $0 < \lambda \leq \Lambda$, $S^n_{\lambda,\Lambda} = \{X \in S^n \mid \lambda I \leq X \leq \Lambda I\}$.

When $H(x,\xi) = \mu(x)|\xi|^m$ with $\mu \in L^q(\Omega)$ for $m \geq 1$, it is already known that the maximum principle for L^p-viscosity solutions holds in [10] under appropriate hypotheses. More precisely, when $m = 1$, $q > n$ and $q \geq p > p_0$, where $p_0 \in [n/2, n)$ is the so-called Escauriaza's constant (see [6] and [5]), the maximum principle holds. On the other hand, when $m > 1$, the maximum principle fails in general (see [10]). However, according to [10], the maximum principle holds even when $m > 1$ if we suppose that $\|f\|_{L^p(\Omega)}$ or $\|\mu\|_{L^q(\Omega)}$ is small.

In this paper, we obtain the maximum principle for L^p-viscosity solutions of (1.2) when $H(x,\xi) = \mu_1(x)|\xi| + \mu_m(x)|\xi|^m$ for $\mu_1, \mu_m \in L^q(\Omega)$ with $q > n$ and $m > 1$ in the elliptic case. Particularly, when $p \in (p_0, n)$, it is not clear how the estimates depend on μ_1 and μ_m. We note that such estimates are important to study further regularity because we will need scaling arguments to establish the Harnack inequality for instance. Moreover, it is necessary to study PDEs with linear and superlinear growth in the first derivatives when we try to show that if $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ is an L^p-viscosity solutions of (1.1), then it is an L^p-strong solutions of (1.1) as in [11].

Here, we remark that if we directly follow the argument in [10] to extremal PDEs (1.2), then we have to suppose that $\|\mu_1\|_{L^q(\Omega)}$ or $\|f\|_{L^p(\Omega)}$ is small in addition to one of $\|\mu_m\|_{L^q(\Omega)}$ and $\|f\|_{L^p(\Omega)}$ is small. Moreover, the dependence on $\|\mu_1\|_{L^q(\Omega)}$, $\|\mu_m\|_{L^q(\Omega)}$ and $\|f\|_{L^p(\Omega)}$ in the estimates would become more complicated than ours in the proceeding sections.

In section 2, we recall the definitions of L^p-viscosity and L^p-strong solutions. Sections 3 is devoted to the study of elliptic PDEs. In Appendix, we show an existence result of L^p-strong solutions for $p \in (p_0, n)$, which was only announced in [10].

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2 Preliminaries

For measurable sets $U \subset \mathbb{R}^n$ and for $1 \leq p \leq \infty$, we denote by $L^p_+(U)$ the set of all nonnegative functions in $L^p(U)$. We will often write $\|\cdot\|_p$ $(1 \leq p \leq \infty)$ instead of $\|\cdot\|_{L^p(U)}$ if there is no confusion. We will use the standard notations from [8].

First of all, we recall the definition of L^p-viscosity solutions of (1.1).

Definition 2.1. We call $u \in C(\Omega)$ an L^p-viscosity subsolution (resp., supersolution) of (1.1) if

$$ess \liminf_{x \to x_0} \{F(x, u(x), D\phi(x), D^2\phi(x)) - f(x)\} \leq 0$$
$$\left(\text{resp.,} \quad ess \limsup_{x \to x_0} \{F(x, u(x), D\phi(x), D^2\phi(x)) - f(x)\} \geq 0\right)$$

whenever for $\phi \in W^{2,p}_{\mathrm{loc}}(\Omega)$, $x_0 \in \Omega$ is a local maximum (resp., minimum) point of $u - \phi$.

A function $u \in C(\Omega)$ is called an L^p-viscosity solution of (1.1) if it is both an L^p-viscosity subsolution and an L^p-viscosity supersolution of (1.1).

We will say u an L^p-subsolution (resp., -supersolution, solution) for an L^p-viscosity subsolution (resp., supersolution, solution) for simplicity. We will also say u an L^p-solution of

$$F(x, u, Du, D^2u) \leq f(x),$$
$$(\text{resp.}, F(x, u, Du, D^2u) \geq f(x)),$$

if it is an L^p-subsolution (resp., -supersolution) of (1.1).

We will use this abbreviation also for L^p-strong sub- and supersolutions below.

Definition 2.2. We call $u \in C(\Omega) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ an L^p-strong subsolution (resp., supersolution) of (1.1) if u satisfies

$$F(x, u(x), Du(x), D^2u(x)) \leq f(x) \quad \text{a.e. in } \Omega,$$
$$(\text{resp.,} \quad F(x, u(x), Du(x), D^2u(x)) \geq f(x) \quad \text{a.e. in } \Omega).$$

Remark 2.3. If u is an L^p-subsolution (resp., L^p-supersolution) of (1.1), then it is also an L^q-subsolution (resp., L^q-supersolution) of (1.1) provided $q \geq p$. However, on the contrary, if u is an L^p-strong subsolution (resp., supersolution) of (1.1), then it is also an L^q-strong subsolution (resp., supersolution) of (1.1) provided $p \geq q$.

3 Elliptic Equation

We always suppose that

$$p > \frac{n}{2}.$$

3.1 Known results for elliptic PDEs

When Ω satisfies the uniform exterior cone condition, it is known (*e.g.* [2]) that there exists $p_0 = p_0(n, \lambda, \Lambda)$ satisfying $\frac{n}{2} \leq p_0 < n$ such that for $p > p_0$, there is a constant $C = C(n, p, \lambda, \Lambda)$ such that if for $f \in L^p(\Omega)$, there is an L^p-strong subsolution $u \in C(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ of

$$\mathcal{P}^-(D^2u) \leq f(x) \quad \text{in } \Omega \tag{3.1}$$

such that $u = 0$ on $\partial\Omega$, and

$$-C\|f^-\|_p \leq u \leq C\|f^+\|_p \quad \text{in } \Omega.$$

Moreover, for each $\Omega' \Subset \Omega$, there is $C' = C'(n, p, \lambda, \Lambda, \mathrm{dist}(\Omega', \partial\Omega)) > 0$ such that

$$\|u\|_{W^{2,p}(\Omega')} \leq C'\|f\|_p.$$

The key tool for it is the following strong solvability of extremal equations while the existence of L^p-strong subsolution of (3.1) was used in [10]. In fact, if we use the strong solvability of (3.1) instead of the following proposition, then we have to suppose that $\|\mu_1\|_q$ is small provided $\|f\|_p$ is not small as mentioned in Introduction.

Since it is easy to obtain the corresponding result for L^p-supersolutions, we only state the result for L^p-subsolutions.

Proposition 3.1 (Proposition 2.6 in [10]). *Let Ω satisfy the uniform exterior cone condition. For*

$$q \geq p > n \quad \text{or} \quad q > p = n,$$

let $f \in L^p_+(\Omega)$ and $\mu_1 \in L^q_+(\Omega)$ satisfy $\mathrm{supp}\,\mu_1 \Subset \Omega$. Then, there exists an L^p-strong subsolution $u \in C(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ of

$$\mathcal{P}^-(D^2u) - \mu_1(x)|Du| \geq f(x) \quad \text{in } \Omega$$

such that $u = 0$ on $\partial\Omega$,

$$-C\exp(\hat{C}\,\|\mu_1\|_n^n)\|f^-\|_n \leq u \leq C\exp(\hat{C}\,\|\mu_1\|_n^n)\|f^+\|_n \quad \text{in } \Omega$$

where $C = C(n, p, \lambda, \Lambda)$ and $\hat{C} = \hat{C}(n, \lambda, \Lambda)$ are positive constants, and

$$\|u\|_{W^{2,p}(\Omega')} \leq C'\exp(\hat{C}\,\|\mu_1\|_n^n)\,\|f\|_{L^p(\Omega)},$$

where for each $\Omega' \Subset \Omega$, $C' = C'(n, p, \lambda, \Lambda, \|\mu_1\|_q, \mathrm{dist}(\Omega', \partial\Omega)) > 0$.

We shall use the following notation since it appears often.

$$\hat{D} = \exp(\hat{C}\|\mu_1\|_n^n).$$

In order to consider the case of $p \in (p_0, n)$, we will use the following maximum principle via the iterated comparison function method in [9] and [10].

Lemma 3.2 (Theorem 2.9 in [10])). *Let $p_0 < p < n < q$. There exist an integer $N = N(n,p,q)$ and $C = C(n,p,q,\lambda,\Lambda) > 0$ such that if $f \in L^p_+(\Omega)$, $\mu_1 \in L^q_+(\Omega)$ and $u \in C(\overline{\Omega})$ is an L^p-solution of*

$$\mathcal{P}^-(D^2u) - \mu_1(x)|Du| \le f(x) \quad in\ \Omega,$$

then

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u + C\left\{\hat{D}\,\|\mu_1\|_q^N + \sum_{k=0}^{N-1}\|\mu_1\|_q^k\right\}\|f\|_p.$$

The strong solvability result in case when $p_0 < p < n < q$ is as follows.

Proposition 3.3. *Let Ω satisfy the uniform exterior cone condition. For*

$$p_0 < p < n < q,$$

let $f \in L^p_+(\Omega)$ and $\mu_1 \in L^q_+(\Omega)$ satisfy $supp\,\mu_1 \Subset \Omega$. Then, there exist an L^p-strong subsolution $u \in C(\overline{\Omega}) \cap W^{2,p}_{\rm loc}(\Omega)$ of

$$\mathcal{P}^-(D^2u) - \mu_1(x)|Du| \ge f(x) \quad in\ \Omega$$

such that $u = 0$ on $\partial\Omega$, and

$$-C\left\{\hat{D}\,\|\mu_1\|_q^N + \sum_{k=0}^{N-1}\|\mu_1\|_q^k\right\}\|f^-\|_p \le u \le C\left\{\hat{D}\,\|\mu_1\|_q^N + \sum_{k=0}^{N-1}\|\mu_1\|_q^k\right\}\|f^+\|_p,$$

for some integer $N = N(n,p,q)$ and $C = C(n,p,\lambda,\Lambda) > 0$. Moreover, for each $\Omega' \Subset \Omega$, there is $C' = C'(n,p,\lambda,\Lambda,\|\mu_1\|_q, dist(\Omega',\partial\Omega)) > 0$ such that

$$\|u\|_{W^{2,p}(\Omega')} \le C'\left\{\hat{D}\,\|\mu_1\|_q^N + \sum_{k=0}^{N-1}\|\mu_1\|_q^k\right\}\|f\|_{L^p(\Omega)}.$$

For the reader's convenience, we will give a proof in Appendix.

3.2 Main results for elliptic PDEs

In this subsection, for a fixed $m > 1$, we consider the following PDE:

$$\mathcal{P}^-(D^2u) - \mu_1(x)|Du| - \mu_m(x)|Du|^m = f(x) \quad \text{in } \Omega. \tag{3.2}$$

In what follows, we shall utilize the same notation of a function $g : U \subset \mathbb{R}^m \to \mathbb{R}$ for its zero-extension outside its domain.

We start with an easy case; $p = q > n$.

Theorem 3.4. *Let $p > n$ and $m > 1$. There exist $\delta = \delta(n, m, p, \lambda, \Lambda) > 0$, and $C = C(n, m, p, \lambda, \Lambda, \|\mu_1\|_q) > 0$ such that if $f \in L^p_+(\Omega)$, $\mu_1 \in L^p_+(\Omega)$, $\mu_m \in L^p_+(\Omega)$,*

$$\hat{D}^m \|f\|_p^{m-1} \|\mu_m\|_p < \delta, \tag{3.3}$$

and $u \in C(\overline{\Omega})$ is an L^p-subsolution of (3.2), *then*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u + C\hat{D}\left(\|f\|_n + \hat{D}^m \|f\|_p^m \|\mu_m\|_n\right),$$

PROOF. In view of Proposition 3.1, we can find an L^p-strong subsolution $v \in C(\overline{B}_3) \cap W^{2,p}_{\mathrm{loc}}(B_3)$ of

$$\mathcal{P}^+(D^2 v) + \mu_1(x)|Dv| \leq -f(x) \quad \text{in } B_3$$

with boundary condition $v = 0$ on ∂B_3, and

$$0 \leq -v \leq C_1 \hat{D} \|f\|_n \quad \text{in } B_3. \tag{3.4}$$

The Sobolev imbedding theorem yields

$$\|Dv\|_{L^\infty(B_2)} \leq \|v\|_{W^{2,p}(B_2)} \leq C_2 \hat{D} \|f\|_p.$$

By setting $w = u + v$ in Ω, it is easy to see that w is an L^p-solution of

$$\mathcal{P}^-(D^2 w) - \mu_1(x)|Dw| - 2^{m-1}\mu_m(x)|Dw|^m \leq 2^{m-1} \|Dv\|^m_{L^\infty(B_{R_1})} \mu_m(x) \quad \text{in } \Omega.$$

Notice that since we used Proposition 3.3, we do not get μ_1 in the right hand side of the above.

In the rest of proof, we follow the argument in [10] though the calculations below are more complicated than those in [10].

For any $\varepsilon > 0$, we find the L^p-strong solution $\zeta_\varepsilon \in C(\overline{B}_2) \cap W^{2,p}_{\mathrm{loc}}(B_2)$ of

$$\mathcal{P}^+(D^2\zeta_\varepsilon) + \mu_1(x)|D\zeta_\varepsilon| \leq -(2^{m-1}C_2^m + 1)\hat{D}^m \|f\|_p^m \mu_m(x) - \varepsilon \leq 0 \quad \text{in } B_2$$

under $\zeta_\varepsilon = 0$ on ∂B_2 such that

$$0 \leq -\zeta_\varepsilon \leq C_3 \hat{D}\left(\hat{D}^m \|f\|_p^m \|\mu_m\|_n + \varepsilon\right) \quad \text{in } B_2. \tag{3.5}$$

Moreover,

$$\|D\zeta_\varepsilon\|_{L^\infty(\Omega)} \leq C_4 \hat{D}\left(\hat{D}^m \|f\|_p^m \|\mu_m\|_p + \varepsilon\right). \tag{3.6}$$

Thus, setting $W_\varepsilon := w + \zeta_\varepsilon$, by (3.4) we verify that W_ε is an L^p-solution of

$$\begin{aligned}\mathcal{P}^-(D^2W_\varepsilon) - \mu_1(x)|DW_\varepsilon| &- 2^{2(m-1)}\mu_m(x)|DW_\varepsilon|^m \\ &\leq \mu_m(x)(2^{2(m-1)}|D\zeta_\varepsilon|^m - \hat{D}^m \|f\|_p^m) - \varepsilon \quad \text{in } \Omega.\end{aligned}$$

Using (3.6), we can find $C_5 > 0$ such that the right hand side of the above is estimated from above by

$$\mu_m(x)\hat{D}^m\left\{C_5\left(\hat{D}^m\|f\|_p^m\|\mu_m\|_p+\varepsilon\right)^m-\|f\|_p^m\right\}-\varepsilon.$$

Hence, taking $\delta = 1/C_5^{1/m} > 0$, we see that if (3.3) holds, then for small $\varepsilon > 0$, W_ε is an L^p-solution of

$$\mathcal{P}^-(D^2W_\varepsilon)-\mu_1(x)|DW_\varepsilon|-2^{2(m-1)}\mu_m(x)|DW_\varepsilon|^m+\varepsilon\le 0 \quad \text{in } \Omega.$$

Therefore, by the definition of L^p-viscosity solutions, we have $W_\varepsilon \le \sup_{\partial\Omega} W_\varepsilon$ in Ω. Hence, by (3.4) and (3.5), we obtain that

$$\begin{aligned}\sup_\Omega u &\le \sup_{\partial\Omega} W_\varepsilon+\sup_\Omega(-v)+\sup_\Omega(-\zeta_\varepsilon)\\ &\le \sup_{\partial\Omega} u+C_6\hat{D}\left(\|f\|_n+\hat{D}^m\|f\|_p^m\|\mu_m\|_n\right)+C_3\hat{D}\varepsilon.\end{aligned}$$

Thus, the conclusion follows by letting $\varepsilon \downarrow 0$. □

Finally, we extend Theorem 3.4 to the case when $p \in (p_0, n]$.

Theorem 3.5. *Let $p_0 < p \le n < q$ and $m > 1$. There exist an integer $N = N(n,m,p,q) \ge 1$, $\delta = \delta(n,m,p,q,\lambda,\Lambda) > 0$ and $C = C(n,m,p,q,\lambda,\Lambda,\|\mu_1\|_q) > 0$ such that if $f \in L^p_+(\Omega)$, $\mu_1 \in L^p_+(\Omega)$ and $\mu_m \in L^p_+(\Omega)$,*

$$p > \frac{nq(m-1)}{mq-n}, \tag{3.7}$$

$$\hat{D}^m\hat{E}_N^m\|f\|_p^{m^N(m-1)}\|\mu_m\|_q^{m^N}<\delta,$$

and $u \in C(\overline{\Omega})$ is an L^p-subsolution of (3.2), then

$$\begin{aligned}\sup_\Omega u \le{}& \sup_{\partial\Omega} u + C\sum_{k=1}^N \hat{E}_k\|\mu_m\|_q^{\frac{m^{k-1}-1}{m-1}}\|f\|_p^{m^{k-1}}\\ &+C\hat{D}\hat{E}_N^m\|f\|_p^{m^N}\|\mu_m\|_q^{\frac{m^N-1}{m-1}}\left\{1+\hat{D}^m\hat{E}_N^{m^N(m-1)}\|\mu_m\|_n\|\mu_m\|_q^{m^N-1}\|f\|_p^{m^N(m-1)}\right\},\end{aligned}$$

where A_j and $\hat{E}_k$ are given by

$$A_j := \hat{D}\|\mu_1\|_q^{N[j]+1}+\sum_{l=0}^{N[j]}\|\mu_1\|_q^l \quad \text{and} \quad \hat{E}_k := \prod_{j=1}^k A_j^{m^{k-j}}.$$

and $N[j]$ $(j = 1, \dots, N)$ satisfying $N[i] \le N[j] \le N$ $(i \le j)$ are constants from Proposition 3.3.

PROOF. In this case, the key of our proof is to use Proposition 3.3.
We define $q_0 = p$, and

$$q_k = \frac{nq_{k-1}q}{n(q_{k-1}+mq) - mq_{k-1}q} \quad \text{for } k \geq 1.$$

Due to (3.7), following the argument in [10], we may choose an integer $N \geq 1$ such that $q_{N-1} \leq n < q_N$. If $q_{N-1} = n$, then we may choose $q_N = q'$ for any $q' \in (n, q)$.

Fix $\frac{diam(\Omega)}{2} < 1 < R_N < \cdots < R_1$. In view of Proposition 3.3, we first find an L^p-strong solution $v_1 \in C(\overline{B}_{R_1}) \cap W^{2,p}_{\text{loc}}(B_{R_1})$ of

$$\mathcal{P}^+(D^2 v_1) + \mu_1(x)|Dv_1| \leq -f(x) \quad \text{in } B_{R_1}$$

with boundary condition $v_1 = 0$ on ∂B_{R_1}, and

$$0 \leq -v_1 \leq CA_1 \|f\|_p \quad \text{in } B_{R_1},$$

and

$$\|Dv_1\|_{L^{p^*}(B_{R_2})} \leq \|v_1\|_{W^{2,p}(B_{R_2})} \leq CA_1 \|f\|_p . \tag{3.8}$$

Setting $w_1 := u + v_1$, we obtain that w_1 is an L^p-solution of

$$\mathcal{P}^-(D^2 w_1) - \mu_1(x)|Dw_1| - 2^{m-1}\mu_m(x)|Dw_1|^m \leq 2^{m-1}\mu_m(x)|Dv_1|^m =: f_2(x) \quad \text{in } \Omega.$$

Moreover, by Hölder's inequality, (3.8) implies

$$\|f_2\|_{L^{q_1}(B_{R_2})} \leq \|\mu_m\|_q \|Dv_1\|^m_{L^{p^*}(B_{R_2})} \leq CA_1^m \|\mu_m\|_q \|f\|_p^m .$$

Next, again in view of Proposition 3.3, we find an L^p-strong solution $v_2 \in C(\overline{B}_{R_2}) \cap W^{2,q_1}_{\text{loc}}(B_{R_2})$ of

$$\mathcal{P}^+(D^2 v_2) + \mu_1(x)|Dv_2| \leq -f_2(x) \quad \text{in } B_{R_2}$$

with $v_2 = 0$ on ∂B_{R_2}. Again

$$0 \leq -v_2 \leq CA_2 \|f_2\|_{L^{q_1}} \quad \text{in } B_{R_2},$$

and

$$\|Dv_2\|_{L^{q_1^*}(B_{R_3})} \leq CA_1^m A_2 \|\mu_m\|_q \|f\|_p^m . \tag{3.9}$$

Hence, $w_2 := w_1 + v_2$ is an L^p-solution of

$$\mathcal{P}^-(D^2 w_2) - \mu_1(x)|Dw_2| - 2^{2(m-1)}\mu_m(x)|Dw_2|^m \leq 2^{2(m-1)}\mu_m(x)|Dv_2|^m =: f_3(x) \quad \text{in } \Omega,$$

and (3.9) implies,

$$\|f_3\|_{L^{q_2}(B_{R_3})} \leq \|\mu_m\|_q \|Dv_2\|^m_{L^{q_1^*}(B_{R_3})} \leq CA_1^{m^2} A_2^m \|\mu_m\|_q^{1+m} \|f\|_p^{m^2} .$$

Inductively, setting $f_k := 2^{(k-1)(m-1)}\mu_m(x)|Dv_{k-1}|^m \in L^{q_{k-1}}(B_{R_k})$, we find the L^p-strong solutions $v_k \in C(\overline{B}_{R_k}) \cap W^{2,q_{k-1}}_{\mathrm{loc}}(B_{R_k})$ of

$$\mathcal{P}^+(D^2v_k) + \mu_1(x)|Dv_k| \le -f_k(x) \quad \text{in } B_{R_k}$$

satisfying $v_k = 0$ on ∂B_{R_k}. Similarly,

$$0 \le -v_k \le CA_k \|f_k\|_{L^{q_{k-1}}(B_{R_k})} \quad \text{in } B_{R_k},$$

and

$$\|f_k\|_{L^{q_{k-1}}(B_{R_k})} \le C \prod_{j=1}^{k-1} A_j^{m^{k-j}} \|\mu_m\|_q^{\frac{m^{k-1}-1}{m-1}} \|f\|_p^{m^{k-1}},$$

$$\|Dv_k\|_{L^{q^*_{k-1}}(B_{R_{k+1}})} \le C \prod_{j=1}^{k} A_j^{m^{k-j}} \|\mu_m\|_q^{\frac{m^{k-1}-1}{m-1}} \|f\|_p^{m^{k-1}}.$$

Therefore, we obtain that $w_N := u + \sum_{k=1}^N v_k$ is an L^p-solution of

$$\mathcal{P}^-(D^2w_N) - \mu_1(x)|Dw_N| - 2^{N(m-1)}\mu_m(x)|Dw_N|^m \le 2^{N(m-1)}\mu_m(x)|Dv_N|^m =: \hat{f}(x) \quad \text{in } \Omega,$$

where $\hat{f} \in L^{p_N}(\Omega)$. Hence, in view of Theorem 3.4, if $\hat{D}\,\|\mu_m\|_q \|\hat{f}\|_{L^{q_N}}^{m-1}$ is small enough, then we get

$$\sup_\Omega w_N \le \sup_{\partial\Omega} w_N + C\hat{D}\left(\|\hat{f}\|_{L^{q_N}} + \hat{D}^m \|\hat{f}\|^m_{L^{q_N}} \|\mu_m\|_n\right).$$

Since $\|\hat{f}\|_{q_N} \le C\hat{E}_N^m \|\mu_m\|_q^{\frac{m^N-1}{m-1}} \|f\|_p^{m^N}$, the results follows. $\square$

Appendix

In this appendix, we give a proof of Proposition 3.3, for the reader's convenience because it was only mentioned in [10]. The proof below is a modification of that in [8].

PROOF. We shall simply write μ for μ_1. Let $\mu_j \in C^\infty(\Omega)$ be such that $\mu_j \to \mu$ in $L^q(\Omega)$ and pointwise a.e. Let $u_j \in C(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ be the unique L^p-strong solution of

$$\mathcal{P}^-(D^2u_j) - \mu_j(x)|Du_j| = f(x) \quad \text{in } \Omega \tag{3.10}$$

with $u = \psi$ on $\partial\Omega$. By Lemma 3.2, (3.2) holds for u_j with μ replaced by μ_j. Since $\mu_j \to \mu$ in $L^q(\Omega)$, we may assume that it holds with μ.

Since we can cover Ω' by a finite number of balls having a fixed radius R, it is enough to show (3.2) for the u_j for B_R instead of Ω'. We will denote the measure of B_R by $|B_R| = \omega_n R^n$, where ω_n is the measure of unit ball B_1. Let $\rho \in (0,1)$ and cut off function $\eta \in C_0^2(B_R)$ be such that $0 \le \eta \le 1$, $\eta = 1$ in $B_{\rho R}$ and $\eta = 0$ in $B_R \backslash B_{\tilde\rho R}$ where $\tilde\rho = (1+\rho)/2$, and

$$|D\eta| \le \frac{4}{(1-\rho)R}, \quad \left\|D^2\eta\right\| \le \frac{16}{(1-\rho)^2R^2}.$$

Setting $v = \eta u_j \in W^{2,p}(B_R)$, and therefore using the estimates of [5], we have

$$\|v\|_{W^{2,p}(B_{\tilde\rho R})} \le C_1 \left\|\mathcal{P}^-(D^2v)\right\|_{L^p(B_{\tilde\rho R})},$$

which implies

$$\|Dv\|_{L^{p^*}(B_{\tilde\rho R})} \le C_2 \|v\|_{W^{2,p}(B_{\tilde\rho R})} \le C_1C_2 \left\|\mathcal{P}^-(D^2v)\right\|_{L^p(B_{\tilde\rho R})}.$$

Then we have

$$\begin{aligned}
&\left\|D^2u_j\right\|_{L^p(B_{\rho R})} \le \left\|D^2v\right\|_{L^p(B_{\tilde\rho R})} \le C_1C_2 \left\|\mathcal{P}^-(D^2v)\right\|_{L^p(B_{\tilde\rho R})}\\
&= C_1C_2 \left\|\mathcal{P}^-(\eta D^2u_j) + 2D\eta \otimes Du_j + u_jD^2\eta\right\|_{L^p(B_{\tilde\rho R})} \qquad (3.11)\\
&\le C_3 \left(\left\|\eta\mathcal{P}^-(D^2u_j)\right\|_{L^p(B_{\tilde\rho R})} + \frac{1}{(1-\rho)R}\|Du_j\|_{L^p(B_{\tilde\rho R})} + \frac{1}{(1-\rho)^2R^2}\|u_j\|_{L^p(B_{\tilde\rho R})}\right).
\end{aligned}$$

By (3.10), it follows that

$$\begin{aligned}
&C_3 \left\|\eta\mathcal{P}^-(D^2u_j)\right\|_{L^p(B_{\tilde\rho R})} \le C_4\|f\|_{L^p(B_{\tilde\rho R})} + C_4\|\eta\mu_jDu_j\|_{L^p(B_{\tilde\rho R})}\\
&\le C_4\|f\|_{L^p(B_{\tilde\rho R})} + C_4\|\mu_jDv\|_{L^p(B_{\tilde\rho R})} + C_4\|\mu_j\|_{L^p(B_{\tilde\rho R})}\frac{\|u_j\|_{L^\infty(\Omega)}}{(1-\rho)R} \qquad (3.12)\\
&\le C_4\|f\|_{L^p(B_{\tilde\rho R})} + C_4\|\mu_jDv\|_{L^p(B_{\tilde\rho R})} + C_5\|\mu_j\|_{L^p(B_{\tilde\rho R})}\frac{\|\psi\|_{L^\infty(\partial\Omega)} + A_1\|f\|_{L^p(\Omega)}}{(1-\rho)R},
\end{aligned}$$

where A_1 is a constant from Theorem 3.5. We now estimate, for $n < q' < q$,

$$\begin{aligned}
C_4\|\mu_jDv\|_{L^p(B_{\tilde\rho R})} &\le C_4\|\mu_j\|_{L^{q'}(B_{\tilde\rho R})}\|Dv\|_{L^{\frac{pq'}{q'-p}}(B_{\tilde\rho R})}\\
&\le C_4C_6(\omega_nR^n)^{\frac{n(q-q')}{qq'}}\|\mu_j\|_{L^q(B_{\tilde\rho R})}\left\|\mathcal{P}^-(D^2v)\right\|_{L^p(B_{\tilde\rho R})}.
\end{aligned}$$

Hence by choosing R small enough, we can show that

$$C_4\|\mu_jDv\|_{L^p(B_{\tilde\rho R})} \le \frac{C_1}{2}\left\|\mathcal{P}^-(D^2v)\right\|_{L^p(B_{\tilde\rho R})}. \qquad (3.13)$$

Combining (3.11), (3.12) and (3.13), we have

$$\begin{aligned}
&\frac{C_1}{2}\left\|\mathcal{P}^-(D^2v)\right\|_{L^p(B_{\tilde\rho R})} \le C_4\|f\|_{L^p(\Omega)}\\
&+\frac{C_7}{(1-\rho)^2R^2}\left((1-\rho)R\left(\|\psi\|_{L^\infty(\partial\Omega)} + A_1\|f\|_{L^p(\Omega)} + \|Du_j\|_{L^p(B_{\tilde\rho R})}\right) + \|u_j\|_{L^p(B_{\tilde\rho R})}\right).
\end{aligned}$$

According to (3.11) again, we have

$$(1-\rho)^2R^2\left\|D^2u_j\right\|_{L^p(B_{\rho R})}\le C_8\left(\|\psi\|_{L^\infty(\partial\Omega)}+A_1\|f\|_{L^p(\Omega)}\right)+C_8\left((1-\tilde\rho)R\|Du_j\|_{L^p(B_{\tilde\rho R})}+\|u_j\|_{L^p(\tilde\rho R)}\right). \tag{3.14}$$

If we introduce norms

$$\Psi_k(v):=\sup_{0<\rho<1}(1-\rho)^kR^k\left\|D^kv\right\|_{L^p(B_{\rho R})},\quad k=0,1,2,$$

then (3.14) gives the inequality

$$\Psi_2(u_j)\le C_8\left(\|\psi\|_{L^\infty(\partial\Omega)}+A_1\|f\|_{L^p(\Omega)}\right)+C_8(\Psi_1(u_j)+\Psi_0(u_j)). \tag{3.15}$$

The $W^{2,p}_{\rm loc}$ estimate follows from the interpolation inequality,

$$\Psi_1\le\varepsilon\Psi_2+\frac{C}{\varepsilon}\Psi_0 \tag{3.16}$$

for any $\varepsilon>0$ where $C=C(n)$, which may found in [8]. Indeed, using (3.16) in (3.15), we get

$$\Psi_2\le C_9\left(\|\psi\|_{L^\infty(\partial\Omega)}+A_1\|f\|_{L^p(\Omega)}+\|u_j\|_{L^p(\Omega)}\right),$$

that is,

$$\left\|D^2u_j\right\|_{L^p(B_{\rho R})}\le\frac{C_9}{(1-\rho)^2R^2}\left(\|\psi\|_{L^\infty(\partial\Omega)}+\|f\|_{L^p(\Omega)}+\|u_j\|_{L^p(\Omega)}\right).$$

The desired estimate (3.2) follows by taking $\rho=1/2$.

Therefore, there exists $u\in W^{2,p}_{\rm loc}(\Omega)$ such that $u_j\rightharpoonup u$ in $W^{2,p}_{\rm loc}(\Omega)$ as $j\to\infty$. Taking a subsequence if necessary, we see that $Du_j\to Du$ a.e.. Thus this implies that $\mu_j|Du_j|\to\mu|Du|$. Since $\mathcal{P}^-$ is concave, we have for a.e. x,

$$\begin{aligned}\mathcal{P}^-(D^2u)&\le\limsup_{j\to\infty}\mathcal{P}^-(D^2u_j)\\&=\limsup_{j\to\infty}\left(\mathcal{P}^-(D^2u_j)-\mu_j(x)|Du_j|+\mu_j(x)|Du_j|\right)\\&=f(x)+\lim_{j\to\infty}\mu_j(x)|Du_j|.\end{aligned}$$

It remains to show that $u\in C(\overline{\Omega})$. By the superadditivity of $\mathcal{P}^-$, we have

$$\mathcal{P}^-(D^2(u_i-u_j))\le\mu_i(x)|Du_i|-\mu_j(x)|Du_j|+f_i(x)-f_j(x)\quad\text{in }\Omega,$$

with $u_i-u_j=0$ on $\partial\Omega$ for $i,j\ge1$. Since ${\rm supp}\mu\Subset\Omega$, we may assume ${\rm supp}\mu_i\subset\Omega'$ and for all $i\ge1$. It is enough to show that

$$\|\mu_i(x)|Du_i|-\mu_j(x)|Du_j|\|_{L^p(\Omega)}\to0\quad\text{as }i,j\to\infty,$$

since the maximum principle will give us that $\sup(u_i - u_j) \to 0$. Indeed, we have

$$\begin{aligned}
&\|\mu_i(x)|Du_i| - \mu_j(x)|Du_j|\|_{L^p(\Omega)} \\
&\leq \|(\mu_i - \mu_j)|Du_i|\|_{L^p(\Omega')} + \|\mu_j|Du_i - Du_j|\|_{L^p(\Omega')} \\
&\leq \|\mu_i - \mu_j\|_{L^n(\Omega')} \|Du_i\|_{L^{p^*}(\Omega')} + \|\mu_j\|_{L^n(\Omega')} \|Du_i - Du_j\|_{L^{p^*}(\Omega')} \\
&\leq C \left(\|\mu_i - \mu_j\|_{L^n(\Omega')} + \|Du_i - Du_j\|_{L^{p^*}(\Omega')} \right) \to 0 ,
\end{aligned}$$

as $i, j \to \infty$. This completes the proof. □

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