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Representation formula of viscosity solutions for parabolic equations via a deterministic two-person game

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1 Introduction

Many PDEs are characterized by deterministic games via the associated value functions. Kohn and Serfaty [12, 14] considered the following 1D heat equation with respect to the backward time,

$$\begin{cases} v_t + v_{xx} = 0 & t < T, \\ v = \psi & t = T. \end{cases} \tag{1.1}$$

Here T is a constant and ψ is a given function. For this equation, they defined the following parametrized value function v^ε which denotes the payoff from one player to the other in the game.

$$\begin{cases} v^\varepsilon(x,t) = \max_{r_1 \in \mathbf{R}} \min_{r_2 = \pm 1} \left\{ v^\varepsilon(x + \sqrt{2}\varepsilon r_2, t + \varepsilon^2) - \sqrt{2}\varepsilon r_1 r_2 \right\}, & \text{if } t < T, \\ v^\varepsilon(x,T) = \psi(x) & \text{if } t = T. \end{cases} \tag{1.2}$$

Here r_1 and r_2 are player's choices and $\varepsilon > 0$ is a small parameter. Commonly, T and ψ are called *maturity time* and *objective function*, respectively. Now let us regard v^ε as the smooth function. Applying Taylor expansion for $v^\varepsilon(x + \sqrt{2}\varepsilon r_2, t + \varepsilon^2)$, we have

$$0 \approx \max_{r_1 \in \mathbf{R}} \min_{r_2 = \pm 1} \left\{ \sqrt{2} r_2 \varepsilon^{-1} (v^\varepsilon_x - r_1) + v^\varepsilon_t + v^\varepsilon_{xx} \right\}.$$

If the player chooses $r_1 = v^\varepsilon_x$, then the above heat equation arises. Thus the limit function $\lim_{\varepsilon \to 0} v^\varepsilon$ is expected to be the solution of (1.1).
For more general equations, Kohn and Serfaty introduced the following value function,

$$\begin{cases} u^\varepsilon(x,t) = \max_{p,X} \min_{w} \left\{ u^\varepsilon(x + \varepsilon w, t + \varepsilon^2) + R^\varepsilon(w,p,X) \right\} & \text{if } t < T, \\ u^\varepsilon(x,T) = \psi(x) & \text{if } t = T. \end{cases} \tag{1.3}$$

The term R^ε is called a running cost and is defined as

$$R^\varepsilon(w,p,X) := -\varepsilon p \cdot w - \frac{\varepsilon^2}{2} \langle Xw, w \rangle + \varepsilon^2 f(p,X).$$

The limit function $\lim_{\varepsilon \to 0} u^\varepsilon$ is expected to converge to a solution of the following equation (see [14]),

$$u_t + f(Du, D^2 u) = 0.$$

Now we will generalize their results to a wider class of PDEs. by introducing the concept of "interest rate" to the value function.

$$u^\varepsilon(x,t) = \Big(\frac{1}{1+\mu\varepsilon^2}\Big) \inf_{p,X} \sup_{w} \Big\{u^\varepsilon(x+\varepsilon w, t+\varepsilon^2) + Q^\varepsilon(w,p,X)\Big\}. \tag{1.4}$$

Here $\mu \geq 0$ is constant, $Q^\varepsilon = R^\varepsilon + \varepsilon^2 H(p)$ and H is uniformly Lipschitz continuous or bounded and uniformly continuous function in $\mathbb{R}^N$. If $\mu = 0$, then the game is replaced by "no rate" problem which corresponds to the case in [14]. Our result shows that the viscosity solution u ([9]) of

$$\begin{cases} \partial_t u - \mu u + F(Du, D^2u) + H(Du) = 0 & \text{in } \mathbb{R}^N \times (-\infty, T), \\ u = \psi & \text{in } \mathbb{R}^N \times \{t = T\} \end{cases} \tag{1.5}$$

is represented by the limit of value function (1.4) as $\varepsilon \to 0$ (i.e., $u = \lim_{\varepsilon\to 0} u^\varepsilon$). In addition, the convergence is uniform. Here $\psi : \mathbb{R}^N \to \mathbb{R}$ is a function belonging to $BUC(\mathbb{R}^N)$ which denotes the set of all bounded and uniform continuous functions in $\mathbb{R}^N$. Note that we impose some appropriate conditions on F. These conditions allow discontinuities for F so that the level set equation of the mean curvature flow is included as an application. In this regard we mention [10, 11] for the related works.

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2 Strategies and Goals of Players

We first describe the setting of the game. There are two players, **A** and **B**. Let x_0 be an initial position of **A** in $\mathbb{R}^N$ ($N \geq 2$) at the starting time T_0, and T ($T_0 < T$) be the final maturity time of the game. In what follows, $\varepsilon \in (0,1)$ is a small parameter denoted by

$$\varepsilon := \sqrt{\frac{T - T_0}{m}}$$

for some integer $m \in \mathbb{N}$ and the function $\psi : \mathbb{R}^N \to \mathbb{R}$ is bounded and uniformly continuous (denoted by $BUC(\mathbb{R}^N)$). The player's choices are followings at the position x_0.

(1) **A** chooses a pair $(p_0, X_0) \in \mathbb{R}^N_* \times \mathcal{S}^N$ with $0 < |p_0| \leq \varepsilon^{-1/4}$ and $|X_0| \leq \varepsilon^{-1/2}$ where $\mathbb{R}^N_* = \mathbb{R}^N \setminus \{0\}$ and $|Z| := \max_{|v|=1} |\langle Zv, v\rangle|$ for $Z \in \mathcal{S}^N$.

(2) For this choice of **A**, **B** chooses a direction $w_0 \in \mathbb{R}^N$ with $|w_0| \leq \varepsilon^{-1/4}$.

(3) **A** moves from x_0 to $x_1 := x_0 + \varepsilon w_0$.

(4) Above steps are repeated m times, until the elapsed time reaches T.

(5) At the maturity time T, for the **A**'s final position $x^\varepsilon(T)$, **A** pays **B** the amount

$$\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^m \psi(x^\varepsilon(T)) + \sum_{i=0}^{m-1} \Big(\frac{1}{1+\mu\varepsilon^2}\Big)^{i+1} Q^\varepsilon(w_i, p_i, X_i) \qquad (\mu \geq 0\ ;\ \text{constant})$$

where p_i, X_i and w_i are respectively choices of **A** and **B** at the position in i-th step.

A and **B** have the opposing goals of minimizing and maximizing the above amount of payoff, respectively. **A**'s optimized payoff is represented by

$$u^\varepsilon(x, T_0) := \inf \sup \Big\{\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^m \psi(x^\varepsilon(T)) + \sum_{i=0}^{m-1} \Big(\frac{1}{1+\mu\varepsilon^2}\Big)^{i+1} Q^\varepsilon(w_i, p_i, X_i)\Big\}, \tag{2.1}$$

where the infimum and supremum are taken over all choices that can be executed until m-th step when starting at x at the time T_0. Players have to take their choices w_i, p_i, X_i at each step so that their purposes are accomplished. We are interested in the limit of $u^\varepsilon(x, T_0)$ as $\varepsilon \to 0$ (i.e., as the total steps $m \to \infty$). Using the dynamic programming, we will begin by considering the characterization of u^ε.

Definition 2.1. Let $\mathcal{J}_\varepsilon$ be the operator denoted by

$$\mathcal{J}_\varepsilon\phi(\cdot) := \Big(\frac{1}{1+\mu\varepsilon^2}\Big) \inf_{p,X} \sup_{w} \Big\{\phi(\cdot + \varepsilon w) + Q^\varepsilon(w,p,X)\Big\} \tag{2.2}$$

for $\phi \in L^\infty(\mathbb{R}^N)$. Here the infimum- supremum are respectively taken over all **A**'s- **B**'s strategies. Then u^ε is defined by

$$\begin{cases} \mathcal{J}_\varepsilon^k\psi(x) = u^\varepsilon(x, T - k\varepsilon^2) & \text{if } 1 \le k \le m, \\ \mathcal{J}_\varepsilon^0 = \mathcal{I} & \text{if } k = 0 \end{cases} \tag{2.3}$$

for $x \in \mathbb{R}^N$ and $\psi \in BUC(\mathbb{R}^N)$ where $\mathcal{J}_\varepsilon^k = \mathcal{J}_\varepsilon \cdots \mathcal{J}_\varepsilon$ and $\mathcal{I}$ is the identity map (cf, [10]).

We mention on the boundedness of u^ε and some properties of $\mathcal{J}_\varepsilon$ in Section 5. Such function u^ε is called the value function of the game with the objective function ψ. Although u^ε is only defined at the discrete time $t = T - k\varepsilon^2$ $(k = 0, 1, \dots m)$, one can consider a natural extension to the continuum time as below.

$$u^\varepsilon(x,t) = \begin{cases} u^\varepsilon(x, T - k\varepsilon^2) & \text{if } T - k\varepsilon^2 \le t < T - (k-1)\varepsilon^2, \\ \psi(x) & \text{if } t = T. \end{cases} \tag{2.4}$$

The difference from [14] is that our case has the interest rate $(1+\mu\varepsilon^2)^{-1}$ in the game so that corresponding PDEs contain 0-order term. Added to this, we consider the modified running cost $Q^\varepsilon = Q^\varepsilon(w, p, X)$.

$$Q^\varepsilon(w,p,X) := -\varepsilon p \cdot w - \frac{\varepsilon^2}{2}\langle Xw, w\rangle + \varepsilon^2 F(p, X) + \varepsilon^2 H(p). \tag{2.5}$$

Here F, H are given functions satisfying suitable conditions (see next section). As a beginning, we will take a formal consideration for the limit of u^ε as $\varepsilon \to 0$ by using (2.1), (2.3) and (2.4). If $u^\varepsilon(x,t) \approx u(x,t) + O(\varepsilon^3)$ for all sufficiently small ε and some smooth function u, then we get the approximate expression

$$\begin{aligned} u(x,t) \approx \Big(\frac{1}{1+\mu\varepsilon^2}\Big) \inf_{p,X} \sup_{w} \Big\{ & u(x,t) + \varepsilon w \cdot (Du(x,t) - p) \\ & + \frac{\varepsilon^2}{2}\langle (D^2u(x,t) - X)w, w\rangle + \varepsilon^2 \partial_t u(x,t) + \varepsilon^2 F(p,X) + \varepsilon^2 H(p)\Big\} + O(\varepsilon^3) \end{aligned}$$

by the Taylor expansion of u and therefore we obtain

$$\begin{aligned} 0 \approx \inf_{p,X} \sup_{w} \Big\{ & \varepsilon^{-1} w \cdot (Du(x,t) - p) + \frac{1}{2}\langle (D^2u(x,t) - X)w, w\rangle \\ & + \partial_t u(x,t) - \mu u(x,t) + F(p,X) + H(p)\Big\} + O(\varepsilon) \end{aligned}$$

where $O(\varepsilon)$ is of the order of ε. Here it is clear that an optimal strategy with respect to w (**B**'s choice) is to take w so that $w \cdot (Du(x,t) - p) = |w \cdot (Du(x,t) - p)|$. If $|w \cdot (Du(x,t) - p)|$ is positive independently of ε, then the right-hand side tends to $+\infty$ as $\varepsilon \to 0$. So the optimal strategy with respect to p (**A**'s choice) is to take $p \approx Du(x,t)$. In addition, if **A** chooses a special strategy $X = D^2u(x,t)$, then

$$\partial_t u(x,t) - \mu u(x,t) + F(Du(x,t), D^2u(x,t)) + H(Du(x,t)) \ge 0$$

holds as $\varepsilon \to 0$ no matter what the choice of w is. Formally, this shows that u is a classical sub (or super) solution of

$$\partial_t U - \mu U + F(DU, D^2U) + H(DU) = 0. \tag{2.6}$$

But we cannot generally expect any smoothness for solutions of (2.6) due to the nonlinearity of F, H. Therefore we consider solutions in the viscosity sense. It is natural that the theory of viscosity solutions is used, since it has the game theoretic backgrounds ([8]). We give a rigorous proof that the above u^ε converges to the viscosity solution of (2.6).

3 Notations and Conditions

We first state a few notations for later use.

Definition 3.1. We say a function $\omega : [0,\infty) \to [0,\infty)$ is a *modulus*, if it is a non-decreasing function with $\lim_{r\to 0}\omega(r) = 0$.

For example, let ϕ be a uniformly continuous function in $\mathbb{R}^N$. Then, the function

$$\omega_\phi(s) := \sup\{|\phi(x) - \phi(y)| \ ; \ |x-y| \le s, \ x, y \in \mathbb{R}^N\} \tag{3.1}$$

is a modulus.

Definition 3.2. Let $\mathcal{M}$ be a metric space and f be a function defined on a subset $\mathcal{M}' \subset \mathcal{M}$ with values in $\mathbb{R} \cup \{\pm\infty\}$. The *upper semi-continuous envelop* f^* and *lower semi-continuous envelop* f_* of f are defined respectively by

$$f^*(z) := \lim_{r\to 0} \sup\{f(\zeta) \ ; \ d_{\mathcal{M}}(z,\zeta) \le r, \ \zeta \in \mathcal{M}'\}, \tag{3.2}$$

$$f_*(z) := \lim_{r\to 0} \inf\{f(\zeta) \ ; \ d_{\mathcal{M}}(z,\zeta) \le r, \ \zeta \in \mathcal{M}'\} \tag{3.3}$$

for any $z \in \overline{\mathcal{M}'}$. Here $d_{\mathcal{M}}$ is the distance function on $\mathcal{M}$, and $\overline{\mathcal{M}'}$ denotes the closure of $\mathcal{M}'$.

The functions f^* and f_* are respectively smallest upper semi-continuous and greatest lower semi-continuous extensions of f on $\overline{\mathcal{M}'}$ and they satisfy $f_* = -(-f)^*$ and $f_* \le f \le f^*$ on $\mathcal{M}'$.

We next state the conditions of F and H.

(F1) $F : \mathbb{R}^N_* \times \mathcal{S}^N \to \mathbb{R}$ is continuous.

(F2) $\lambda_0 := \sup_p |F(p, O)| < \infty$ and $\inf_p F(p, X) = F_*(0, X)$, where $O \in \mathcal{S}^N$ is the zero matrix.

(F3) There exists the positive constant λ_1 such that

$$F(p,X) - F(p,Y) \le \frac{\lambda_1^2}{2}\mathcal{E}^+(X-Y)$$

where $\mathcal{E}^+ : \mathcal{S}^N \to [0,\infty)$ is defined by

$$\mathcal{E}^+(\cdot) := \max\{0, \ \mathcal{E}(\cdot)\}.$$

(F4) For any $r, R > 0$, there exists a modulus $\omega_{r,R}$ such that

$$|F(p,X) - F(q,X)| \le \omega_{r,R}(|p-q|), \qquad \text{if } |p|, |q| \ge r, |X| \le R.$$

(F5) $-\infty < F_*(0,O) = F^*(0,O) < \infty$.

(H) There exists the positive constant λ_2 such that

$$|H(p) - H(q)| \le \lambda_2|p-q|.$$

Remark 3.3. From **(F2)** and **(F3)**, one can see that F has at most linear growth (and at least linear decay). In fact, there exists the constant $C = C(\lambda_0, \lambda_1)$ such that

$$|F(p,X)| \le C(1+|X|) \qquad \text{for } (p,X) \in \mathbb{R}^N_* \times \mathcal{S}^N. \tag{3.4}$$

In addition, $-F$ is (degenerate) elliptic, since $-F(\cdot,Y) \leq -F(\cdot,X)$ if $Y \geq X$ from **(F3)**. In **(H)**, we can replace "Lipschitz" by "Hölder" and also treat the case $H \in BUC(\mathbb{R}^N)$.

Now, consider the following terminal value problem.

$$\begin{cases} \partial_t u - \mu u + F(Du, D^2u) + H(Du) = 0 & \text{in } \mathbb{R}^N \times (T_0, T), \\ u(x,T) = \psi(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{TP}$$

By replacing t with $T-\tau$ and setting $v(\cdot,\tau) := u(\cdot, T-\tau)$, we may regard the terminal value problem (TP) as the usual initial value problem

$$\begin{cases} \partial_\tau v = -\mu v + F(Dv, D^2v) + H(Dv) & \text{in } \mathbb{R}^N \times (0, T_1), \\ v(x,0) = \psi(x) & \text{in } \mathbb{R}^N \end{cases} \tag{IP}$$

where $T_1 := T - T_0 > 0$. Let us give some examples of (TP).

Example 3.4. (*First order equation*)

$$\partial_t u - \mu u + H(Du) = 0.$$

Example 3.5. (*Level set equation*)

$$\partial_t u + \Big(\Delta u - \langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|}\rangle\Big) + V|Du| = 0.$$

Here V is a constant.

These examples satisfy conditions **(F1)–(F5), (H)**. In particular, Example 3.5 is the level set equation of the motion of mean curvature plus the velocity V which represents the uniform velocity.

4 Representation Theorem

Before giving the statement of main theorem, let us start with defining the *relaxed limits* of u^ε. Let (x,t) be a point in $\mathbb{R}^N \times [T_0, T]$. For $\delta > 0$, we define the set $S^\delta = S^\delta(x,t)$ as follows.

$$S^\delta(x,t) := \big\{(y,s) \in \mathbb{R}^N \times [T_0,T]\ ;\ |x-y| \leq \delta,\ |t-s| \leq \delta\big\}.$$

Definition 4.1. For $(x,t) \in \mathbb{R}^N \times [T_0,T]$, the upper relaxed limit $\overline{u}$ and lower relaxed limit $\underline{u}$ are defined by

$$\overline{u}(x,t) := \lim_{\delta \to 0} \sup_{\varepsilon<\delta,\ S^\delta(x,t)} u^\varepsilon(y,s), \tag{4.1}$$

$$\underline{u}(x,t) := \lim_{\delta \to 0} \inf_{\varepsilon<\delta,\ S^\delta(x,t)} u^\varepsilon(y,s). \tag{4.2}$$

These limits are called relaxed limits and the advantage is that their limits always exist with the values in $\mathbb{R} \cup \{\pm\infty\}$. In addition, $\overline{u}$ and $\underline{u}$ are respectively upper and lower semi-continuous. So if $\overline{u} = \underline{u}$ $(= u)$, then u is continuous, and u^ε locally and uniformly converges to u as $\varepsilon \to 0$. Our main result is the following.

Theorem 4.2. *Assume that $\psi \in BUC(\mathbb{R}^N)$ and (F1)–(F5), (H) hold. Then, there exists the unique viscosity solution $u \in BUC(\mathbb{R}^N \times [T_0,T])$ of (TP). In addition,*

$$u(x,t) = \lim_{\varepsilon \to 0} \mathcal{J}_\varepsilon^n \psi(x) \tag{4.3}$$

for $x \in \mathbb{R}^N$. Here $n = n(\varepsilon,t)$ is the non-negative integer such that $T - n\varepsilon^2 \leq t < T - (n-1)\varepsilon^2$ for $t \in [T_0,T]$.

This theorem implies that problem (TP) is globally solvable. Theorem 4.2 follows from the following propositions.

Proposition 4.3. *Let ψ be a function of $BUC(\mathbb{R}^N)$. Then $\overline{u}$, $\underline{u} \in BUC(\mathbb{R}^N \times [T_0, T])$ with $\overline{u}(\cdot, T) = \underline{u}(\cdot, T) = \psi(\cdot)$.*

Proposition 4.4. *The function $\overline{u}$ is a viscosity subsolution of (TP).*

Proposition 4.5. *The function $\underline{u}$ is a viscosity supersolution of (TP).*

From Proposition 4.3, 4.4 and 4.5, we obtain the existence of a viscosity sub- and supersolution such that they belong to $BUC(\mathbb{R}^N \times [T_0, T])$ and their initial value are identical. So we can apply the comparison theorem for $\overline{u}$ and $\underline{u}$ (see Section 7). Consequently we have the inequality $\overline{u} \le \underline{u}$ in $\mathbb{R}^N \times (T_0, T]$ which implies the locally uniform convergence of u^ε as $\varepsilon \to 0$ and the continuity of its limit. To prove Proposition 4.3, we need some lemmas. Lemma 4.6 is the key in this paper to prove the other lemmas and propositions. We are going to prove it in Appendix.

Lemma 4.6. *Let (q, Y) be a pair in $\mathbb{R}^N \times \mathcal{S}^N$ and let R_0 be a fixed constant such that $|q|, |Y| \le R_0$. Assume that **(F1)–(F4)**, **(H)** hold. If $|q| \ge K^{-1}$ ($K \in \mathbb{N}$), then there exists $\varepsilon_1 = \varepsilon_1(K, R_0, N, \lambda_0, \lambda_1, \lambda_2)$ such that for any $(p, X) \in \mathbb{R}^N_* \times \mathcal{S}^N$ with $|p| \le \varepsilon^{-1/4}, |X| \le \varepsilon^{-1/2}$ there exists $\overline{w} = \overline{w}(\varepsilon, p, q, X, Y)$ with $|\overline{w}| \le \varepsilon^{-1/4}$ such that*

$$Q^\varepsilon(\overline{w}, p, X) \ge Q^\varepsilon_*(\overline{w}, q, Y) - h_1(\varepsilon^{1/4})\varepsilon^2 \tag{4.4}$$

holds whenever $\varepsilon \le \varepsilon_1$. If $|q| \le K^{-1}$, then there exists $\varepsilon_2 = \varepsilon_2(R_0, N, \lambda_0, \lambda_1, \lambda_2)$ such that for any $(p, X) \in \mathbb{R}^N_* \times \mathcal{S}^N$ with $|p| \le \varepsilon^{-1/4}, |X| \le \varepsilon^{-1/2}$ there exists $\overline{w} = \overline{w}(\varepsilon, p, q, X, Y)$ with $|\overline{w}| \le \varepsilon^{-1/4}$ such that

$$Q^\varepsilon(\overline{w}, p, X) \ge Q^\varepsilon_*(\overline{w}, 0, Y) - h_2(\varepsilon^{1/4})\varepsilon^2 \tag{4.5}$$

holds whenever $\varepsilon \le \varepsilon_2$. Here h_1, h_2 are given by

$$h_1(r) := \omega_{1/2K, R_0}(r) + \lambda_2 r, \qquad h_2(r) := \lambda_2 r \tag{4.6}$$

*for $r \ge 0$ where ω is the modulus as in **(F4)** and λ_2 is the constant as in **(H)**.*

Since $h_1(r) \ge h_2(r)$, we set $h^\varepsilon_K := h_1(\varepsilon^{1/4})$ to simplify.

Lemma 4.7. *Let ψ be a C^2-function whose derivatives are bounded up to second order. We set $E^\varepsilon(x, y, k) := \mathcal{J}^k_\varepsilon \psi(x) - \mathcal{J}^k_\varepsilon \psi(y)$ for $x, y \in \mathbb{R}^N$ and $k = 0, 1, \dots m$. Then,*

$$|E^\varepsilon(x, y, k)| \le L\Big(\frac{1}{1 + \mu\varepsilon^2}\Big)^k |x - y| \tag{4.7}$$

holds if $\varepsilon \le \varepsilon'$. Here L is the Lipschitz constant of ψ and $\varepsilon' = \varepsilon'(\psi, N, \lambda_0, \lambda_1, \lambda_2)$.

Lemma 4.7 yields the Lipschitz continuity of $\mathcal{J}^k_\varepsilon \psi$ whenever ψ is C^2.

Lemma 4.8. *Let ψ be a function as in Lemma 4.7. We set $E^\varepsilon(x, k) := \mathcal{J}^{k-1}_\varepsilon \psi(x) - \mathcal{J}^k_\varepsilon \psi(x)$ for $x \in \mathbb{R}^N$ and $k = 1, \dots m$. Then, there exists a positive constant C such that*

$$|E^\varepsilon(x, k)| \le C\Big(\frac{1}{1 + \mu\varepsilon^2}\Big)^k \varepsilon^2 \tag{4.8}$$

holds in $\varepsilon \le \varepsilon'$. Here $C = C(\psi, \lambda_0, \lambda_1, \lambda_2)$ and ε' is the small number as same as Lemma 4.7.

We remark that Lemma 4.8 shows $u^\varepsilon(x, \cdot)$ is Lipschitz continuous with respect to the discrete time $t = T - k\varepsilon^2$ ($k = 0, 1, \dots m$). In the next section, we will prove Lemma 4.7 and 4.8. The proof of Lemma 4.6 is relegated to Appendix.

5 Proofs of Lemmas

Before giving the proof of Lemma 4.7, we prove the boundedness of u^ε in the case ψ is C^2 and its derivatives are bounded. Let ψ be a C^2-function whose derivatives are bounded up to second order. We prove that there exists a positive number C such that

$$||\mathcal{J}_\varepsilon^k \psi||_{L^\infty} \le ||\psi||_{L^\infty} + C \tag{5.1}$$

for each $k = 0, 1, \dots m$. We specify the dependence of C later. At first, we show the upper bound of $\mathcal{J}_\varepsilon\psi(\cdot) = u^\varepsilon(\cdot, T - \varepsilon^2)$. Applying (2.2) and the *mean value theorem*, we obtain

$$(1+\mu\varepsilon^2)\mathcal{J}_\varepsilon\psi(x) = \inf_{p,X}\sup_{w}\Big\{\psi(x) + \varepsilon w\cdot(D\psi(x)-p) + \frac{\varepsilon^2}{2}\langle (D^2\psi(x') - X)w, w\rangle + \varepsilon^2 F(p,X) + \varepsilon^2 H(p)\Big\},$$

where $x' = x + \varepsilon\theta w$ for some $\theta \in (0,1)$, and the infimum and supremum are taken over $0 < |p| \le \varepsilon^{-1/4}$, $|X| \le \varepsilon^{-1/2}$ and $|w| \le \varepsilon^{-1/4}$. Since $D\psi$ and $D^2\psi$ are bounded, we consider the amount $C_0[\psi]$ depending only on ψ as follows.

$$C_0[\psi] := \max\big[\ ||\psi||_{L^\infty}, ||D\psi||_{L^\infty}, ||D^2\psi||_{L^\infty}\ \big]. \tag{5.2}$$

Since the inequality

$$\sup_{y\in\mathbb{R}^N} |\langle D^2\psi(y)v, v\rangle| \le C_0[\psi]\langle v, v\rangle$$

holds for any $v \in \mathbb{R}^N$, we have

$$(1+\mu\varepsilon^2)\mathcal{J}_\varepsilon\psi(x) \le \inf_{p,X}\sup_{w}\Big\{\psi(x) + \varepsilon w\cdot(D\psi(x)-p) + \frac{\varepsilon^2}{2}\langle (C_0[\psi]I - X)w, w\rangle + \varepsilon^2 F^*(p,X) + \varepsilon^2 H(p)\Big\}.$$

Here $I \in \mathcal{S}^N$ denotes the identity. Since $|w| \le \varepsilon^{-1/4}$, we have

$$(1+\mu\varepsilon^2)\mathcal{J}_\varepsilon\psi(x) \le \inf_{p,X}\sup_{w}\Big\{\psi(x) + \varepsilon^{3/4}|D\psi(x)-p| + \frac{\varepsilon^{3/2}}{2}\mathcal{E}^+(C_0[\psi]I - X) + \varepsilon^2\sup_{q\in\mathbb{R}^N} F^*(q,X) + \varepsilon^2 H(p)\Big\}. \tag{5.3}$$

Let ε be small enough so that $C_0[\psi] \le \varepsilon^{-1/4}$, then player **A** can choose the choice $(p, X) = (D\psi(x), C_0[\psi]I)$ in (5.3). Thus we obtain

$$\begin{aligned}(1+\mu\varepsilon^2)\mathcal{J}_\varepsilon\psi(x) &\le \psi(x) + \varepsilon^2\sup_{q\in\mathbb{R}^N} F^*(q, C_0[\psi]I) + \varepsilon^2|H(D\psi(x))| \\ &\le \psi(x) + \varepsilon^2 C(1 + C_0[\psi]) + \varepsilon^2(|H(0)| + \lambda_2 C_0[\psi]).\end{aligned}$$

Here C is the constant in Remark 3.3. Consequently the following inequality holds for the constant $C' = C'(\mu, \lambda_0, \lambda_1, \lambda_2)$, if $C_0[\psi] \le \varepsilon^{-1/4}$.

$$\mathcal{J}_\varepsilon\psi(x) - \psi(x) \le C'(1 + C_0[\psi])\Big(\frac{1}{1+\mu\varepsilon^2}\Big)\varepsilon^2. \tag{5.4}$$

Next, we show the lower bound of $\mathcal{J}_\varepsilon\psi(\cdot) = u^\varepsilon(\cdot, T-\varepsilon^2)$. Similar to the above arguments, for any w with $|w| \leq \varepsilon^{-1/4}$ we have

$$(1+\mu\varepsilon^2)\mathcal{J}_\varepsilon\psi(x) \geq \inf_{p,X}\Big\{\psi(x) + \varepsilon w\cdot(D\psi(x)-p) + \frac{\varepsilon^2}{2}\langle(-C_0[\psi]I - X)w, w\rangle + \varepsilon^2 F(p,X) + \varepsilon^2 H(p)\Big\}. \quad (5.5)$$

Applying (4.4) and (4.5) in Lemma 4.6 with $q = D\psi(x), Y = -C_0[\psi]I$ and choosing an appropriate $w = \overline{w}(\varepsilon, p, q, X, Y)$, we have

$$(1+\mu\varepsilon^2)\mathcal{J}_\varepsilon\psi(x) \geq \Big\{\psi(x) + \varepsilon^2 F_*(D\psi(x), -C_0[\psi]I) + \varepsilon^2 H(D\psi(x)) - h_1^\varepsilon\varepsilon^2\Big\}$$

if $|D\psi(x)| \geq 1$ and we have

$$(1+\mu\varepsilon^2)\mathcal{J}_\varepsilon\psi(x) \geq \Big\{\psi(x) + \varepsilon^2 F_*(0, -C_0[\psi]I) + \varepsilon^2 H(0) - h_1^\varepsilon\varepsilon^2\Big\}$$

if $|D\psi(x)| \leq 1$ for all sufficiently small $\varepsilon \leq \min[\varepsilon_1, \varepsilon_2]$ with $R_0 := C_0[\psi]$ and $K = 1$. Here we recall that ε_1, ε_2 are small numbers as in Lemma 4.6 and $h_1^\varepsilon = \omega_{1/2,R_0}(\varepsilon) + \lambda_2\varepsilon$. As same as (5.4), we have

$$\mathcal{J}_\varepsilon\psi(x) - \psi(x) \geq -C'(1 + C_0[\psi] + h_1^\varepsilon)\Big(\frac{1}{1+\mu\varepsilon^2}\Big)\varepsilon^2. \quad (5.6)$$

Combining (5.4) and (5.6), consequently we obtain

$$|\mathcal{J}_\varepsilon\psi(x) - \psi(x)| \leq C'(1 + C_0[\psi] + h_1^\varepsilon)\Big(\frac{1}{1+\mu\varepsilon^2}\Big)\varepsilon^2. \quad (5.7)$$

The formula (5.7) also shows that (4.8) in Lemma 4.8 holds for $k = 1$, when $\psi \in C^2(\mathbb{R}^N)$ and $C_0[\psi] < \infty$. Let us set

$$C^\varepsilon[\psi, K] := C'(1 + C_0[\psi] + h_K^\varepsilon) \quad (5.8)$$

to simplify. Here $C' = C'(\mu, \lambda_0, \lambda_1, \lambda_2)$. We will show the boundedness of $\mathcal{J}_\varepsilon^k\psi(x) = u^\varepsilon(x, T - k\varepsilon^2)$ for each $k = 0, 1, \ldots m$. To prove it, we set

$$S_k^\varepsilon := C^\varepsilon[\psi, 1]\sum_{i=1}^{k}\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^i\varepsilon^2 \quad (5.9)$$

and suppose that

$$|\mathcal{J}_\varepsilon^k\psi(x) - \psi(x)| \leq S_k^\varepsilon \quad (5.10)$$

holds for any $x \in \mathbb{R}^N$ if $1 \leq k \leq n$ (note that it is clear in the case $n = 1$ from (5.7)). Then, we obtain

$$\begin{aligned}
\mathcal{J}_\varepsilon^{n+1}\psi(x) &= \Big(\frac{1}{1+\mu\varepsilon^2}\Big)\inf_{p,X}\sup_{w}\Big\{\mathcal{J}_\varepsilon^n\psi(x+\varepsilon w) + Q^\varepsilon(w,p,X)\Big\}\\
&\leq \Big(\frac{1}{1+\mu\varepsilon^2}\Big)\inf_{p,X}\sup_{w}\Big\{\psi(x+\varepsilon w) + Q^\varepsilon(w,p,X)\Big\} + \Big(\frac{1}{1+\mu\varepsilon^2}\Big)S_n^\varepsilon\\
&= \mathcal{J}_\varepsilon\psi(x) + \Big(\frac{1}{1+\mu\varepsilon^2}\Big)S_n^\varepsilon\\
&\leq \psi(x) + C^\varepsilon[\psi,1]\Big(\frac{1}{1+\mu\varepsilon^2}\Big)\varepsilon^2 + \Big(\frac{1}{1+\mu\varepsilon^2}\Big)S_n^\varepsilon\\
&= \psi(x) + S_{n+1}^\varepsilon
\end{aligned}$$

and similarly

$$\mathcal{J}_\varepsilon^{n+1}\psi(x) \geq \psi(x) - S_{n+1}^\varepsilon.$$

In addition, we see $S_k^\varepsilon \leq S_m^\varepsilon$ and verify that the sum of geometric series $S_m^\varepsilon \leq C_\mu^\varepsilon$ by the elementary calculations. Here C_μ^ε denoted by

$$C_\mu^\varepsilon = \begin{cases} C^\varepsilon[\psi,1](T-T_0) & \text{if } \mu = 0, \\ C^\varepsilon[\psi,1]\dfrac{1-e^{-\mu(T-T_0)}}{\mu} & \text{if } \mu > 0. \end{cases} \tag{5.11}$$

Note that C_μ^ε is bounded independent of $\varepsilon \leq \min[\varepsilon_1, \varepsilon_2]$ with $R_0 := C_0[\psi]$ and $K = 1$. So we conclude the formula (5.1) with $C = C_\mu^\varepsilon$.

Now we will prove Lemma 4.7 and mention the continuity of value function.

Proof of Lemma 4.7. Let us set $A_k(x) := \mathcal{J}_\varepsilon^k\psi(x+\varepsilon w) + Q^\varepsilon(w,p,X)$ for $p \neq 0$. Then the formula

$$A_k(x) - A_k(y) = \mathcal{J}_\varepsilon^k\psi(x+\varepsilon w) - \mathcal{J}_\varepsilon^k\psi(y+\varepsilon w) = E^\varepsilon(x+\varepsilon w, y+\varepsilon w, k) \tag{5.12}$$

holds for any choices p, X and w of players. When $k=0$, we have

$$|A_0(x) - A_0(y)| = |E^\varepsilon(x+\varepsilon w, y+\varepsilon w, 0)| \leq L|x-y|$$

for any $x, y \in \mathbb{R}^N$ where L is the Lipschitz constant of ψ ($L \leq C_0[\psi]$). Suppose that $|E^\varepsilon(x+\varepsilon w, y+\varepsilon w, k)|$ is bounded with respect to w with $|w| \leq \varepsilon^{-1/4}$ for $k = 0, \ldots n$ ($0 \leq n \leq m-1$). Then we have

$$A_n(x) - A_n(y) \leq \sup_w E^\varepsilon(x+\varepsilon w, y+\varepsilon w, n) < \infty \tag{5.13}$$

and

$$A_n(x) - A_n(y) \geq \inf_w E^\varepsilon(x+\varepsilon w, y+\varepsilon w, n) > -\infty \tag{5.14}$$

for each $x, y \in \mathbb{R}^N$ by inductive assumptions. Therefore we obtain

$$\sup_w A_n(x) - \sup_w A_n(y) \leq \sup_w E^\varepsilon(x+\varepsilon w, y+\varepsilon w, n) \tag{5.15}$$

and

$$\sup_w A_n(x) - \sup_w A_n(y) \geq \inf_w E^\varepsilon(x+\varepsilon w, y+\varepsilon w, n), \tag{5.16}$$

since the right-hand side of (5.13) and (5.14) are independent of w. Similarly, since the right-hand side of (5.15) and (5.16) are independent of p, X, we conclude that

$$\inf_{p,X}\sup_w A_n(x) - \inf_{p,X}\sup_w A_n(y) \leq \sup_w E^\varepsilon(x+\varepsilon w, y+\varepsilon w, n) \tag{5.17}$$

and

$$\inf_{p,X}\sup_w A_n(x) - \inf_{p,X}\sup_w A_n(y) \geq \inf_w E^\varepsilon(x+\varepsilon w, y+\varepsilon w, n) \tag{5.18}$$

hold. From the formula (2.2), one can see $\inf_{p,X}\sup_w A_n(z) = (1+\mu\varepsilon^2)\mathcal{J}_\varepsilon^{n+1}\psi(z)$ for $z \in \mathbb{R}^N$ (it is well-defined from the previous section). So we have

$$|E^\varepsilon(x,y,n+1)| \leq \left(\frac{1}{1+\mu\varepsilon^2}\right)\sup_w |E^\varepsilon(x+\varepsilon w, y+\varepsilon w, n)|. \tag{5.19}$$

By the induction, we have the conclusion of Lemma 4.7. □

Arguing as same as above, we can prove Lemma 4.8.

Proof of Lemma 4.8. From the formula (5.7), the conclusion of the lemma holds for $k = 1$. If we set $A_k(x) := \mathcal{J}_\varepsilon^k \psi(x+\varepsilon w) + Q^\varepsilon(w,p,X)$ for $p \neq 0$, then

$$A_{k-1}(x) - A_k(x) = \mathcal{J}_\varepsilon^{k-1}\psi(x+\varepsilon w) - \mathcal{J}_\varepsilon^k \psi(x+\varepsilon w) = E^\varepsilon(x+\varepsilon w, k) \tag{5.20}$$

holds for $x \in \mathbb{R}^N$. Suppose that

$$|E^\varepsilon(y,k)| \leq C^\varepsilon[\psi,1]\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^k \varepsilon^2 \tag{5.21}$$

holds for any $y \in \mathbb{R}^N$ and $k = 1, \dots n$ $(1 \leq n \leq m-1)$. From the above conditions, we have

$$A_{n-1}(x) - A_n(x) \leq \sup_w E^\varepsilon(x+\varepsilon w, n) < \infty$$

and

$$A_{n-1}(x) - A_n(x) \geq \inf_w E^\varepsilon(x+\varepsilon w, n) > -\infty$$

for any choices p, X and w of players. Arguing as same as the previous lemma, we have

$$\inf_{p,X}\sup_w A_{n-1}(x) - \inf_{p,X}\sup_w A_n(x) \leq \sup_w E^\varepsilon(x+\varepsilon w, n) \tag{5.22}$$

and

$$\inf_{p,X}\sup_w A_{n-1}(x) - \inf_{p,X}\sup_w A_n(x) \geq \inf_w E^\varepsilon(x+\varepsilon w, n). \tag{5.23}$$

Since $\inf_{p,X}\sup_w A_k(z) = (1+\mu\varepsilon^2)\mathcal{J}_\varepsilon^{k+1}\psi(z)$ for $z \in \mathbb{R}^N$, we conclude

$$|E^\varepsilon(x, n+1)| \leq \Big(\frac{1}{1+\mu\varepsilon^2}\Big) \sup_w |E^\varepsilon(x+\varepsilon w, n)| \tag{5.24}$$

for $x \in \mathbb{R}^N$. Consequently we have the conclusion of Lemma 4.8 by the induction. □

Now the proofs of Lemma 4.7 and 4.8 are completed.

6 Proofs of Propositions

Our purpose in this section is to state the properties of $\mathcal{J}_\varepsilon$ and to give proofs of Proposition 4.3, 4.4 and 4.5.

In the previous section, we only consider the case $\psi \in C^2(\mathbb{R}^N)$ and its derivatives are bounded up to second order. Actually, we can extend the conclusions of Lemma 4.7 and 4.8 to the case $\psi \in BUC(\mathbb{R}^N)$. Before stating it, we remark on the operator $\mathcal{J}_\varepsilon$.

Lemma 6.1. *Let ϕ, ϕ' be a function in $L^\infty(\mathbb{R}^N)$. Then, following properties hold.*

(a) $\mathcal{J}_\varepsilon : L^\infty(\mathbb{R}^N) \longrightarrow L^\infty(\mathbb{R}^N)$.
(b) If $\phi \leq \phi'$ *a.e, then* $\mathcal{J}_\varepsilon\phi \leq \mathcal{J}_\varepsilon\phi'$ *a.e.*
(c) For $c \in \mathbb{R}$, $\mathcal{J}_\varepsilon(\phi + c) = \mathcal{J}_\varepsilon\phi + (1+\mu\varepsilon^2)^{-1}c$.

Proof of Lemma 6.1. If $\mathcal{J}_\varepsilon\phi$ is well-defined for $\phi \in L^\infty(\mathbb{R}^N)$, then (b) and (c) are clear from (2.2). So we only prove (a). Assume that $\phi \in L^\infty(\mathbb{R}^N)$. Then we have the upper bound

$$\begin{aligned}(1+\mu\varepsilon^2)\mathcal{J}_\varepsilon\phi(x) &\leq ||\phi||_{L^\infty} + \inf_{p,X}\sup_w Q^\varepsilon(w,p,X),\\ &\leq ||\phi||_{L^\infty} + (F^*(0,O) + H(0))\varepsilon^2\end{aligned}$$

for all ε. And the lower bound

$$\begin{aligned}(1+\mu\varepsilon^2)\mathcal{J}_\varepsilon\phi(x) &\geq -||\phi||_{L^\infty} + \inf_{p,X}\sup_{w} Q^\varepsilon(w,p,X)\\ &\geq -||\phi||_{L^\infty} + Q^\varepsilon_*(\overline{w},0,O) - h_1^\varepsilon\varepsilon^2,\\ &= -||\phi||_{L^\infty} + (F_*(0,O)+H(0)-h_1^\varepsilon)\varepsilon^2\end{aligned}$$

holds for all $\varepsilon \leq \varepsilon_2$ where the second inequality comes from Lemma 4.6 and ε_2 and $\overline{w}$ are as in Lemma 4.6 with $q=0$, $Y=O$ and $R_0=1$. Since we can choose $K=1$, we have

$$||\mathcal{J}_\varepsilon\phi||_{L^\infty} \leq ||\phi||_{L^\infty} + C\varepsilon^2 \tag{6.1}$$

for all sufficiently small ε and $\phi \in L^\infty(\mathbb{R}^N)$. Here the constant C depends only on λ_0 and $H(0)$. Consequently property (a) is proved. By the induction, in addition,

$$||\mathcal{J}_\varepsilon^k\phi||_{L^\infty} \leq ||\phi||_{L^\infty} + C(T-T_0) \tag{6.2}$$

holds (due to $k\varepsilon^2 \leq m\varepsilon^2 = T-T_0$). □

Now we prove that relaxed limits $\overline{u}$ and $\underline{u}$ are uniformly continuous with spacial variables in the case $\psi \in BUC(\mathbb{R}^N)$ too. From property (a) and (6.2), we can see $\mathcal{J}_\varepsilon^k\psi$ is well-defined. To get the analogous inequality of Lemma 4.7 in the case ψ is not differentiable, for a parameter $\delta > 0$ we introduce the regularization $\psi_\delta^\pm \in C^2(\mathbb{R}^N)$ of ψ such that they satisfy

$$\psi - \delta \leq \psi_\delta^- \leq \psi \leq \psi_\delta^+ \leq \psi + \delta \quad \text{in } \mathbb{R}^N \tag{6.3}$$

and their derivatives are bounded up to second order. From Lemma 4.7, we have the estimate

$$|\mathcal{J}_\varepsilon\psi_\delta^\pm(x) - \mathcal{J}_\varepsilon\psi_\delta^\pm(y)| \leq L_\delta\Big(\frac{1}{1+\mu\varepsilon^2}\Big)|x-y| \tag{6.4}$$

for $x, y \in \mathbb{R}^N$ and sufficiently small ε. Here L_δ is the maximum of the Lipschitz constants of ψ_δ^+ and ψ_δ^-. The estimate (6.4) shows that $\mathcal{J}_\varepsilon\psi_\delta^\pm \in UC(\mathbb{R}^N)$. In addition, we see that $\mathcal{J}_\varepsilon\psi_\delta^\pm$ are bounded by previous arguments. From (2.2), (6.3) and the properties of $\mathcal{J}_\varepsilon$, we obtain

$$\begin{aligned}\mathcal{J}_\varepsilon\psi \leq &\mathcal{J}_\varepsilon\psi_\delta^+ \leq \mathcal{J}_\varepsilon\psi + \Big(\frac{1}{1+\mu\varepsilon^2}\Big)\delta,\\ \mathcal{J}_\varepsilon\psi - \Big(\frac{1}{1+\mu\varepsilon^2}\Big)\delta \leq &\mathcal{J}_\varepsilon\psi_\delta^- \leq \mathcal{J}_\varepsilon\psi.\end{aligned}$$

Hence we obtain

$$\mathcal{J}_\varepsilon\psi - \delta \leq \mathcal{J}_\varepsilon\psi_\delta^- \leq \mathcal{J}_\varepsilon\psi \leq \mathcal{J}_\varepsilon\psi_\delta^+ \leq \mathcal{J}_\varepsilon\psi + \delta, \tag{6.5}$$

since $(1+\mu\varepsilon^2)^{-1} \leq 1$. Combining (6.4) and (6.5), we conclude that

$$|\mathcal{J}_\varepsilon\psi(x) - \mathcal{J}_\varepsilon\psi(y)| \leq L_\delta\Big(\frac{1}{1+\mu\varepsilon^2}\Big)|x-y| + \delta \tag{6.6}$$

for $x, y \in \mathbb{R}^N$ whenever $\varepsilon \leq \varepsilon'$. Here $\varepsilon' = \varepsilon'(\psi_\delta^\pm, \lambda_0, \lambda_1, \lambda_2)$ is sufficiently small number. Inductively, we have the generalized inequality of (4.7)

$$|\mathcal{J}_\varepsilon^k\psi(x) - \mathcal{J}_\varepsilon^k\psi(y)| \leq L_\delta\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^k|x-y| + \delta. \tag{6.7}$$

Next, we will construct the modified estimate of (4.8) in Lemma 4.8 as before. Assume that $\psi \in BUC(\mathbb{R}^N)$. For the regularizations $\psi_\delta^\pm$ of ψ as before, the estimate

$$|\mathcal{J}_\varepsilon^k\psi_\delta^\pm(x) - \mathcal{J}_\varepsilon^{k-1}\psi_\delta^\pm(x)| \leq C^\varepsilon[\psi_\delta^\pm]\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^k\varepsilon^2 \tag{6.8}$$

holds for each k and all sufficiently small ε from Lemma 4.8 where $C^\varepsilon[\psi_\delta^\pm] := \max\left[C^\varepsilon[\psi_\delta^+, 1], C^\varepsilon[\psi_\delta^-, 1]\right]$. If $0 \le i \le j \le m$, then we have

$$\mathcal{J}_\varepsilon^i \psi_\delta^+ - \mathcal{J}_\varepsilon^j \psi_\delta^+ \le S_j^\varepsilon(\delta) - S_i^\varepsilon(\delta) \tag{6.9}$$

and

$$\mathcal{J}_\varepsilon^i \psi_\delta^- - \mathcal{J}_\varepsilon^j \psi_\delta^- \ge -\big(S_j^\varepsilon(\delta) - S_i^\varepsilon(\delta)\big). \tag{6.10}$$

Here $S_k^\varepsilon(\delta)$ is denoted by

$$S_k^\varepsilon(\delta) := C^\varepsilon[\psi_\delta^\pm] \sum_{l=1}^{k} \Big(\frac{1}{1+\mu\varepsilon^2}\Big)^l \varepsilon^2.$$

In addition, one can verify that

$$S_j^\varepsilon(\delta) - S_i^\varepsilon(\delta) \le C^\varepsilon[\psi_\delta^\pm]\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^{i+1} (j-i)\varepsilon^2 \tag{6.11}$$

holds for $0 \le i \le j \le m$. On the other hand, from (6.5) we obtain

$$\mathcal{J}_\varepsilon^j \psi - \mathcal{J}_\varepsilon^i \psi \le \mathcal{J}_\varepsilon^j \psi_\delta^+ - \mathcal{J}_\varepsilon^i \psi_\delta^+ + \delta, \tag{6.12}$$

$$\mathcal{J}_\varepsilon^j \psi - \mathcal{J}_\varepsilon^i \psi \ge \mathcal{J}_\varepsilon^j \psi_\delta^- - \mathcal{J}_\varepsilon^i \psi_\delta^- - \delta. \tag{6.13}$$

Consequently the estimate

$$|\mathcal{J}_\varepsilon^j \psi(x) - \mathcal{J}_\varepsilon^i \psi(x))| \le C^\varepsilon[\psi_\delta^\pm]\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^{i+1} (j-i)\varepsilon^2 + \delta \tag{6.14}$$

holds for $0 \le i \le j \le m$ and all sufficiently small ε from (6.9)–(6.13). Notice that (6.14) is the modified estimate of (4.8). Now we give the proof of Proposition 4.3 by using (6.7) and (6.14).

Proof of Proposition 4.3. For any $t, s \in [T_0, T]$ with $t \le s$, there exist i, j such that $0 \le i \le j \le m$ and

$$T - j\varepsilon^2 \le t < T - (j-1)\varepsilon^2, \quad T - i\varepsilon^2 \le s < T - (i-1)\varepsilon^2$$

hold. From (2.4), one can see $u^\varepsilon(x,t) = \mathcal{J}_\varepsilon^j \psi(x)$ and $u^\varepsilon(y,s) = \mathcal{J}_\varepsilon^i \psi(y)$ for $x, y \in \mathbb{R}^N$. From (6.6), (6.14) and the triangle inequality, we can estimate as follows.

$$|u^\varepsilon(x,t) - u^\varepsilon(y,s)| \le C^\varepsilon[\psi_\delta^\pm](j-i)\varepsilon^2 + L_\delta|x-y| + 2\delta. \tag{6.15}$$

Set $C_0[\psi_\delta^\pm] := \max\left[C_0[\psi_\delta^+],\ C_0[\psi_\delta^-]\right]$. Notice that $L_\delta \le C_0[\psi_\delta^\pm]$ by the definition of $C_0[\cdot]$ in (5.2). Since $i\varepsilon^2 \le T - t + \varepsilon^2$ and $-j\varepsilon^2 \le s - T$ hold, we have

$$|u^\varepsilon(x,t) - u^\varepsilon(y,s)| \le C^\varepsilon[\psi_\delta^\pm](|x-y| + |s-t| + \varepsilon^2) + 2\delta \tag{6.16}$$

for all sufficiently small ε so that $\varepsilon \le \varepsilon'$ where $\varepsilon' = \varepsilon'(R_0, N, \lambda_0, \lambda_1, \lambda_2)$ with $R_0 := C_0[\psi_\delta^\pm]$. Now we fix $\delta > 0$ in the formula (6.16). And then considering the relaxed limit $\overline{u}$ of u^ε, we have

$$|\overline{u}(x,t) - \overline{u}(y,s)| \le C[\psi_\delta^\pm](|x-y| + |s-t|) + 2\delta \tag{6.17}$$

where $C[\psi_\delta^\pm, 1]$ is denoted by

$$C[\psi_\delta^\pm] := \lim_{\varepsilon \to 0} C^\varepsilon[\psi_\delta^\pm] = C'(1 + C_0[\psi_\delta^\pm]). \tag{6.18}$$

Note that $\lim_{\varepsilon\to 0} h_1^\varepsilon = 0$ holds. Then we have

$$|\overline{u}(x,t) - \overline{u}(y,s)| \le C[\psi_\delta^\pm](|x-y| + |s-t|) + 2\delta. \tag{6.19}$$

Finally, we define $\omega_0 : [0, \infty) \to [0, \infty)$ by

$$\omega_0(r) := \inf_{\delta>0}(C[\psi_\delta^\pm]r + 2\delta) \tag{6.20}$$

for $r \geq 0$, then ω_0 is the modulus of continuity of $\overline{u}$, i.e., we get the estimate

$$|\overline{u}(x,t) - \overline{u}(y,s)| \leq \omega_0(|x-y| + |t-s|). \tag{6.21}$$

In addition, taking $s = T$ in (6.16) and arguing as above, we conclude that $\overline{u} = \psi$ at $t = T$. The same holds for the case $\underline{u}$. Therefore $\overline{u}$, $\underline{u} \in UC(\mathbb{R}^N \times [T_0, T])$. Consequently we get the conclusion of Proposition 4.3. □

Finally, we will prove Proposition 4.4 and Proposition 4.5. At first, we give the definition of *(viscosity) sub- supersolutions* of (TP). Note that the following definitions are different from the usual, since our problem (TP) is the time backward case.

Definition 6.2. We call a function $u : \mathbb{R}^N \times (T_0, T] \to \mathbb{R}$ subsolution of (TP), if u satisfies the followings. Let ϕ be a smooth function on $\mathbb{R}^N \times (T_0, T)$.

(i) $u^* < \infty$ in $\mathbb{R}^N \times (T_0, T)$.
(ii) If $u^* - \phi$ has a local maximum at $(x_0, t_0) \in \mathbb{R}^N \times (T_0, T)$, then

$$\partial_t\phi - \mu u^* + F^*(D\phi, D^2\phi) + H(D\phi) \geq 0 \tag{6.22}$$

holds at (x_0, t_0).
(iii)

$$u^*(x, T) \leq \psi(x) \tag{6.23}$$

holds for $x \in \mathbb{R}^N$.

Supersolutions are also defined as above.

Definition 6.3. We call a function $u : \mathbb{R}^N \times (T_0, T] \to \mathbb{R}$ supersolution of (TP), if u satisfies (i) and (ii). Let ϕ be a smooth function on $\mathbb{R}^N \times (T_0, T)$.

(i) $u_* > -\infty$ in $\mathbb{R}^N \times (T_0, T)$.
(ii) If $u_* - \phi$ has a local minimum at $(x_0, t_0) \in \mathbb{R}^N \times (T_0, T)$, then

$$\partial_t\phi - \mu u_* + F_*(D\phi, D^2\phi) + H(D\phi) \leq 0 \tag{6.24}$$

holds at (x_0, t_0).
(iii)

$$u_*(x, T) \geq \psi(x) \tag{6.25}$$

holds for $x \in \mathbb{R}^N$.

Without loss of generality, we can replace "local" by "strict local" and assume that the strict local maximum (minimum) value is 0. In fact, if we replace the function ϕ by

$$\widetilde{\phi}(x,t) := \phi(x,t) + |x - x_0|^4 + |t - t_0|^2 + (u^* - \phi)(x_0, t_0),$$

then, $\widetilde{\phi}$ satisfies (6.22) and $u^* - \widetilde{\phi}$ realizes the strict local maximum 0 at (x_0, t_0). The same holds for the case of supersolution.

Proof of Proposition 4.4. Since $\overline{u}(\cdot, t)$, $\underline{u}(\cdot, t) \in BUC(\mathbb{R}^N)$ for any $t \in [T_0, T]$ and they are continuous (i.e., $\overline{u} = \overline{u}^*$ and $\underline{u} = \underline{u}_*$) $\overline{u} = \underline{u} = \psi$ at $t = T$, the condition (i) and (iii) in Definition 6.2, 6.3 are already satisfied. Therefore we only check whether they satisfy the condition (ii).

We assume that the condition (ii) does not hold. Then there exist a positive constant θ_0 and a smooth function ϕ, such that the following holds at the strict local maximal point $(x_0,t_0) \in \mathbb{R}^N \times (T_0,T)$ of $\overline{u}-\phi$.

$$\partial_t\phi - \mu\overline{u} + F^*(D\phi, D^2\phi) + H(D\phi) \le -\theta_0 < 0 \qquad \text{in } \overline{B}_0. \tag{6.26}$$

Here $\overline{B}_0 \subset \mathbb{R}^N \times (T_0,T)$ is a sufficiently small closed ball centered at (x_0,t_0), and $\max_{\overline{B}_0}(\overline{u}-\phi) = 0$.

Let (x,t) be a point in $\overline{B}_0$. From (2.3) and the Taylor expansion of ϕ, we have

$$\begin{aligned} u^\varepsilon(x,t) \le \frac{1}{1+\mu\varepsilon^2} \inf_{p,X}\sup_{w} \Big\{ &(u^\varepsilon-\phi)(x+\varepsilon w, t+\varepsilon^2) \\ &+ \phi(x,t) + \varepsilon w\cdot(D\phi(x,t)-p) + \varepsilon^2\partial_t\phi(x,t) \\ &+ \frac{\varepsilon^2}{2}\langle (D^2\phi(x,t)-X)w,w\rangle + \varepsilon^2F^*(p,X) + \varepsilon^2H(p)\Big\} + C\varepsilon^{9/4}. \end{aligned}$$

Here C is a positive constant depending only on the C^3 norm of ϕ in a sufficiently small neighborhood of $\overline{B}_0$ (note that $|w| \le \varepsilon^{-1/4}$).

Taking the special choices $p = D\phi(x,t)$ and $X = D^2\phi(x,t)$ of player **A**, the inequality

$$\begin{aligned} (u^\varepsilon-\phi)(x,t) \le \frac{1}{1+\mu\varepsilon^2}\sup_{w}\Big\{ &(u^\varepsilon-\phi)(x+\varepsilon w, t+\varepsilon^2) \\ &+ \varepsilon^2\{\partial_t\phi(x,t) - \mu\phi(x,t) \\ &+ F^*(D\phi(x,t), D^2\phi(x,t)) + H(D\phi(x,t))\}\Big\} + C\varepsilon^{9/4}. \end{aligned}$$

holds whenever $\|D\phi\|_{L^\infty(B_0)} \le \varepsilon^{-1/4}$, $\|D^2\phi\|_{L^\infty(B_0)} \le \varepsilon^{-1/2}$. From (6.26) and holding $\overline{u}-\phi \le 0$ in $\overline{B}_0$, we have

$$((u^\varepsilon)^*-\phi)(x,t) \le \frac{1}{1+\mu\varepsilon^2}\Big\{((u^\varepsilon)^*-\phi)(x+\varepsilon w_0, t+\varepsilon^2) + (C\varepsilon^{1/4}-\theta_0)\varepsilon^2\Big\}$$

for all $(x,t) \in \overline{B}_0$, where $w_0 = w_0^\varepsilon(x,t)$ is a vector with $|w_0| \le \varepsilon^{-1/4}$ which gives the supremum of $((u^\varepsilon)^*-\phi)(x+\varepsilon w, t+\varepsilon^2)$. Since $(u^\varepsilon)^*-\phi$ is upper semi-continuous on compact set, such w_0 exists. Hence we have the estimate

$$((u^\varepsilon)^*-\phi)(x,t) \le \frac{1}{1+\mu\varepsilon^2}((u^\varepsilon)^*-\phi)(x+\varepsilon w_0, t+\varepsilon^2) \tag{6.27}$$

for all sufficiently small ε such that $C\varepsilon^{1/4} \le \theta_0$. Let $X_0^\varepsilon = (x_0^\varepsilon, t_0^\varepsilon)$ be a point such that $\lim_{\varepsilon\to 0} u^\varepsilon(X_0^\varepsilon) = \overline{u}(x_0,t_0)$ (if the need arises, we take an appropriate subsequence). Then $X_0^\varepsilon \in B_0$ for all sufficiently small ε. Now we define for each ε $X_k^\varepsilon = (x_k^\varepsilon, t_k^\varepsilon)$ as follows.

$$X_k^\varepsilon = X_{k-1}^\varepsilon + (\varepsilon w_0^\varepsilon(X_{k-1}^\varepsilon), \varepsilon^2) \qquad 1 \le k \le m.$$

From (6.27), we obtain

$$((u^\varepsilon)^*-\phi)(X_{k-1}^\varepsilon) \le \frac{1}{1+\mu\varepsilon^2}((u^\varepsilon)^*-\phi)(X_k^\varepsilon) \tag{6.28}$$

if $X_1^\varepsilon, \dots X_k^\varepsilon \in \overline{B}_0$. So we have

$$(u^\varepsilon-\phi)(X_0^\varepsilon) \le \Big(\frac{1}{1+\mu\varepsilon^2}\Big)^k((u^\varepsilon)^*-\phi)(X_k^\varepsilon). \tag{6.29}$$

Let $\mathcal{P}$ be the projection from $\overline{B}_0$ onto $[T_0,T]$. Then, there exists $\delta_0 > 0$ such that $\mathcal{P}\overline{B}_0 = [t_0-\delta_0, t_0+\delta_0] \subset (T_0,T)$. Choosing the sufficiently small δ_0, in advance, we can suppose that $t_0 + 5\delta_0 < T$ and $4\delta_0 < (T-T_0)$. Furthermore we choose ε so that $|t_0^\varepsilon - t_0| \le \delta_0$. If we take $n = n^\varepsilon$ such that

$3\delta_0 \le n\varepsilon^2 \le 4\delta_0$ for sufficiently small ε, then one can verify that $t_n^\varepsilon \notin \mathcal{P}\overline{B}_0$ (i.e., $X_n^\varepsilon \notin \overline{B}_0$). Indeed, we obtain

$$t_0 + 2\delta_0 \le t_0 - \delta_0 + n\varepsilon^2 \le t_n^\varepsilon \le t_0 + \delta_0 + n\varepsilon^2 \le t_0 + 5\delta_0. \tag{6.30}$$

In addition, $n \le m$, since $n\varepsilon^2 \le 4\delta_0 < m\varepsilon^2 = T - T_0$. There exists the minimal number $K \le n$ such that $X_K^\varepsilon \in \overline{B}_0$ and $X_{K+1}^\varepsilon \notin \overline{B}_0$, since $X_0^\varepsilon \in \overline{B}_0$ and $X_n^\varepsilon \notin \overline{B}_0$. Applying these properties to (6.29), we obtain

$$(u^\varepsilon - \phi)(X_0^\varepsilon) \le \Big(\frac{1}{1+\mu\varepsilon^2}\Big)^K ((u^\varepsilon)^* - \phi)(X_K^\varepsilon). \tag{6.31}$$

We can let X_K^ε converges to some point $X' \in \overline{B}_0 \backslash \{X_0\}$ as $\varepsilon \to 0$ (i.e., $m \to \infty$) by taking an appropriate subsequence. Note that the limit of $(1+\mu\varepsilon^2)^{-K}$ as $m \to \infty$ (with taking a subsequence) is positive and less than 1, since $K \le n \le m$. In fact, we have the estimate

$$0 < e^{-\mu(T-T_0)} \le \Big(1 + \mu\frac{T-T_0}{m}\Big)^{-m} \le \Big(1 + \mu\frac{T-T_0}{m}\Big)^{-K} \le 1 \tag{6.32}$$

(note that $\varepsilon^2 = (T - T_0)/m$). Consequently there exists a constant $\alpha \in (0, 1]$ such that

$$0 = (\overline{u} - \phi)(x_0, t_0) \le \alpha(\overline{u} - \phi)(X') \tag{6.33}$$

for every cases, this is because $\limsup_{\varepsilon\to 0}(u^\varepsilon)^*(X_K^\varepsilon) \le \overline{u}(X')$ by the definition of $\overline{u}$. Therefore we get a contradiction, since our assumption is that $\overline{u} - \phi$ has the strict local maximum in $\overline{B}_0$ ($(x_0, t_0) \ne X' \in \overline{B}_0$). Now the proof of Proposition 4.4 is completed. □

Proof of Proposition 4.5. Next we will show that $\underline{u}$ is a supersolution of (TP). As same as before, assume that $\underline{u}$ is not a supersolution. Then there exist a positive constant θ_0 and a smooth function ϕ, such that the following property holds at the strict local minimal point $(x_0, t_0) \in \mathbb{R}^N \times (T_0, T)$ of $\underline{u} - \phi$.

$$\partial_t\phi - \mu\underline{u} + F_*(D\phi, D^2\phi) + H(D\phi) \ge \theta_0 > 0 \qquad \text{in } \overline{B}_0. \tag{6.34}$$

Here $\overline{B}_0 \subset \mathbb{R}^N \times (T_0, T)$ is a sufficiently small closed ball centered at (x_0, t_0), and $\min_{\overline{B}_0}(\underline{u} - \phi) = 0$. From (2.3) and the Taylor expansion of ϕ, we have

$$\begin{aligned} u^\varepsilon(z) \ge \frac{1}{1+\mu\varepsilon^2} \inf_{p,X} \sup_w \Big\{ &(u^\varepsilon - \phi)(z + \zeta_\varepsilon(w)) + \phi(z) \\ &+ \varepsilon w \cdot (D\phi(z) - p) + \frac{\varepsilon^2}{2}\langle (D^2\phi(z) - X)w, w\rangle \\ &+ \varepsilon^2\partial_t\phi(z) + \varepsilon^2 F(p, X) + \varepsilon^2 H(p) - C\varepsilon^{9/4}\Big\}. \end{aligned} \tag{6.35}$$

Here we set $z := (x, t) \in \overline{B}_0$ and $\zeta_\varepsilon(w) := (\varepsilon w, \varepsilon^2)$, and the positive constant C depends only on the C^3 norm of ϕ in $\overline{B}_0$. We take a sufficiently large constant $R_0 > 0$ so that $||D\phi||_{L^\infty(B_0)}, ||D^2\phi||_{L^\infty(B_0)} \le R_0$.

At first, we consider the case $D\phi(z_0) = D\phi(x_0, t_0) \ne 0$. In advance, if we choose a sufficiently small $\overline{B}_0$, then there exists a positive number η_0 such that $|D\phi| \ge \eta_0 > 0$ in $\overline{B}_0$. Hence there exists a sufficiently large $j_0 \in \mathbb{N}$ such that $|D\phi| \ge \eta_0 \ge j_0^{-1}$ holds in $\overline{B}_0$. Applying Lemma 4.6 to (6.35), there exists $\overline{w}$ such that

$$\begin{aligned} u^\varepsilon(z) \ge \frac{1}{1+\mu\varepsilon^2} \inf_{p,X} \Big\{ &(u^\varepsilon - \phi)(z + \zeta_\varepsilon(\overline{w})) + \phi(z) \\ &+ \varepsilon^2\partial_t\phi(z) + \varepsilon^2 F_*(D\phi(z), D^2\phi(z)) \\ &+ \varepsilon^2 H(D\phi(z)) - h_{j_0}^\varepsilon\varepsilon^2 - C\varepsilon^{9/4}\Big\} \end{aligned} \tag{6.36}$$

holds. So we obtain

$$(u^\varepsilon - \phi)(z) \geq \frac{1}{1+\mu\varepsilon^2} \inf_{p,X} \Big\{ (u^\varepsilon - \phi)(z + \zeta_\varepsilon(\overline{w})) - \varepsilon^2 \mu \underline{u}(z) + \varepsilon^2 \partial_t \phi(z) + \varepsilon^2 F_*(D\phi(z), D^2\phi(z)) + \varepsilon^2 H(D\phi(z)) - h^\varepsilon_{j_0}\varepsilon^2 - C\varepsilon^{9/4} \Big\} \quad (6.37)$$

from (6.34) and holding $-\phi \geq -\underline{u}$ in $\overline{B}_0$. Thereby the estimate

$$(u^\varepsilon - \phi)(z) \geq \frac{1}{1+\mu\varepsilon^2} \inf_{p,X} \Big\{ (u^\varepsilon - \phi)(z + \zeta_\varepsilon(\overline{w})) + C^\varepsilon(j_0)\varepsilon^2 \Big\} \quad (6.38)$$

holds for sufficiently small ε where

$$C^\varepsilon(j_0) := \theta_0 - h^\varepsilon_{j_0} - C\varepsilon^{1/4}. \quad (6.39)$$

Note that $((u^\varepsilon)_* - \phi)(z + \zeta_\varepsilon(w))$ is lower semi-continuous on compact set with respect to w. And $((u^\varepsilon)_* - \phi)(z + \zeta_\varepsilon(\overline{w}))$ is bounded with respect to p, X $(0 < |p| \leq \varepsilon^{-1/4}, |X| \leq \varepsilon^{-1/2})$. In addition, since $|\overline{w}| \leq \varepsilon^{-1/4}$, taking an appropriate subsequence of (p_n, X_n) which approximates the infimum, we can find at least one $w^\varepsilon_0(z) := \lim_{i\to\infty} \overline{w}(\varepsilon, z, p_{n_i}, X_{n_i})$ such that

$$((u^\varepsilon)_* - \phi)(z + \zeta_\varepsilon(w^\varepsilon_0(z))) = \inf_{p,X}((u^\varepsilon)_* - \phi)(z + \zeta_\varepsilon(\overline{w}))$$

and $|w^\varepsilon_0(z)| \leq \varepsilon^{-1/4}$. For this $w_0 = w^\varepsilon_0(z)$, we have the bound

$$((u^\varepsilon)_* - \phi)(z) \geq \frac{1}{1+\mu\varepsilon^2}((u^\varepsilon)_* - \phi)(z + \zeta_\varepsilon(w_0)) + \frac{C^\varepsilon(j_0)\varepsilon^2}{1+\mu\varepsilon^2} \quad (6.40)$$

Next, we consider the case $D\phi(z_0) = 0$. Let $\mathcal{F} : \overline{B}_0 \to \mathbb{R}$ be the function denoted by

$$\mathcal{F}(\cdot) := \partial_t\phi(\cdot) - \mu\underline{u}(\cdot) + F_*(0, D^2\phi(\cdot)) + H(D\phi(\cdot)). \quad (6.41)$$

Then we can assume that $\mathcal{F}(z) \geq \theta_0$ for any $z \in \overline{B}_0$. From (6.35), we have

$$(u^\varepsilon - \phi)(z) \geq \frac{1}{1+\mu\varepsilon^2} \inf_{p,X} \sup_{w} \Big\{ (u^\varepsilon - \phi)(z + \zeta_\varepsilon(w)) - \varepsilon^2\mu\underline{u}(z) + \varepsilon w \cdot (D\phi(z) - p) + \frac{\varepsilon^2}{2}\langle (D^2\phi(z) - X)w, w\rangle + \varepsilon^2\partial_t\phi(z) + \varepsilon^2 F(p, X) + \varepsilon^2 H(p) - C\varepsilon^{9/4} \Big\}. \quad (6.42)$$

Applying Lemma 4.6, there exists $\overline{w}$ such that (1) if $|D\phi(z)| \geq j^{-1}$,

$$(u^\varepsilon - \phi)(z) \geq \frac{1}{1+\mu\varepsilon^2} \Big\{ (u^\varepsilon - \phi)(z + \zeta_\varepsilon(\overline{w})) + \varepsilon^2\partial_t\phi(z) - \varepsilon^2\mu\underline{u}(z) + \varepsilon^2 F(D\phi(z), D^2\phi(z)) + \varepsilon^2 H(D\phi(z)) - h^\varepsilon_j\varepsilon^2 - C\varepsilon^{9/4} \Big\} \quad (6.43)$$

yields and (2) if $|D\phi(z)| \leq j^{-1}$,

$$(u^\varepsilon - \phi)(z) \geq \frac{1}{1+\mu\varepsilon^2} \Big\{ (u^\varepsilon - \phi)(z + \zeta_\varepsilon(\overline{w})) + \varepsilon^2\mathcal{F}(z) - h^\varepsilon_j\varepsilon^2 - C\varepsilon^{9/4} \Big\} \quad (6.44)$$

yields where $j \in \mathbb{N}$. From **(F2)**,

$$F(D\phi(z), D^2\phi(z)) \geq F_*(0, D^2\phi(z))$$

holds in the case (1). Hence (6.44) holds for every $j \in \mathbb{N}$ and every cases. Hence there exists a vector $w_0 = w_0^\varepsilon(z)$ such that $|w_0| \leq \varepsilon^{-1/4}$ and

$$((u^\varepsilon)_* - \phi)(z) \geq \frac{1}{1+\mu\varepsilon^2}((u^\varepsilon)_* - \phi)(z + \zeta_\varepsilon(w_0)) + \frac{C^\varepsilon(j)\varepsilon^2}{1+\mu\varepsilon^2} \tag{6.45}$$

holds for any $z \in \overline{B}_0$. For fixed j, we can take ε is small enough so that $C^\varepsilon(j) \geq 0$, and set $v^\varepsilon(z) := ((u^\varepsilon)_* - \phi)(z)$ for $z \in \overline{B}_0$. Then we obtain

$$v^\varepsilon(z) \geq \frac{1}{1+\mu\varepsilon^2} v^\varepsilon(z + \zeta_\varepsilon(w_0)). \tag{6.46}$$

Let $X_0^\varepsilon = (x_0^\varepsilon, t_0^\varepsilon)$ be a point such that $X_0^\varepsilon \to X_0 := (x_0, t_0)$ as $\varepsilon \to 0$ and $\lim_{\varepsilon\to 0}(u^\varepsilon - \phi)(X_0^\varepsilon) = (\underline{u} - \phi)(X_0)$ and $X_k^\varepsilon = (x_k^\varepsilon, t_k^\varepsilon)$ be the sequence defined by

$$X_k^\varepsilon = X_{k-1}^\varepsilon + \zeta_\varepsilon(w_0^\varepsilon(X_{k-1}^\varepsilon)).$$

In advance, we take ε be small enough so that $X_0^\varepsilon \in \overline{B}_0$. From (6.46), if $X_1^\varepsilon, \cdots X_k^\varepsilon \in \overline{B}_0$, we have

$$v^\varepsilon(X_0^\varepsilon) \geq \left(\frac{1}{1+\mu\varepsilon^2}\right)^k v^\varepsilon(X_k^\varepsilon). \tag{6.47}$$

By the exactly same way as the previous proposition, we verify that there exists the minimal number $K \in \mathbb{N}$ such that $K \leq m$ and $X_K^\varepsilon \in \overline{B}_0$, $X_{K+1}^\varepsilon \notin \overline{B}_0$. Since (6.47) also holds for this number K, we obtain

$$v^\varepsilon(X_0^\varepsilon) \geq \left(\frac{1}{1+\mu\varepsilon^2}\right)^K v^\varepsilon(X_K^\varepsilon). \tag{6.48}$$

Since $(1+\mu\varepsilon^2)^{-K} \to \alpha \in (0,1]$ as $m \to \infty$, taking an appropriate subsequence, we get the following estimate as same as the previous proposition.

$$0 = (\underline{u} - \phi)(X_0) \geq \alpha(\underline{u} - \phi)(X_0'). \tag{6.49}$$

Here $X_0' \in \overline{B}_0 \setminus \{X_0\}$. This inequality implies that $\underline{u} - \phi$ has at least two minimal point in $\overline{B}_0$. Consequently we get a contradiction. Now the proof of Proposition 4.5 is completed. □

7 Construction of Viscosity Solution

Let Ω be a domain in $\mathbb{R}^N$ and $\partial_p Q$ be the parabolic boundary of $Q = \Omega \times (0,T)$ (i.e., $\partial_p Q = \partial\Omega \times [0,T) \cup \Omega \times \{t = 0\}$). If $\Omega = \mathbb{R}^N$, the parabolic boundary of Q is defined by $\mathbb{R}^N \times \{t = 0\}$. Assume that the function G satisfies following conditions.

(1) $G : [0,T] \times \mathbb{R} \times \mathbb{R}_*^N \times \mathcal{S}^N \to \mathbb{R}$ is continuous.

(2)
$$G(t,r,p,X) \leq G(t,r,p,Y) \qquad \text{for } X \geq Y,\ X, Y \in \mathcal{S}^N$$
and $t \in [0,T]$, $r \in \mathbb{R}$, $p \in \mathbb{R}_*^N$.

(3) $-\infty < G_*(t,r,0,O) = G^*(t,r,0,O) < \infty$.

(4) For some constant c_0,
$$r \mapsto G(t,r,p,X) + c_0 r$$
is a non-decreasing function.

Theorem 7.1 ([16, Theorem 3.1.4]). *Let u and v be respectively a sub- and supersolution of*

$$\partial_t u + G(t, u, Du, D^2u) = 0 \qquad \text{in } Q.$$

Assume that u and $-v$ are bounded from above on Q. Assume that

$$\lim_{\delta\to 0} \sup\{u^*(x,t) - v_*(y,s) \; ; \; |x-y| \le \delta, \; |t-s| \le \delta, \\ \operatorname{dist}((x,t), \partial_p Q) \le \delta, \; \operatorname{dist}((y,s), \partial_p Q) \le \delta, \\ (x,t), (y,s) \in \overline{\Omega} \times [0, T']\} \le 0 \tag{7.1}$$

for each $T' \in (0,T)$ and that $u^* > -\infty$, $v_* < \infty$ on $\partial_p Q$. Then

$$\lim_{\delta\to 0} \sup\{u^*(x,t) - v_*(y,s) \; ; \; |x-y| \le \delta, \; |t-s| \le \delta, \\ (x,t), (y,s) \in \overline{\Omega} \times [0, T']\} \le 0 \tag{7.2}$$

for each $T' \in (0,T)$.

Setting $G(t,r,p,X) = -F(p,X) - H(p) + \mu r$ (independent of t) and changing of variables with respect to the time, one can see that our conditions **(F1)**–**(F4)** and **(H)** satisfy the above conditions (1)–(4). By the contribution of this theorem, we obtain the uniquely existence of the viscosity solution and its uniform continuity. Indeed, let $T_0' \in (T_0, T)$ be an arbitrary-fixed constant and δ be a positive number. Then the following estimates yield for any $(x,t), (y,s) \in \mathbb{R}^N \times [T_0', T]$ such that $|T-t| \le \delta$, $|T-s| \le \delta$, $|t-s| \le \delta$ and $|x-y| \le \delta$.

$$\begin{aligned} \overline{u}(x,t) - \underline{u}(y,s) &= \big(\overline{u}(x,t) - \overline{u}(x,T)\big) + \big(\overline{u}(x,T) - \underline{u}(y,T)\big) + \big(\underline{u}(y,T) - \underline{u}(y,s)\big) \\ &\le \omega_0(T-t) + \big(\psi(x) - \psi(y)\big) + \omega_0(T-s) \\ &\le 3\omega_0(\delta) \end{aligned}$$

where ω_0 is the modulus of continuity of $\overline{u}$ and $\underline{u}$. In addition, since $\overline{u}$ and $\underline{u}$ are respectively a viscosity sub- and supersolution, the assumption (7.1) is satisfied. Hence (7.2) holds. Consequently we have the comparison inequality

$$\overline{u} \le \underline{u} \quad \text{in } \mathbb{R}^N \times [T_0', T] \tag{7.3}$$

for any $T_0' \in (T_0, T)$. Generally, $\underline{u} \le \overline{u}$ in $\mathbb{R}^N \times [T_0', T]$ holds from their definitions. Therefore $\overline{u} = \underline{u}$ yields. If we set $u = \overline{u} = \underline{u}$, then u is the viscosity solution of (TP) which belongs to the class $BUC(\mathbb{R}^N \times (T_0, T])$. This shows that the value function u^ε uniformly converges to u as $\varepsilon \to 0$ on any compact set in $\mathbb{R}^N \times (T_0, T]$. So the conclusion of Theorem 4.2 holds.

Remark 7.2. Actually, u can be extended as the viscosity solution in $\mathbb{R}^N \times [T_0, T]$, since it is well-defined at $t = T_0$ (see [16, Theorem 3.2.10]). Furthermore Theorem 7.1 implies the uniqueness of viscosity solution which has the uniform continuity. Consequently our viscosity solution $u = \lim_{\varepsilon\to 0} u^\varepsilon$ of (TP) is unique.

8 Proof of Key Lemma

In this section, we give a sketch of the proof. To obtain (4.4) and (4.5), we prove that the following properties hold for each cases. Assume that $(q,Y) \in \mathbb{R}^N \times \mathcal{S}^N$ with $|q|$, $|Y| \le R_0$. For any $(p,X) \in \mathbb{R}^N_* \times \mathcal{S}^N$ such that $|p| \le \varepsilon^{-1/4}$, $|X| \le \varepsilon^{-1/2}$ and $p \ne q$, $X \ne Y$, there exists $\overline{w} = \overline{w}(\varepsilon, p, q, X, Y)$ such that $|\overline{w}| \le \varepsilon^{-1/4}$ and

$$\varepsilon^{-1}\overline{w} \cdot (q-p) + \frac{1}{2}\langle (Y-X)\overline{w}, \overline{w}_0 \rangle + F(p,X) + H(p) \ge F(q,Y) + H(q) - h_1(\varepsilon^{1/4}) \tag{8.1}$$

holds for any $\varepsilon \leq \varepsilon_1$, if $|q| \geq 1/K$ and

$$\varepsilon^{-1}\overline{w} \cdot (q-p) + \frac{1}{2}\langle (Y-X)\overline{w}, \overline{w}\rangle + F(p,X) + H(p) \geq F_*(0,Y) + H(q) - h_2(\varepsilon^{1/4}) \tag{8.2}$$

holds for any $\varepsilon \leq \varepsilon_2$, if $|q| \leq 1/K$. Here $K \in \mathbb{N}$ is an arbitrary-fixed number, $\varepsilon_1 = \varepsilon_1(K, R_0, \lambda_0, \lambda_1, \lambda_2)$, $\varepsilon_2 = \varepsilon_2(R_0, N, \lambda_0, \lambda_1, \lambda_2)$ and then h_1 is the modulus depending on K, λ_2 and R_0, on the other hand, h_2 is the modulus depending on λ_2 and R_0.

Proof of Lemma 4.6. In what follows, we set the maximum eigenvalue of $Z \in \mathcal{S}^N$ as $\mathcal{E}(Z)$ to simplify. Assume that $p \neq q$ and $X \neq Y$. Using unit eigenvectors $\xi_0, \xi_1, \ldots \xi_{N-1} \in \mathbb{R}^N$ of $Y - X$, we can represent w with $|w| \leq \varepsilon^{-1/4}$ by

$$w = \sum_{i=0}^{N-1} s_i \xi_i$$

where $s_i \in \mathbb{R}$ $(i = 0, 1, \ldots N-1)$ with $s_0^2 + \cdots + s_{N-1}^2 \leq \varepsilon^{-1/2}$. In particular, let ξ_0 be the unit eigenvector which gives the maximum eigenvalue of $Y - X$. Thus $\varepsilon^{-2}Q^\varepsilon(w, p, X)$ is rewritten by

$$\varepsilon^{-1}s_0\xi_0 \cdot (q-p) + \varepsilon^{-1}\sum_{i=1}^{N-1} s_i\xi_i \cdot (q-p) + \frac{1}{2}s_0^2\mathcal{E}(Y-X)$$
$$+ \frac{1}{2}\sum_{i=1}^{N-1} s_i^2\langle (Y-X)\xi_i, \xi_i\rangle + F(p,X) + H(p). \tag{8.3}$$

Case 1. The case $|q| \geq 1/K$ for $K \in \mathbb{N}$.

(1-I) If $|p-q| \leq \varepsilon^{1/4}$, then we have $|p| \geq 1/2K$ for all sufficiently small ε so that $\varepsilon \leq C_1K^{-4}(C_1 = 16^{-1})$.

In the case $\mathcal{E}(Y-X) > 0$ (i.e., $\mathcal{E}^+(Y-X) = \mathcal{E}(Y-X)$), we take $|s_0| = \lambda_1$ and $s_i = 0$ for $i = 1, \ldots N-1$ in the formula (8.3) where $\lambda_1 \leq \varepsilon^{-1/4}$. Then it is rewritten by

$$\varepsilon^{-1}\lambda_1|\xi_0 \cdot (q-p)| + \frac{\lambda_1^2}{2}\mathcal{E}^+(Y-X) + F(p,X) + H(p). \tag{8.4}$$

Note that choosing an appropriate sign of s_0, we let the term $s_0\xi_0 \cdot (q-p)$ be non-negative. From **(F3)**, one can verify that for any $p \in \mathbb{R}^N_*$,

$$\frac{\lambda^2}{2}\mathcal{E}^+(Y-X) + F(p,X) \geq F(p,Y) \tag{8.5}$$

holds. From **(F4)** and **(H)**, in addition, we have the following estimates for the terms of F and H, since $|p| \geq 1/2K$.

$$F(p,Y) \geq F(q,Y) - \omega_0(\varepsilon^{1/4}), \tag{8.6}$$
$$H(p) \geq H(q) - \lambda_2\varepsilon^{1/4} \tag{8.7}$$

where $\omega_0 = \omega_{1/2K,R_0}$ is the modulus depending only on K and R_0, on the other hand, λ_2 is the Lipschitz constant of H. Substituting (8.5), (8.6) and (8.7) for (8.4), the formula (8.4) is estimated by

$$F(q,Y) + H(q) - h_1(\varepsilon^{1/4}) \tag{8.8}$$

from below where $h_1(s) = \omega_0(s) + \lambda_2 s$.

In the case $\mathcal{E}(Y-X)\le 0$ (i.e., $\mathcal{E}^+(Y-X)=0$ or $Y\le X$), we take $s_i=0$ for $i=0,1,\dots N-1$ in the formula (8.3). One can verify that $F(p,X)\ge F(p,Y)$ for any $p\in\mathbb{R}^N_*$ holds, since $-F$ is (degenerate) elliptic (see Remark 3.3). From (8.6) and (8.7), we see that it is also estimated by (8.8) from below in this case too. Consequently we have the formula (4.5) whenever $\varepsilon\le C_1K^{-4}$ in the case **(1-I)**. Here the positive constant C_1 also depends only on λ_1.

(1-II) If $|p-q|\ge\varepsilon^{1/4}$, then we can represent $(q-p)/|q-p|$ by

$$\frac{q-p}{|q-p|}=\sum_{i=0}^{N-1} r_i\xi_i \tag{8.9}$$

where $r_i\in\mathbb{R}$ with $r_0^2+r_1^2+\cdots+r_{N-1}^2=1$. Let us divide this case into two parts.

(i) The case $|\xi_0\cdot(q-p)|\ge(3\lambda_2/\lambda_1)\varepsilon^{1/2}$.

If $\mathcal{E}(Y-X)>0$, then we choose s_i so that $|s_0|=\lambda_1$, $s_i=0$ $(i=1,\dots N-1)$ and obtain the same formula as in (8.4). From the assumption in this case, we can estimate (8.4) by

$$3\lambda_2\varepsilon^{-1/2}+\frac{\lambda_1^2}{2}\mathcal{E}^+(Y-X)+F(p,X)+H(p) \tag{8.10}$$

from below, since $\varepsilon^{-1/2}\ge\varepsilon^{-1/4}$ by **(F3)**.

$$\begin{aligned}(8.10)\ &\ge 3\lambda_2\varepsilon^{-1/2}+F(p,Y)+H(p),\\ &\ge 3\lambda_2\varepsilon^{-1/2}-C(1+R_0)+H(q)-\lambda_2|p-q|,\\ &\ge 3\lambda_2\varepsilon^{-1/2}-C(1+R_0)+H(q)-2\lambda_2\varepsilon^{-1/4}.\end{aligned} \tag{8.11}$$

Note that $|F(p,Y)|\le C(1+R_0)$ holds for any $p\in\mathbb{R}^N_*$ from Remark 3.3, and if $R_0\le\varepsilon^{-1/4}$, then we obtain $|p-q|\le|p|+|q|\le 2\varepsilon^{-1/4}$. Here $C=C(\lambda_0,\lambda_1)$. The formula (8.11) is estimated by

$$\lambda_2\varepsilon^{-1/4}-C(1+R_0)+H(q) \tag{8.12}$$

from below. In addition, if ε is small enough so that $\lambda_2\varepsilon^{-1/4}\ge 2C(1+R_0)$, then we have the bound as follows.

$$(8.12)\ \ge C(1+R_0)+H(q)\ge F(q,Y)+H(q) \tag{8.13}$$

for all $\varepsilon\le C_2(1+R_0)^{-4}$ where the positive constant C_2 depends only on λ_0, λ_1 and λ_2.

If $\mathcal{E}(Y-X)\le 0$, we choose s_i so that $s_0=\varepsilon^{1/4}\lambda_1$, $s_i=0$ $(i=1,\dots N-1)$, and substitute these for (8.3). Then, the formula (8.3) is estimated by

$$3\lambda_2\varepsilon^{-1/4}+\frac{\lambda_1^2}{2}\varepsilon^{1/2}\mathcal{E}(Y-X)+F(p,X)+H(p) \tag{8.14}$$

from below. We verify that $\mathcal{E}(Y-X)\ge-(R_0+\varepsilon^{-1/2})$ and $F(p,X)\ge F(p,Y)$ hold, so the following estimates yield

$$\begin{aligned}(8.14)\ &\ge 3\lambda_2\varepsilon^{-1/4}-\frac{\lambda_1^2}{2}(\varepsilon^{1/2}R_0+1)+F(p,Y)+H(p),\\ &\ge 3\lambda_2\varepsilon^{-1/4}-C(1+R_0)+F(p,Y)+H(q)-2\lambda_2\varepsilon^{-1/4},\\ &=\lambda_2\varepsilon^{-1/4}-C(1+R_0)+F(p,Y)+H(q),\\ &\ge F(q,Y)+H(q)\end{aligned}$$

for any $\varepsilon \le C_2(1+R_0)^{-4}$ as same as the case **(i-a)**. Therefore we obtain the formula (4.5) in the case **(i)**.

(ii) The case $|\xi_0 \cdot (q-p)| \le (3\lambda_2/\lambda_1)\varepsilon^{1/2}$.

From (8.9) and the assumptions, we have the bound for r_0 as follows.

$$|r_0| = \left| \frac{\xi_0 \cdot (q-p)}{|q-p|} \right| \le \frac{3\lambda_2 \varepsilon^{1/4}}{\lambda_1} \quad (=: c_0 \varepsilon^{1/4}). \tag{8.15}$$

Since $r_0^2 + \cdots + r_{N-1}^2 = 1$, we have the inequality

$$1 - c_0^2 \varepsilon^{1/2} \le |r_1| + |r_2| + \cdots + |r_{N-1}| \tag{8.16}$$

where we take ε so that $c_0^2 \varepsilon^{1/2} < 1/2$, in advance. This inequality implies that there exists at least one number j_0 such that

$$|r_{j_0}| \ge \frac{1 - c_0^2 \varepsilon^{1/2}}{N-1} > \frac{1}{2N}. \tag{8.17}$$

Now we take s_i so that $s_i = 0$ $(i \ne 0, j_0)$ in the formula (8.3). Then we can rewrite it as follows.

$$\begin{aligned} &\varepsilon^{-1} s_0 \xi_0 \cdot (q-p) + \varepsilon^{-1} s_{j_0} \xi_{j_0} \cdot (q-p) + \frac{s_0^2}{2} \mathcal{E}(Y-X) \\ &\qquad + \frac{s_{j_0}^2}{2} \langle (Y-X)\xi_{j_0}, \xi_{j_0} \rangle + F(p,X) + H(p). \end{aligned} \tag{8.18}$$

We choose s_0 so that

$$|s_0| = \begin{cases} \lambda_1 & \text{if } \mathcal{E}(Y-X) > 0, \\ 0 & \text{if } \mathcal{E}(Y-X) \le 0 \end{cases} \tag{8.19}$$

and $s_0 \xi_0 \cdot (q-p) \ge 0$, in addition, take $|s_{j_0}| = \lambda_1 \varepsilon^{1/4}$ so that $s_{j_0} \xi_{j_0} \cdot (q-p) \ge 0$. Then the formula (8.18) is estimated by

$$\varepsilon^{-3/4} \lambda_1 |r_{j_0}| |q-p| + \frac{\lambda_1^2}{2} \varepsilon^{1/2} \langle (Y-X)\xi_{j_0}, \xi_{j_0} \rangle + F(p,Y) + H(p) \tag{8.20}$$

from below. Note that $|r_{j_0}|$ has the bound (8.17) and $|q-p| \ge \varepsilon^{1/4}$, then the following inequalities hold.

$$\begin{aligned} (8.20) &\ge \frac{\lambda_1 \varepsilon^{-1/2}}{2N} - C(\varepsilon^{1/2} R_0 + 1) + F(p,Y) + H(q) - 2\lambda_2 \varepsilon^{-1/4}, \\ &\ge \lambda_2 \varepsilon^{-1/4} - C(1+R_0) + F(p,Y) + H(q), \\ &\ge F(q,Y) + H(q). \end{aligned}$$

Here ε is small enough such that $(\lambda_1/2N)\varepsilon^{-1/2} \ge 3\lambda_2 \varepsilon^{-1/4}$ (i.e., $\varepsilon \le C_3$ where C_3 depends only on λ_0, λ_1 and λ_2) and $\varepsilon \le C_2(1+R_0)^{-4}$ hold. In particular, since $|q| \ge 1/K > 0$, we see $F(q,Y) = F_*(q,Y)$. Consequently if we set $\varepsilon_1 = \min\{C_1 K^{-4}, C_2(1+R_0)^{-4}, C_3\}$, then the formula (4.4) holds with $h_1(s) = \omega_0(s) + \lambda_2 s$ in the **Case 1.**

Case 2. The case $|q| \le 1/K$ for $K \in \mathbb{N}$.

Arguing the same as **Case 1**, we can have the estimate (8.2). Finally, we consider the case of $p = q$ or $X = Y$ for $q \in \mathbb{R}^N_*$. We can choose the sequences $\{p_k\} \subset \mathbb{R}^N_*$ and $\{X_n\} \subset S^N$ such that $p_k \to q$, $X_n \to Y$ as $k, n \to \infty$, respectively. Now let us set $w_k^n = w_0(\varepsilon, p_k, q, X_n, Y)$. Then $\{w_k^n\}$ has a subsequence which converges to some point as $k \to \infty$ or as $n \to \infty$. In the formula (4.5), since right-hand side is independent of p, X, we verify that in the case of $p = q$ or $X = Y$, the conclusion of the lemma holds by taking $\overline{w}$ as the limit of a subsequence $\{w_k^n\}$ as $k \to \infty$ or $n \to \infty$. □

References

[1] H. Ishii, On Uniqueness and Existence of Viscosity Solutions of Fully Nonlinear Second-Order Elliptic PDE's, in *Comm. Pure Appl. Math.*, **42** (1989) 15-45.

[2] H. Ishii and P.-L. Lions, Viscosity Solutions of Fully Nonlinear Second-Order Elliptic Partial Differential Equations, in *J. Diff. Equa.*, **83** (1990) 26-78.

[3] L. A. Caffarelli and X. Cabré, Fully Nonlinear Elliptic Equations, Colloquium Publications **43** American Mathematical Society (1995).

[4] L. C. Evans and J. Spruck, Motion of Level Sets by Mean Curvature. I, in *J. Diff. Geom.*, **33** (1991) 635-681.

[5] L. C. Evans, Partial Differential Equations, GSM **19** American Mathematical Society (1998).

[6] L. C. Evans, Convergence of an Algorithm for Mean Curvature Motion, in *Indiana Univ. Math.*, **42** (1993) 533-557.

[7] M. Bardi and I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Birkhäuser (1997).

[8] M. G. Crandall and P.-L. Lions, Viscosity Solutions of Hamilton–Jacobi Equations, in *Trans. Amer. Math. Soc.*, **277** (1983) 1-42.

[9] M. G. Crandall, H. Ishii and P.-L. Lions, User's Guide to Viscosity Solutions of Second Order Partial Differential Equations, in *Bull. Amer. Math. Soc.*, **27** (1992) 1-67.

[10] M. Rudd, Game-Theoretic Schemes for Generalized Curvature Flows in the Plane (2007) preprint.

[11] R. Buckdahn, P. Cardaliaguet and M. Quincampoix, A representation formula for the mean curvature motion, in *SIAM J. Math. Anal.*, **33** (2001) 827-846.

[12] R. V. Kohn and S. Serfaty, A deterministic-control-based approach to motion by curvature, in *Comm. Pure Appl. Math.*, **59** (2006) No. 3 344-407.

[13] R. V. Kohn, Parabolic PDEs and Deterministic Games, The presentation in ICIAM07-Zurich-July 2007., *Courant Institute, NYU,*
web site http://www.math.nyu.edu/faculty/kohn/papers/iciam-talk.pdf.

[14] R. V. Kohn and S. Serfaty, "Second-order PDE's and Deterministic Games," to appear in Proceedings ICIAM07.

[15] S. Koike, A Beginner's Guide to the Theory of Viscosity Solutions, MSJ Memoirs **13** Mathematical Society of Japan (2004).

[16] Y. Giga, Surface Evolution Equations -A Level Set Approach-, Birkhäuser **99** Monographs in Mathematics (2006).

[17] Y. G. Chen, Y. Giga and S. Goto, Uniqueness and Existence of Viscosity Solutions of Generalized Mean Curvature Flow Equations, in *J. Diff. Geom.*, **33** (1991) 749-786.

[18] Y. Giga and Q. Liu, A Remark on the Discrete Deterministic Game Approach for Curvature Flow Equations, in *Eprint Series of Department of Mathematics, Hokkaido University* (2008).