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A Survey on Axiomatic Development of Lexicographic Expected Utility *

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Abstract

The objective of this article is to survey the axiomatic development of the decision making under uncertainty. We review two well-known models of decision making: Subjective Expected Utility Model and The Criterion of Admissibility.

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1 Introduction

The objective of this article is to survey the axiomatic development of the decision making under uncertainty. We review two well-known models of decision making: Subjective Expected Utility Model and The Criterion of Admissibility.

Admissibility is a criterion of rationality that is widely used in decision and game theory.[1] Roughly, it is the requirement that "weakly dominated" actions should not be taken. In other words, one action should be preferred to another if the outcome

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[1]See for example, Arrow [1], Luce and Raiffa [12], Kohlberg and Mertens [9], and Brandenberger, Friedenberg, and Keisler [3].

of the first action is at least as good as that of the second action for each state, and strictly better for at least one state.

In the theory of subjective probability, Savage derives unique probability over objective states from preference and provides an axiomatic foundation for subjective expected utility theory. Subjective expected utility models satisfy admissibility only if there is no null state. This assumption is restrictive because such preferences would rule out pure strategy equilibria in games.[2] In an Anscombe–Aumann framework, Blume, Brandenburger, and Dekel [2] (henceforth, BBD) develop a non-Archimedean subjective probability model that allows for both the criterion of admissibility and "null" events, although not in the sense of Savage. In their model, the agent has a lexicographic hierarchy of subjective probabilities over objective states and may think that some states are "infinitely less relevant" than others. Unless two actions are indifferent in terms of all states in the first hierarchy, the agent does not care about outcomes in the other states. The agent thinks of "null" states as infinitely less relevant, but does not entirely exclude them from consideration.

A restrictive feature of BBD is the exogenous state space. Kreps [10, 11] shows how the ranking of menus of alternatives reveals subjective uncertainty. Building on that, Dekel, Lipman, and Rustichini [4] (henceforth, DLR) endogenize the state space in an Archimedean framework. DLR take preference over menus of lotteries as a primitive and derive a unique subjective state space, corresponding to possible future preferences over lotteries. Higashi and Hyogo [8] provide a non-Archimedean model with subjective states, which in principle enables us to use admissibility criterion based on the subjective state space.

2 Expected Utility Models with Objective State Space

2.1 Anscombe-Aumann Model

Anscombe-Aumann Model include the following primitives:

- Ω: finite set of *objective* states
- B: finite set of prizes, let $|B| = B$[3]
- $\Delta(B)$: set of probability measures over B, it is compact metric under the weak convergence topology; a generic element is denoted by β and referred to as a *lottery*
- $\mathcal{H}$: set of Anscombe-Aumann acts $h : \Omega \to \Delta(B)$

[2] In complete information games, one can think of states as other agents' pure strategy profiles.
[3] The set of prizes need not to be finite. Under slight modification, all results remain the same.

- preference $\succsim^*$ is defined on $\mathcal{H}$

Here, the state space is exogenously given. In other words, it is assumed observable by the modeler. Hence, we call it the objective state space.

The following are the main axioms in Anscombe-Aumann Model.

Axiom 1 (Order). $\succsim^*$ is complete and transitive.

Note that $\mathcal{H}$ is a mixture space under componentwise mixture operation.

Axiom 2 (Independence). For all $h, h', h'' \in \mathcal{H}$ and $\lambda \in (0,1)$,

$$h \succsim^* h' \Leftrightarrow \lambda h + (1-\lambda)h'' \succsim^* \lambda h' + (1-\lambda)h''.$$

Axiom 3 (Nontriviality). There exist x and x' such that $x \succ x'$.

Let $h_E h''$ denote a mapping such that $h_E h''(\omega) = h(\omega)$ if $\omega \in E$ and $h_E h''(\omega) = h''(\omega)$ otherwise. We define the notion of conditional preferences $\succsim^*_E$ for every $E \subset \Omega$.

Definition 1. $h \succsim^*_E h'$ if, for some $h'' \in \mathcal{H}$, $h_E h'' \succsim^* h'_E h''$.

The next axiom needs the notion of null event.

Definition 2 (Null event). The event $E \subset \Omega$ is null if $h \sim_E h'$ for all $h, h' \in \mathcal{H}$.

Axiom 4 (State Independence). For all non-null states $\omega, \omega' \in \Omega$ and $p, q \in \Delta(B)$, $p \succsim^*_\omega q$ if and only if $p \succsim^*_{\omega'} q$

The following is a weaker version of continuity:

Axiom 5 (Archimedean Property). If $h \succ^* h' \succ^* h''$, then there exists $0 < \alpha < \beta < 1$ such that $\beta h + (1-\beta)h'' \succ^* h' \succ^* \alpha h + (1-\alpha)h''$.

With all the above axioms, we have the following result.

Theorem 2.1. *$\succsim^*$ satisfies Order, Independence, Nontriviality, State Independence, and Archimedean Property if and only if there is an affine function $u : \Delta(B) \to R$ and a probability measure p over Ω such that*

$$h \succsim^* h' \Leftrightarrow \sum_{\omega \in \Omega} p(\omega) u(h(\omega)) \geqq \sum_{\omega \in \Omega} p(\omega) u(h'(\omega)).$$

2.2 The BBD Model

BBD weakens Archimedean Property so that it holds only on conditional preferences over a state.

Axiom 6 (Conditional Archimedean Property). For each $\omega \in \Omega$, if $h \succ^*_\omega h' \succ^*_\omega h''$, then there exists $0 < \alpha < \beta < 1$ such that $\beta h + (1-\beta)h'' \succ^*_\omega h' \succ^*_\omega \alpha h + (1-\alpha)h''$.

By this weakening of Archimedean Property, a numerical expected utility representation is not always possible. Thus we have a lexicographic expected utlility representation.

Theorem 2.2. *$\succsim^*$ satisfies Order, Independence, Nontriviality, State Independence, and Conditional Archimedean Property if and only if there is an affine function $u : \Delta(B) \to R$ and a hierarchy of probability measures $\{p_k\}_{k=1}^K$ over Ω such that*

$$h \succsim^* h' \Leftrightarrow (\sum_{\omega \in \Omega} p_k(\omega)u(h(\omega)))_{k=1}^K \geqq_L (\sum_{\omega\in\Omega} p_k(\omega)u(h'(\omega)))_{k=1}^K.$$

3 Expected Utility Models with Subjective State Space

3.1 The DLR Model

DLR include the following primitives:

- B: finite set of prizes, let $|B| = B$
- $\Delta(B)$: set of probability measures over B, it is compact metric under the weak convergence topology; a generic element is denoted by β and referred to as a *lottery*
- $\mathcal{X}$: set of closed nonempty subsets of $\Delta(B)$, it is endowed with the Hausdorff topology; a generic element is denoted by x and called a *menu*[4]
- preference $\succsim$ is defined on $\mathcal{X}$

Note that a state space is not exogenously given here. Instead, we define preference over menus. The interpretation is as follows: At time 0 (ex ante), the agent chooses a menu according to $\succsim$. At time 1 (ex post), a subjective state is realized and then she chooses a lottery out of the previously chosen menu. Note that the ex post stage is not a primitive of the formal model. However, since the agent is forward looking, her ex ante choice of menus reflects her subjective perception of states. Therefore, preference $\succsim$ over menus reveals a subjective state space.

The following are the main axioms in DLR.

[4]DLR do not restrict menus to be closed. If we allow any subset to be a menu, then we have to modify the definition of *critical set*. Under slight modification, all results remain the same.

Axiom 7 (Order). $\succsim$ is complete and transitive.

We define the mixture of two menus for a number $\lambda \in [0, 1]$ by

$$\lambda x + (1 - \lambda)x' = \{\lambda\beta + (1 - \lambda)\beta' | \beta \in x, \ \beta' \in x'\}.$$

The following is a version of the Independence Axiom adapted to a model with preference over menus.

Axiom 8 (Independence). For all $x, y, z \in \mathcal{X}$ and $\lambda \in (0, 1)$,

$$x \succsim y \Leftrightarrow \lambda x + (1 - \lambda)z \succsim \lambda y + (1 - \lambda)z.$$

Axiom 9 (Nontriviality). There exist x and x' such that $x \succ x'$.

Axiom 10 (Continuity). For every menu x, the sets $\{x' \in \mathcal{X} | x' \succsim x\}$ and $\{x' \in \mathcal{X} | x \succsim x'\}$ are closed.

The next axiom is introduced by Dekel, Lipman, and Rustichini [5] (henceforth DLR2) to ensure, together with the other axioms, the finiteness of the state space. Let $\text{conv}(x)$ denote the convex hull of x.

Definition 3. A set $x' \subset \text{conv}(x)$ is *critical for* x if for all menus y with $x' \subset \text{conv}(y) \subset \text{conv}(x)$, we have $y \sim x$.

Axiom 11 (Finiteness). Every menu has a finite critical subset.

The intuition is that when the agent faces a menu and contemplates future contingencies, she cares about only finitely many possibilities. Note that the set of states she cares about could depend on the menu. Therefore, this axiom does not imply finiteness of the subjective state space by itself.

Now, we explain a finite state space version of DLR's model. Let S be a state space. A function $U : \Delta(B) \times S \to R$ is a *state-dependent utility function* if $U(\beta, s)$ has an expected utility form, that is, for $\beta \in \Delta(B)$,

$$U(\beta, s) = \sum_{b \in B} \beta(b) U(b, s).$$

Consider the functional form $W : \mathcal{X} \to R$ defined by

$$W(x) = \sum_{s \in S} \mu(s) \max_{\beta \in x} U(\beta, s), \tag{1}$$

where μ is a measure on S.

Note that S is just an index set though we call it the state space. Given the pair (S, U), define the ex post preference $\succsim_s^*$ over $\Delta(B)$ by

$$\beta \succsim_s^* \beta' \Leftrightarrow U(\beta, s) \geq U(\beta', s),$$

and let

$$P(S,U) = \{\succsim_s^* | s \in S\}.$$

Following DLR, we refer to the set of ex post preferences $P(S,U)$ as the *subjective state space.*

In general, there are many functional forms (1) that represent the same preference on $\mathcal{X}$. In order to obtain the uniqueness property, DLR concentrate on "relevant" subjective states: given a representation of the form (1), a state s is *relevant* if there exist menus x and y such that $x \not\sim y$ and that for every $s' \neq s$, $\max_{\beta \in x} U(\beta, s') = \max_{\beta \in y} U(\beta, s')$.

Definition 4. A *finite additive representation* (S, U, μ) is a tuple consisting of a nonempty finite state space S, a state-dependent utility function $U : \Delta(B) \times S \to R$, and a measure μ such that (i) $\succsim$ is represented by the functional form $W : \mathcal{X} \to R$, (ii) every state $s \in S$ is relevant, and (iii) if $s \neq s'$, then $\succsim_s^* \neq \succsim_{s'}^*$.

DLR and DLR2 prove

Theorem 3.1. *$\succsim$ satisfies Order, Independence, Nontriviality, Continuity, and Finiteness if and only if it has a finite additive representation.*

Corollary 3.2. *Suppose $\succsim$ has a finite additive representation. Then all finite additive representations of $\succsim$ have the same subjective state space.*

Axiom 12 (Monotonicity). If $x \subset x'$, then $x' \succsim x$.

Monotonicity states that the agent values the flexibility of having more options. The consequence of Monotonicity is the following.

Corollary 3.3. *$\succsim$ satisfies Monotonicity and the axioms in Theorem 3.1 if and only if it has a finite additive representation with a positive measure μ.*

3.2 Continuity and a hierarchy of hypotheses

In this section, we argue that the axiom Continuity is not always compelling.

The intuition against Continuity is as follows: Suppose that a menu x is strictly preferred to a menu x'. Consider an agent who perceives some subjective contingencies and who has, in her mind, several hypotheses about these contingencies. Think of a hypothesis as a (singed) measure over contingencies that is used to weight the valuation of outcomes across states.[5] She may view one hypothesis is "infinitely less relevant" than another. Think of this as being captured by a hierarchy of hypotheses. Then there is a critical level k^* such that x and x' are indifferent according to each hypothesis at level k less than k^*, but x is strictly better than x' according to the

[5] As explained later, a hypothesis in the formal model does not corresponds to beliefs about states, and thus we refer instead to "weights".

hypothesis at level k^*. Now consider a "small" variation of x, denoted by x_ϵ. Then she should rank x_ϵ strictly better than x' using only the contingencies derived by the hypothesis at level k^*. However, the critical level for comparing x' and x_ϵ may be different than k^*; x' could be better than x_ϵ according to the hypothesis at the new critical level. Therefore the small deviation might change the ranking between the menus.

The following examples are provided to illustrate this intuition.

Example: Consider an agent who used to like peanut butter very much, but who has an allergy to peanut now. Moreover, when she chooses an orange, she will pick the one which is more likely to be sweet.

There are three alternatives: the first one is an orange o_ϵ which turns out to be sweet with probability $0.9 + \epsilon$ and sour with $0.1 - \epsilon$; the second one is an orange o which turns out to be sweet with probability 0.9 and sour with 0.1; the last one is bread with peanut butter, which is denoted by p.

Then she may have the following ranking: for every $\epsilon \in (0, 0.1]$

$$\{o_\epsilon\} \succ \{o, p\} \succ \{o\}. \tag{2}$$

The intuition is that she has two hypotheses for her allergy: the first is that allergy continues, and the second is that her allergy disappears. However, she thinks that it is infinitely less relevant to take into account the possibility that her allergy disappears. That is, she would rank the two hypotheses hierarchically in her mind.

First, consider the first and second menus. Since flexibility provided by bread with peanut butter is irrelevant in the primary hypothesis, the ranking of the first and second menus follows the taste of orange. Hence, the agent prefers the first menu to the second one.

Next, consider the second and third menus. At first, the agent uses the primary hypothesis to rank the menus. Since two menus are indifferent in the primary hypothesis, the ranking of menus in the secondary hypothesis is relevant for her choice among menus. Thus she wants to retain the opportunity to have peanut butter. The agent prefers the second menu to the third one.

Ranking (2) violates Continuity.

3.3 The Higashi and Hyogo Model

In the previous section, the difficulties for Continuity arise out of the strict preference relation. Therefore, we impose "continuity" only for indifference sets.

Axiom 13 (Indifference Continuity). For every menu x, the indifference set $\{x' \in \mathcal{X} | x' \sim x\}$ is closed.

There is no corresponding axiom in BBD. The reason is that BBD assume that the state space is exogenous and finite. In Higashi and Hyogo, the state space is derived endogenously from preference.

Since we weaken Continuity, a numerical representation is not always possible. We consider a lexicographic representation that compares a vector of utilities assigned to a menu by $\geq_L$.[6] More formally, let S and $U : \Delta(B) \times S \to R$ be a state space and a state-dependent utility function. Consider the vector-valued function $V : \mathcal{X} \to R^K$ defined by

$$V(x) = \left(\sum_{s \in S} \mu_k(s) \max_{\beta \in x} U(\beta, s) \right)_{k=1}^{K}, \tag{3}$$

where $\{\mu_k\}_{k=1}^K$ is a hierarchy of measures. This vector-valued function is the counterpart of the DLR functional form (1). As in DLR, we concentrate on "relevant" subjective states: given a representation of the form (3), a state s is *relevant* if there exist menus x and y such that $x \not\sim y$ and that for every $s' \neq s$, $\max_{\beta \in x} U(\beta, s') = \max_{\beta \in y} U(\beta, s')$.

Definition 5. A *lexicographic representation* $(S, U, \{\mu_k\}_{k=1}^K)$ is a tuple consisting of a nonempty finite state space S, a state dependent-utility function $U : \Delta(B) \times S \to R$, and a hierarchy $\{\mu_k\}_{k=1}^K$ of measures such that

$$\text{(i) } x \succsim y \Leftrightarrow \left(\sum_{s \in S} \mu_k(s) \max_{\beta \in x} U(\beta, s) \right)_{k=1}^{K} \geq_L \left(\sum_{s \in S} \mu_k(s) \max_{\beta \in y} U(\beta, s) \right)_{k=1}^{K},$$

(ii) every state $s \in S$ is relevant, and (iii) if $s \neq s'$, then $\succsim_s^* \neq \succsim_{s'}^*$.

The integer K is referred to as the *length* (of the hierarchy).

Higashi and Hyogo [8] shows the following:

Theorem 3.4. $\succsim$ *satisfies Order, Independence, Nontriviality, Indifference Continuity, and Finiteness if and only if it has a lexicographic representation.*

For interpretation, note that the ex post behavior is as in DLR: a state s in S will be realized at the beginning of time 1. Then she will choose the best alternative out of the previously chosen menu according to the ex post utility function $U(\cdot, s)$. Moreover, she anticipates this ex post behavior at time 0. The difference from DLR is how she perceives subjective contingencies ex ante. The agent has a hierarchy of measures in her mind. Each level of the hierarchy represents her hypothesis about how she should allow for the future contingencies ex ante. The measure μ_1 indicates her primary hypothesis. She has a secondary hypothesis, which is represented by μ_2. If menus are indifferent according to her primary hypothesis, she compares them according to her secondary hypothesis. She has a tertiary hypothesis, which is represented by μ_3, and

[6] For $a, b \in R^K$, $a \geq_L b$ if and only if whenever $b_k > a_k$, there is a $j < k$ such that $a_j > b_j$.

so on. Since μ_k enters into the ranking of any two menus x and y only if x and y are indifferent according to $\mu_1, \cdots, \mu_{k-1}$, the measure μ_k is relevant but may be thought of as being "infinitely less relevant" than $\mu_1, \cdots, \mu_{k-1}$.

To further illustrate the meaning of "μ_k is infinitely less relevant than μ_{k-1}," consider the special case where there is no overlap among the supports of μ_k's. Suppose that s_{k-1} and s_k belong to $\text{supp}(\mu_{k-1})$ and $\text{supp}(\mu_k)$ respectively. Consider two menus x and y such that the agent expects the same ex post utilities at all states except s_{k-1} and s_k. Then the ex post ranking between x and y at s_{k-1} determines the ex ante ranking regardless of the ex post ranking at s_k. This leads us to say "s_k is infinitely less relevant than s_{k-1}." In the Archimedean case, as in DLR, every state is either relevant or not. Higashi and Hyogo admits a richer comparison between subjective states. That is, there may be a state which is relevant but infinitely less relevant than another state.

Uniqueness of the representation does not hold in general. For example, the μ_k's are not uniquely determined by preference, just as in DLR. Secondly, there may be redundancies in the hierarchy [2, p. 66].

To express the uniqueness properties of the representation, define for each $k = 1, \ldots, K$,

$$P_k(S, U, \{\mu_k\}_{k=1}^{K}) = \{\succsim_s^* \,|\, s \in \cup_{j=1}^{k}\text{supp}(\mu_j)\} \subset P(S, U).$$

Following DLR, we can think of $\{P_k(S, U, \{\mu_k\}_{k=1}^{K})\}_{k=1}^{K}$ as a hierarchy of (incomplete) subjective state spaces. Note that there is a lexicographic representation with minimal length K, denoted by K^*. To avoid the redundancies, we concentrate on lexicographic representations of minimal length K^*.

Corollary 3.5. *Suppose that* $\succsim$ *admits a lexicographic representation. Let* $(S, U, \{\mu_k\}_{k=1}^{K^*})$ *and* $(S', U', \{\mu'_k\}_{k=1}^{K^*})$ *be lexicographic representations of* $\succsim$ *with the minimal length* K^*. *Then, for* $k = 1, \cdots, K^*$,

$$P_k(S, U, \{\mu_k\}_{k=1}^{K^*}) = P_k(S', U', \{\mu'_k\}_{k=1}^{K^*}).$$

The next axiom is the difference between Higashi and Hyogo's model and DLR's finite additive representation.

Axiom 14 (Upper Semicontinuity). For every menu x, the upper contour set $\{x' \in \mathcal{X} | x' \succsim x\}$ is closed.

Theorem 3.6. $\succsim$ *has a lexicographic representation and satisfies Upper Semicontinuity if and only if it has a finite additive representation (as in DLR).*

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