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Gröbner bases on projective bimodules and the Hochschild cohomology *

Part IV. (Co)homology

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This is a continuation of the previous papers [3], [4] and [5]. We develop the theory of Gröbner bases on projective modules over an algebra based on a well-ordered semigroup. We construct resolutions of modules admitting Gröbner bases. This gives an effective way to compute the (co)homology of such modules.

14 Suitable orders

Let $S = B \cup \{0\}$ be a well-ordered reflexive semigroup with 0 and K be a commutative ring with 1. Let $F = K \cdot B$ be the K-algebra based on B and let I be a (two-sided) ideal of F. Let $A = F/I$ be the quotient algebra of F by I and $\rho : F \to A$ be the natural surjection. We fix a (reduced) Gröbner basis G of I.

Let X be an left edged set and $F{\cdot}X$ be the projective left F-module generated by X. Assume that a left compatible well-order $>$ on $B \cdot X$ is given and it is extended to a partial order $\succ$ on $F \cdot X$ in a natural way. The leading term of $f \in F \cdot X$ with respect to $\succ$ is denoted by $\mathrm{lt}(f)$.

Let H be a set of monic left uniform elements of $F \cdot X$, which is considered to be a left edged set. Let $F{\cdot}H$ be the projective left F-module generated by H. For $h \in H$, $[h]$ denotes the formal generator of $F \cdot H$ corresponding to $h \in H$.

We define a (strict) partial order $>'$ on $B \cdot X$ as follows. For $x[h], x'[h'] \in B \cdot H$, such that $x \cdot \mathrm{lt}(h) \neq 0$ and $x' \cdot (h') \neq 0$, define $x[h] >' x'[h']$ if and only if

(i) $x \cdot \mathrm{lt}(h) > x' \cdot \mathrm{lt}(h')$, or
(ii) $x \cdot \mathrm{lt}(h) = x' \cdot \mathrm{lt}(h')$ and $x > x'$.

Clearly, this partial order is well founded. Let $L'(H)$ (resp. $L''(H)$) be the K-subspace of $F \cdot H$ spanned by

$$\{x[h] \in B \cdot X \mid x \cdot \mathrm{lt}(h) \neq 0\} \quad (\text{resp. } \{x[h] \in B \cdot X \mid x \cdot \mathrm{lt}(h) = 0\}).$$

*This is a preliminary report and the details appear elsewhere.

Easily we see that $L''(H)$ is an F-submodule of $F \cdot H$ and

$$F \cdot H = L'(H) \oplus L''(H)$$

holds.

The partial order $>'$ is total on $\{x[h] \in B \cdot X \mid x \cdot \mathrm{lt}(h) \neq 0\}$, and is extended to a partial order $\succ'$ on $L'(H)$ in the same way as we did on $F \cdot X$. The partial order $\succ'$ satisfies the following weak compatibility. For $f, g \in L'(H)$ and $a, b \in B$

(1) $f \succ' g, axw \neq 0 \Rightarrow (af)' \succ' (ag)'$, and

(2) $a > b, axw \neq 0 \Rightarrow (af)' \succ' (bf)'$,

where $(af)', (ag)'$ and $(bf)'$ are the projections of af, ag and bf to $L'(H)$ respectively.

A left compatible well-order $>$ on $B \cdot H = \{x[h] \mid x \in B, h \in H\}$ is *suitable*, if

(i) it extends the partial order order $>'$ on $L'(H)$, and

(ii) $x[h] \succ \int(xt)$ for any $x \in B$ and $h = w\xi - t \in H$,

where $\succ$ is the partial order on $F \cdot H$ naturally extended from $>$. So, a well-order $>$ on $F \cdot H$ is suitable, if for any $x, x', a, b \in B$ and $h = w\xi - t, h' = w'\xi' - t' \in H$ with $w, w' \in B$, $\xi, \xi' \in X$ and $t, t' \in F \cdot X$, the following conditions are satisfied:

(iii) $x[h] > x'[h'], yx \neq 0, yx' \neq 0 \Rightarrow yx[h] > yx'[h']$,

(iv) $a > b$, $ax \neq 0$, $bx \neq 0 \Rightarrow ax[h] > bx[h]$.

(v) $xw \neq 0, x'w' \neq 0$, $xw > x'w'$ or $(xw = x'w', x > x') \Rightarrow x[h] > x'[h']$,

and (ii) above.

Remark that if $xw \neq 0$, the inequality $x[h] \succ \int(xt)$ in (ii) follows from (iii).

If the base semigroup S is coherent, that is, $xy \neq 0$ for any $x, y \in B$ with $\tau(x) = \sigma(y)$, then $F \cdot H = L'(H)$ and $>'$ is a total order on $B \cdot H$, and hence $>'$ itself is suitable. We do not know the general condition for the existence of a suitable well-order. In the next section we assume that $>$ is a suitable well-order on $B \cdot H$, and it is extended to a partial order $\succ$ on $F \cdot H$. For a nonzero $f \in LF \cdot H$, $\mathrm{lt}(f)$ denotes the maximal term of f with respect to $\succ$, and set $\mathrm{rt}(f) = f - \mathrm{lt}(f)$.

15 Gröbner basis made from critical pairs and critical z-elements

Let $h = w\xi - t, h' = w'\xi - t' \in H$ and $x, x' \in B$ such that $xw = x'w' \neq 0$, the appearance (x, w) is at the immediate right of (x', w') in xw and x and x' are left coprime, then we have the critical pair of the first kind and the element

$$c_1 = x[h] - x'[h'] + \int(x \cdot t) - \int(x't')$$

in (11.1) ([5]). Since $(x, w\xi) > (x', w'\xi)$, $x[h] > x'[h']$ by (iii) above. Moreover, $x[h] \succ \int(xt)$ and $x'[h'] \succ \int(x't')$ by (ii) (or (iii)). Thus, $\mathrm{lt}(c_1) = z[h]$.

Let $u - v \in G$ and $x, y, y' \in B$ such that $xw = yuy' \neq 0$, $(x, w\xi)$ is rightmost in $xw\xi$, (y, u, y') is rightmost in xw, and x and y are coprime, then we have the cortical pair of the second kind and the element

$$c_2 = x[h] + \int(x \cdot t) - \int(yvy'\xi)$$

in (11.2). We have $x[h] \succ \int(xt)$ and $x[h] \succ \int(yv'y'\xi)$ because $xw\xi \succ xt$ and $xw\xi \succ yvy'\xi$. Thus, $\mathrm{lt}(c_2) = z[h]$.

Let $z \in B$ be such that $xw = 0$, then we have a z-element zt. This situation is *critical*, if there is no nonidempotential left factor y of z; $z = yz'$ such that $y'w = 0$. In this case we call zt a *critical z-element*, and we have the element

$$c_3 = z[h] + \int(z \cdot t)$$

in (11.3) made from a critical z-element. We see $\mathrm{lt}(c_3) = z[h]$ by (ii).

Let C be the set of the elements c_1, c_2 made from critical pairs together with the elements c_3 made form critical z-elements.

Let $\delta : F \cdot H \longrightarrow F \cdot X$ be the morphism of left F-modules defined by $\partial_1([h]) = h$ for $h \in H$, and let $\rho : F \cdot X \longrightarrow A \cdot X$ be the canonical surjection. Let $\mathcal{K} = \mathrm{Ker}(\delta \circ \rho)$.

Theorem 15.1. *If H is a Gröbner basis on $F \cdot X$ and $>$ is a suitable well-order on $B \cdot H$, then the set C is a Gröbner basis on $F \cdot H$ of the kernel $\mathcal{K}$ modulo G.*

Under the existence of a suitable order we can strengthen Theorem 11.3 in [5] as follows. Remark that the set C here excludes z-elements that are not critical.

Corollary 15.2. *If H is a Gröbner basis and $\succ$ is a suitable order on $F \cdot H$, then C generates $\mathcal{K}$ modulo G.*

16 Projective resolutions

Let M be a left A-module defined by a Gröbner basis H on the projective left A-module $A \cdot X$ generated by a left edged set X, that is, $M \cong F \cdot X / L^{\ell}(H, G)$, where $L^{\ell}(H, G)$ is the submodule of $F \cdot X$ generated by H modulo G. We assume that there is a suitable order $>$ on $B \cdot H$.

Let C be the Gröbner basis on $F \cdot H$ made from critical pairs and critical z-elements in the previous section. Considering C to be a left edged set, we have the projective left A-module $A \cdot C$. Let $\partial' : A \cdot C \longrightarrow A \cdot H$ be the morphism of left A-modules defined by

$$\partial'([c]) = c$$

for $c \in C$. Let $\eta : A \cdot X \longrightarrow M$ be the canonical surjection. Since H generates $L^{\ell}(H, G)$ and C generates the kernel $\mathrm{Ker}(\rho \circ \delta)$ modulo G, we have

Theorem 16.1. *The sequence*

$$A \cdot C \xrightarrow{\partial'} A \cdot H \xrightarrow{\partial} A \cdot X \xrightarrow{\eta} M \to 0$$

is exact.

Suppose that a suitable well-order can be defined on the projective left F-module $F \cdot C$, then we have the Gröbner basis D on $F \cdot C$ made from critical pairs and critical z-elements with respect to C and G and a morphism ∂'' : $A \cdot D \to A \cdot C$ defined by $\partial''([d]) = d$. If we can repeat this construction (that is, if a suitable well-order exists at every step), then we can construct a projective resolution of M.

Corollary 16.2. *Let M be a left A-module defined by a Gröbner basis X_1 on the projective left A-module $A \cdot X_0$ generated by a left edged set X_0. If at every step above, a suitable well-order exists, we have a projective resolution of M:*

$$\to A \cdot X_n \xrightarrow{\partial_n} A \cdot X_{n-1} \to \cdot \to A \cdot X_1 \xrightarrow{\partial_1} A \cdot X_0 \xrightarrow{\eta} M \to 0.$$

Suppose that F has an identity element 1 and A is supplemented with a morphism $\epsilon : A \to K$. Let X be a generating set of nonidenmotents of B, then $\{ a - \epsilon(\rho(a)) \cdot 1 \mid a \in X \}$ forms a Gröbner basis for $\mathrm{Ker}(\epsilon)$ modulo G. Starting with this Gröbner basis, we can construct a projective resolution of K and we can compute the (co) homology of the algebra A (or the semigroup S).

17 Bimodules and the Hochschild cohomology

The enveloping semigroup $S^e = (B \times B) \cup \{0\}$ of $S = B \cup \{0\}$ is a well-ordered reflexive semigroup, in which the product and the order are given as

$$(x, y) \cdot (x', y') = (xx', y'y),$$

and

$$(x, y) \succ (x', y') \Leftrightarrow x \succ x' \text{ or } (x = x' \text{ and } y \succ y')$$

for $x, y, x', y' \in B$, respectively. The enveloping algebra $A^e = A \otimes_K A^o$ of $A = F/I$ is isomorphic to the quotient F^e/I^e, where $I^e = I \otimes F + F \otimes I$, and the set

$$G^e = \{ g \otimes 1, 1 \otimes g \mid g \in G \}.$$

is a Gröbner basis of the ideal I^e. An F-bimodule (resp. A-bimodule) is naturally a left F^e-module (resp. left A^e-module).

Let X be an edged set and

$$F \cdot X \cdot F = \bigoplus_{\xi \in X} F\sigma(\xi) \times \tau(\xi)F$$

be the projective F-bimodule generated by X and let H be a set of monic uniform elements of $F \cdot X \cdot F$. We have three kinds of critical pairs with respect to H modulo G. Let $h = w\xi z - t, h' = w'\xi z' - t' \in H$, $u - v \in G$ and $x, y, x', y' \in B$.

First suppose that $xw = x'w' \neq 0$ and $zy = z'y' \neq 0$, x and x' are left coprime, y and y' are right coprime, and the appearance of $w\xi z$ in the context (x, y) is immediate right of the appearance of $w'\xi z'$ in the context (x', y'). Then we have a critical pair $(xty, x't'y')$ of the first kind and the element

$$c_1 = x[h]y - x'[h']y' + \int(xty) + \int(x't'y').$$

of the projective F-bimodule $F \cdot H \cdot F$ generated by H. Next suppose that $xw = yuy' \neq 0$, u is rightmost in xw, $w\xi$ is rightmost in $xw\xi$ and x and y are left coprime. Then, we have a critical pair $(xt, yvy'\xi w')$ of the second kind, and an element

$$c_2 = x[h] - \int(yvy'\xi z) + \int(xt)$$

of $F \cdot H \cdot F$. Dually suppose that $zx = y'uy \neq 0$, u is leftmost in zx, ξz is leftmost in $xw\xi$, and x and y are right coprime. Then, we have a critical pair $(tx, w\xi y'vy)$ of the third kind, and an element

$$c_3 = [h]x - \int(w\xi y'vy) + \int(tx)$$

of $F \cdot H \cdot F$. If $xw = 0$ but $xt \neq 0$ and there is no nonidempotential left factor y of x; $x = yx'$ such that $x'w = 0$, we have a critical z-element xt and an element

$$c_4 = x[h] + \int(xt).$$

If $zx = 0$ but $tx \neq 0$ and there is no nonidempotential right factor y of x; $x = x'y$ such that $zx' = 0$, we have a critical z-element tx and an element

$$c_5 = [h]x + \int(tx).$$

Let C be the collection of all elements c_1, c_2, c_3, c_4 and c_5 above, and let $A \cdot C \cdot A$ be the projective A-bimodule generated by C.

Let $\delta : F \cdot H \cdot F \to F \cdot X \cdot F$ be the morphisms of left F-bimodules defined by $\delta([h]) = h$ for $h \in H$, and let $\rho : F \cdot X \cdot F \to A \cdot X \cdot A$ be the canonical surjection. Let M be the A-bimodule defined by H modulo G, that is, $M = A \cdot X \cdot A/L(M, G)$, where $L(M, G)$ is the subbimodule of $A \cdot X \cdot A$ generated by $\rho(M)$. Let $\partial : A \cdot H \cdot A \to A \cdot X \cdot A$ and $\partial' : A \cdot C \cdot A \to A \cdot H \cdot A$ be the morphisms of A-bimodules defined by $\partial([h]) = h$ and $\partial'([c]) = c$.

Theorem 17.1. *If H is a Gröbner basis on $F \cdot X \cdot F$ and $>$ is a suitable well-order on $B \cdot H \cdot B$, then the set C is a Gröbner basis on $F \cdot H \cdot F$ of the kernel of $\rho \circ \delta$ modulo G. Moreover we have an exact sequence of A-bimodules:*

$$A \cdot C \cdot A \xrightarrow{\partial'} A \cdot H \cdot A \xrightarrow{\partial} A \cdot X \cdot A \xrightarrow{\eta} M \to 0$$

Corollary 17.2. *Let M be an A-bimodule defined by a Gröbner basis X_1 on the projective left F-bimodule $F \cdot X_0 \cdot F$ generated by a left edged set X_0. If at every step above, a suitable well-order exists, we have a projective A-bimodule resolution of M:*

$$\to A \cdot X_n \cdot A \xrightarrow{\partial_n} A \cdot X_{n-1} \cdot A \to \cdots \to A \cdot X_1 \cdot A \xrightarrow{\partial_1} A \cdot X_0 \cdot A \xrightarrow{\eta} M \to 0.$$

Let E be the set of all idempotents in B, and let X be a generating set of nonidempotents of B. Considering them as edged sets we have projective F-bimodules $F \cdot E \cdot F, F \cdot X \cdot F$ and A-bimodules $A \cdot E \cdot A, A \cdot X \cdot A$ generated by them. We have an augmentation map $\epsilon : F \cdot E \cdot F \to F$ and $\bar{\epsilon} : A \cdot E \cdot A \to A$ defined by $\epsilon([e]) = e$ and $\bar{\epsilon}([e]) = e$ for $e \in E$.

Let

$$H = \{ a[\tau(a)] - [\sigma(a)]a \mid a \in X \}.$$

Then, H is a Gröbner basis on $F \cdot E \cdot F$ for $\mathrm{Ker}(\epsilon)$. In this way we have an exact sequence

$$A \cdot X \cdot A \xrightarrow{\partial} A \cdot E \cdot A \xrightarrow{\epsilon} M \to 0,$$

where the morphism ∂ is defied by $\partial([a]) = a[\tau(a)] - [\sigma(a)]a \quad (a \in X)$. Thus, if under the existence of suitable order in every step, we can construct a projective A-bimodule resolution of A. This gives a way to compute the Hochschild cohomology of the algebra A.

18 Examples

Since the free monoid Σ^* is well-ordered and coherent, its submonids are well-ordered and coherent. So, the existence of suitable order is guaranteed in every step of construction. In this section we pick up some easy submonoids of Σ^* and compute the (co)homology (other examples can be found in [1], [2]).

Example 18.1. Let B be the submonoid of $\{a\}^*$ generated by $X = \{a^2, a^3\}$. B is isomorphic to the additive monoid $\mathbb{N} \setminus \{1\}$ of natural numbers excluding 1. Let $F = K \cdot B$ be the algebra based on $B \cup \{0\}$. We have an augmentation map $\epsilon : F \cdot [\,] \cdot F \to F$ given by $\epsilon([\,]) = 1$, and a Gröbner basis

$$\{ \alpha_1 = a^2[\,] - [\,]a^2, \beta_1 = a^3[\,] - [\,]a^3 \}$$

of $\mathrm{Ker}(\epsilon)$. Let $X = \{\alpha, \beta\}$ and define a morphism $\partial_1 : F \cdot X \cdot F \to F \cdot [\,] \cdot F$ by $\partial_1([\alpha]) = \alpha_1$, and $\partial_1([\beta]) = \beta_1$.

From the equation $a^3 \cdot a^2 = a^2 \cdot a^3$ we have a critical pair of the first kind $(a^3[\,]a^2, a^2[\,]a^3)$ and an element

$$\alpha_2 = a^3[\alpha] - [\alpha]a^3 - a^2[\beta] + [\beta]a^2$$

of $F \cdot X \cdot F$. From the equation $(a^2)^2 \cdot a^2 = a^3 \cdot a^3$ we have another critical pair of first kind $(a^4[\,]a^2, a^3[\,]a^3)$ and an element

$$\beta_2 = a^4[\alpha] + a^2[\alpha]a^2 + [\alpha]a^4 - a^3[\beta] - [\beta]a^3$$

of $F \cdot X \cdot F$. There is no critical pairs of the other kinds because the Gröbner basis G on F is empty. There is no z-element either because S is coherent. Hence, these two elements form a Gröbner basis of $\mathrm{Ker}(\partial_1)$. We have a morphism $\partial_2 : F \cdot X \cdot F \to F \cdot X \cdot F$ given by $\partial_2([\alpha]) = \alpha_2$ and $\partial_2([\beta]) = \beta_2$. Note that $\mathrm{lt}(\alpha_2) = a^3[\alpha]$ and $\mathrm{lt}(\beta_2) = a^4[\alpha]$.

From the equation $a^3 \cdot a^3 = a^2 \cdot a^4$ we have an element

$$\alpha_3 = a^3[\alpha] + [\alpha]a^3 - a^2[\beta] + [\beta]a^2,$$

and from the equation $(a^2)^2 \cdot a^3 = a^3 \cdot a^4$ we have an element

$$\beta_3 = a^4[\alpha] + a^2[\alpha]a^2 + [\alpha]a^4 - a^3[\beta] + [\beta]a^3.$$

They form a Gröbner basis of $\mathrm{Ker}(\partial_2)$. continuing this calculation we can construction a free bimodule resolution of F:

$$\to A \cdot X \cdot A \xrightarrow{\partial_n} A \cdot X \cdot A \to \cdots \to A \cdot X \cdot A \xrightarrow{\partial_1} A \cdot [\,] \cdot A \xrightarrow{\eta} F,$$

where ∂_n is given by

$$\partial_1([\alpha]) = a^2[\,] - [\,]a^2, \ \partial_1([\beta]) = a^3[\,] - [\,]a^3,$$

$$\partial_n([\alpha]) = a^3[\alpha] + (-1)^{n-1}[\alpha]a^3 - a^2[\beta] + [\beta]a^2$$

and

$$\partial_n([\beta]) = a^4[\alpha] + a^2[\alpha]a^2 + [\alpha]a^4 - a^3[\beta] + (-1)^{n-1}[\beta]a^3$$

for $n \geq 2$.

From this resolution we can compute the Hochschild cohomology of F as follows. Here, K is a field of characteristic p.

$$H^0(F) = F,$$

$$H^1(F) = \begin{cases} F & \text{if } p = 2 \text{ or } 3 \\ \oplus_{i \geq 2} K \cdot a^i & \text{otherwise.} \end{cases}$$

Let $n \geq 2$. If $p = 2$,

$$H^n(F) = K \oplus K \cdot a^2 \oplus K \cdot a^3 \oplus K \cdot a^5.$$

if $p = 3$,

$$H^n(F) = K \oplus K \cdot a^2 \oplus K \cdot a^4,$$

and if $p \neq 2, 3$,

$$H^n(F) = \begin{cases} K \oplus K \cdot a^2 & \text{if } n \text{ is even} \\ K \cdot (2a^2, 3a^3) \oplus K \cdot (2a^3, 3a^4) & \text{if } n \text{ is odd.} \end{cases}$$

Example 18.2. Let B be the submonoid of $\{a, b\}^*$ generated by $X = \{ab, ba, aba\}$, and let $S = B \cup \{0\}$ and $F = K \cdot B$ is the algebra based on S. We have an augmentation $\epsilon : F \cdot [\,] \cdot F \to F$ given by $\epsilon([\,]) = 1$. We have a Gröbner basis

$$\{\, ab[\,] - [\,]ab, \ ba[\,] - [\,]ba, \ aba[\,] - [\,]aba \,\},$$

of $\mathrm{Ker}(\epsilon)$ and a differential map

$$\partial_1 : F \cdot X \cdot F \to A \cdot [\,] \cdot A$$

with

$$\partial_1([ab]) = ab[\,] - [\,]ab,\ \partial_1([ba]) = ba[\,] - [\,]ba,$$
$$\partial_1([aba]) = aba[\,] - [\,]aba.$$

X is not a code because we have a word equation $(aba)ba = ab(aba)$. From this equation we have a critical pair $(aba[\,]ba, ab[\,]aba)$, and we obtain a Gröbner basis of $\mathrm{Ker}(\partial_1)$:

$$\{\, aba[ba] + [aba]ba - ab[aba] - [ab]aba \,\}.$$

In this way we get a free bi-module resolution of F:

$$0 \to F \cdot \{ababa\} \cdot F \xrightarrow{\partial_2} F \cdot X \cdot F \xrightarrow{\partial_1} F \cdot [\,] \cdot F \xrightarrow{\epsilon} F,$$

where

$$\partial_2([ababa]) = aba[ba] + [aba]ba - ab[aba] - [ab]aba.$$

F is supplemented with $\epsilon : F \to K$ defined by $\epsilon(ab) = \epsilon(ba) = \epsilon(aba) = 0$. Tensoring with the F-module K on the right, we have a minimal free left resolution of K:

$$0 \to F \cdot \{ababa\} \xrightarrow{\bar{\partial}_2} F \cdot X \xrightarrow{\bar{\partial}_1} F \xrightarrow{\bar{\epsilon}} K,$$
$$\bar{\partial}_1([ab]) = ab,\ \bar{\partial}_1([ba]) = ba,\ \bar{\partial}_1([aba]) = aba,$$
$$\bar{\partial}_2([ababa]) = aba[ba] - ab[aba].$$

The Betti number $b_2 = \dim_K(\mathrm{Tor}_2^F(K, K)) = 1$ seems reflect the ambiguity of X; how distant from codes.

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