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The q-Onsager algebra

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This article gives a summary of the finite-dimesional irreducible representations of the q-Onsager algebra, which are treated in detail in [3].

Fix a nonzero scalar $q\in\mathbb{C}$ which is not a root of unity. Let $\mathcal{A}=\mathcal{A}_q$ denote the associative $\mathbb{C}$-algebra with 1 defined by genarators z, z^* subject to the relations

$$(\mathrm{TD})\left\{\begin{array}{lll}[z,\ z^2z^*-\beta z\,z^*z+z^*z^2] &=& \delta\,[z,\ z^*],\\ [z^*,\ z^{*2}z-\beta z^*z\,z^*+z\,z^{*2}] &=& \delta\,[z^*,\ z],\end{array}\right.$$

where $\beta=q^2+q^{-2}$ and $\delta=-(q^2-q^{-2})^2$. (TD) can be regarded as a q-analogue of the Dolan-Grady relations and we call $\mathcal{A}$ the *q-Onsager algebra.* We classify the finite-dimensional irreducible representations of $\mathcal{A}$. All such representations are explicitly constructed via embeddings of $\mathcal{A}$ into the $U_q(sl_2)$-loop algebra. As an application, tridiagonal pairs of q-Racah type over $\mathbb{C}$ are classified in the case where q is not a root of unity.

The $U_q(sl_2)$-loop algebra $\mathcal{L}=U_q(L(sl_2))$ is the associative $\mathbb{C}$-algebra with 1 generated by $e_i^+, e_i^-, k_i, k_i^{-1}$ $(i=0,1)$ subject to the relations

$$\begin{aligned} k_0k_1 &= k_1k_0=1,\\ k_ik_i^{-1} &= k_i^{-1}k_i=1,\\ k_ie_i^{\pm}k_i^{-1} &= q^{\pm2}e_i^{\pm},\\ k_ie_j^{\pm}k_i^{-1} &= q^{\mp2}e_j^{\pm}\quad (i\neq j),\\ [e_i^+,e_i^-] &= \frac{k_i-k_i^{-1}}{q-q^{-1}},\\ [e_i^+,e_j^-] &= 0\quad (i\neq j),\end{aligned}$$

$$[e_i^{\pm},(e_i^{\pm})^2e_j^{\pm}-(q^2+q^{-2})e_i^{\pm}e_j^{\pm}e_i^{\pm}+e_j^{\pm}(e_i^{\pm})^2]=0\quad (i\neq j).$$

Note that if we replace $k_0k_1=k_1k_0=1$ in the defining relations for $\mathcal{L}$ by $k_0k_1=k_1k_0$, then we have the quantum affine algebra $U_q(\widehat{sl}_2)$: $\mathcal{L}$ is isomorphic to the quotient algebra of $U_q(\widehat{sl}_2)$ by the two-sided ideal generated by k_0k_1-1.

Proposition 1 *For arbitrary nonzero $s,t\in\mathbb{C}$, there exists an algebra homomorphism $\varphi_{s,t}$ from $\mathcal{A}$ to $\mathcal{L}$ that sends z, z^* to*

$$\begin{aligned} z_t(s) &= x(s)+t\,k(s)+t^{-1}k(s)^{-1},\\ z_t^*(s) &= y(s)+t^{-1}k(s)+t\,k(s)^{-1},\end{aligned}$$

respectively, where

$$\begin{aligned} x(s) &= \alpha(se_0^+ + s^{-1}e_1^-k_1) \quad \text{with } \alpha = -q^{-1}(q-q^{-1})^2, \\ y(s) &= se_0^-k_0 + s^{-1}e_1^+, \\ k(s) &= sk_0. \end{aligned}$$

Moreover $\varphi_{s,t}$ *is injective.*

We give an overview of finite-dimensional representations of $\mathcal{L}$ that we need to state our explicit construction of irreducible $\mathcal{A}$-modules via $\varphi_{s,t}$. For $a \in \mathbb{C}$ ($a \neq 0$) and $\ell \in \mathbb{Z}$ ($\ell > 0$), $V(\ell, a)$ denotes the *evaluation module* of $\mathcal{L}$, i.e., $V(\ell, a)$ is an $(\ell+1)$-dimensional vector space over $\mathbb{C}$ with a basis $v_0, v_1, \ldots, v_\ell$ on which $\mathcal{L}$ acts as follows:

$$\begin{aligned} k_0 v_i &= q^{2i-\ell}\, v_i, \\ k_1 v_i &= q^{\ell-2i} v_i, \\ e_0^+ v_i &= a\, q\, [i+1]\, v_{i+1}, \\ e_0^- v_i &= a^{-1}q^{-1}[\ell-i+1]\, v_{i-1}, \\ e_1^+ v_i &= [\ell-i+1]\, v_{i-1}, \\ e_1^- v_i &= [i+1]\, v_{i+1}, \end{aligned}$$

where $v_{-1} = v_{\ell+1} = 0$ and $[j] = [j]_q = (q^j - q^{-j})/(q-q^{-1})$. $V(\ell, a)$ is an irreducible $\mathcal{L}$-module. We call $v_0, v_1, \ldots, v_\ell$ a *standard basis.*

Let Δ denote the *coproduct* of $\mathcal{L}$: the algebra homomorphism from $\mathcal{L}$ to $\mathcal{L} \otimes \mathcal{L}$ defined by

$$\begin{aligned} \Delta(k_i^{\pm 1}) &= k_i^{\pm 1} \otimes k_i^{\pm 1}, \\ \Delta(e_i^+) &= k_i \otimes e_i^+ + e_i^+ \otimes 1, \\ \Delta(e_i^- k_i) &= k_i \otimes e_i^- k_i + e_i^- k_i \otimes 1. \end{aligned}$$

Given $\mathcal{L}$-modules V_1, V_2, the tensor product $V_1 \otimes V_2$ becomes an $\mathcal{L}$-module via Δ. Given a set of evaluation modules $V(\ell_i, a_i)$ $(1 \leq i \leq n)$ of $\mathcal{L}$, the tensor product

$$V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$$

makes sense as an $\mathcal{L}$-module without being affected by the parentheses for the tensor product because of the coassociativity of Δ.

With an evaluation module $V(\ell, a)$ of $\mathcal{L}$, we associate the set $S(\ell, a)$ of scalars $a\,q^{-\ell+1}$, $a\,q^{-\ell+3}, \cdots, a\,q^{\ell-1}$:

$$S(\ell, a) = \{a\,q^{2i-\ell+1} \mid 0 \leq i \leq \ell-1\}.$$

The set $S(\ell, a)$ is called a *q-string* of length ℓ. Two q-strings $S(\ell, a)$, $S(\ell', a')$ are said to be *adjacent* if $S(\ell, a) \cup S(\ell', a')$ is a longer q-string, i.e., $S(\ell, a) \cup S(\ell', a') = S(\ell'', a'')$ for some ℓ'', a'' with $\ell'' > \max\{\ell,\ \ell'\}$. It can be easily checked that $S(\ell, a)$, $S(\ell', a')$ are adjacent if and only if $a^{-1}a' = q^{\pm i}$ for some

$$i \in \{|\ell-\ell'|+2,\ |\ell-\ell'|+4, \cdots, \ell+\ell'\}.$$

Two q-strings $S(\ell, a)$, $S(\ell', a')$ are defined to be *in general position* if they are not adjacent, i.e., if either

(i) $S(\ell, a) \cup S(\ell', a')$ is not a q-string,
or
(ii) $S(\ell, a) \subseteq S(\ell', a')$ or $S(\ell, a) \supseteq S(\ell', a')$.

A multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q-strings is said to be *in general position* if $S(\ell_i, a_i)$ and $S(\ell_j, a_j)$ are in general position for any i, j $(i \neq j, 1 \leq i \leq n, 1 \leq j \leq n)$. The following fact is well-known and easy to prove. Let Ω be a finite multi-set of nonzero scalars from $\mathbb{C}$. Then there exists a multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q-strings in general position such that

$$\Omega = \bigcup_{i=1}^n S(\ell_i, a_i)$$

as multi-sets of nonzero scalars. Moreover such a multi-set of q-strings is uniquely determined by Ω.

With a tensor product $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ of evaluation modules $V(\ell_i, a_i)$ $(1 \leq i \leq n)$, we associate the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q-strings. The following (i), (ii), (iii) are well-known [1]:

(i) A tensor product $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ of evaluation modules is irreducible as an $\mathcal{L}$ -module if and only if the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q-strings is in geneal position.

(ii) Set $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell'_1, a'_1) \otimes \cdots V(\ell'_{n'}, a'_{n'})$ and assume that V, V' are both irreducible as an $\mathcal{L}$-module. Then V, V' are isomorphic as $\mathcal{L}$-modules if and only if the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n$, $\{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ coincide, i.e., $n = n'$ and $\ell_i = \ell'_i$, $a_i = a'_i$ for all i $(1 \leq i \leq n)$ with a suitable reordering of $S(\ell'_1, a'_1), \cdots, S(\ell'_n, a'_n)$.

(iii) Every nontrivial finite-dimensional irreducible $\mathcal{L}$-module of type (1,1) is isomorphic to some $V(\ell_1, a_1) \otimes \cdots V(\ell_n, a_n)$.

Two multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n, \{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ of q-strings are defined to be *equivalent* if there exists $\varepsilon_i \in \{\pm 1\}$ $(1 \leq i \leq n)$ such that $\{S(\ell_i, a_i^{\varepsilon_i})\}_{i=1}^n$ and $\{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ coincide, i.e., $n = n'$ and $\ell_i = \ell'_i$, $a_i^{\varepsilon_i} = a'_i$ for all i $(0 \leq i \leq n)$ with a suitable reordering of $S(\ell'_1, a'_1), \cdots, S(\ell'_n, a'_n)$. A multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q-strings is defined to be *strongly in general position* if any multi-set of q-strings equivalent to $\{S(\ell_i, a_i)\}_{i=1}^n$ is in general position, i.e., the multi-set $\{S(\ell_i, a_i^{\varepsilon_i})\}_{i=1}^n$ is in general position for any choice of $\varepsilon_i \in \{\pm 1\}$ $(1 \leq i \leq n)$.

Lemma 1 *Let Ω be a finite multi-set of nonzero scalars from $\mathbb{C}$ such that c and c^{-1} appear in Ω in pairs, i.e., c and c^{-1} have the same multiplicity in Ω for each $c \in \Omega$, where we understand that if 1 or -1 appears in Ω, it has even multiplicity. Then there exists a multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q-strings strongly in general position such that*

$$\Omega = \bigcup_{i=1}^n \left(S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1})\right)$$

as multi-sets of nonzero scalars. Such a multi-set of q-strings is uniquely determined by Ω *up to equivalence.*

For an $\mathcal{L}$-module V, let ρ_V denote the representation of $\mathcal{L}$ afforded by the $\mathcal{L}$-module V. Then $\rho_V \circ \varphi_{s,t}$ is a representation of $\mathcal{A}$.

Theorem 1 *The following (i), (ii), (iii) hold.*

(*i*) *For an* $\mathcal{L}$*-module* $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ *and nonzero* $s, t \in \mathbb{C}$*, the representaiton* $\rho_V \circ \varphi_{s,t}$ *of* $\mathcal{A}$ *is irreducible if and only if*

(*i*.1) *the multi-set* $\{S(\ell_i, a_i)\}_{i=1}^{n}$ *of q-strings is strongly in general position,*

(*i*.2) *none of* $-s^2$, $-t^2$ *belongs to* $S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1})$ *for any* i $(1 \le i \le n)$,

(*i*.3) *none of the four scalars* $\pm st$, $\pm st^{-1}$ *equals* q^i *for any* $i \in \mathbb{Z}$ $(-d+1 \le i \le d-1)$.

(*ii*) *For* $\mathcal{L}$*-modules* $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_{n'}, a'_{n'})$ *and* (s,t), $(s',t') \in (\mathbb{C}\backslash\{0\}) \times (\mathbb{C}\backslash\{0\})$*, set* $\rho = \rho_V \circ \varphi_{s,t}$ *and* $\rho' = \rho_{V'} \circ \varphi_{s',t'}$*. Assume that the representations* ρ, ρ' *of* $\mathcal{A}$ *are both irreducible. Then they are isomorphic as representations of* $\mathcal{A}$ *if and only if the multi-sets* $\{S(\ell_i, a_i)\}_{i=1}^{n}$, $\{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ *are equivalent and*

$$(s',t') \in \{\pm(s,t),\ \pm(t^{-1}, s^{-1}),\ \pm(t,s),\ \pm(s^{-1}, t^{-1})\}.$$

(*iii*) *Every nontrivial finite-dimensional irreducible representation of* $\mathcal{A}$ *is isomorphic to* $\rho_V \circ \varphi_{s,t}$ *for some* $\mathcal{L}$*-module* $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ *and* $(s,t) \in (\mathbb{C}\backslash\{0\}) \times (\mathbb{C}\backslash\{0\})$.

Let $A, A^* \in \mathrm{End}(V)$ be a TD-pair [2] with eigenspaces $\{V_i\}_{i=0}^{d}$, $\{V_i^*\}_{i=0}^{d}$ respectively. Then we have the split decomposition:

$$V = \bigoplus_{i=0}^{d} U_i,$$

where

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d).$$

By [2, Corollary 5.7], it holds that

$$\dim U_i = \dim V_i = \dim V_i^* \quad (0 \le i \le d),$$

and

$$\dim U_i = \dim U_{d-i} \qquad (0 \le i \le d).$$

Note that $\dim U_i$ is invariant under standardization of A, A^* by affine transformations $\lambda A + \mu I$, $\lambda^* A^* + \mu^* I$. For an $\mathcal{L}$-module

$$V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n),$$

set $A = \rho_V \circ \varphi_{s,t}(z)$, $A^* = \rho_V \circ \varphi_{s,t}(z^*)$. If the conditions (i.1), (i.2), (i.3) in Theorem 1 hold, then A, A^* are a standardized TD-pair of q-Racah type. Every standardized TD-pair A, A^* of q-Racah type arises in this way. The split decomposition of V for A, A^* coincides with the eigenspace decomposition of the element k_0 of $\mathcal{L}$ acting on V. Thus the generating function for $\dim U_i$

$$g(\lambda) = \sum_{i=0}^{d} (\dim U_i)\, \lambda^i$$

is given as follows.

Proposition 2 ([2, Conjecture 13.7])

$$g(\lambda) \;=\; \prod_{i=1}^{n} (1 + \lambda + \lambda^2 + \cdots + \lambda^{\ell_i}).$$

A TD-pair A, A^* is called a *Leonard pair* if $\dim U_i = 1$ for all i $(0 \leq i \leq d)$. A standardized TD-pair A, A^* of q-Racah type is a Leonard pair if and only if it is afforded by an evaluation module. In view of this fact, a standardized TD-pair A, A^* of q-Racah type is regarded as a 'tensor product of Leonard pairs'.

References

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