数理解析研究所講究録
第 1658 巻 2009 年 1-6

On the Existence of Ground State and Asymptotic Fields of Relativistic Quantum Electrodynamics

Toshimitsu TAKAESU (Kyushu univ.)

1 Introduction

This paper is a short review on the results of the spectral analysis of Quantum electrodynamics (**QED**) obtained in [5, 7, 19]. QED describes the system of Dirac fields interacting with radiation fields. By introducing ultraviolet and spatial cutoffs, it is seen that QED Hamiltonian is an operator on a boson-fermion Fock space. In [5, 7], they considered the radiation field quantized in the Coulomb gauge and the Dirac field quantized with an external potential. On the other hand, in [19], the radiation field quantized in the Coulomb gauge is also considered, but the Dirac field without external potentials is investigated. It is proven that for sufficiently small values of coupling constants the ground state exists under the infrared regularity condition [5, 7, 19]. It is noted that the existence of the ground state is not trivial, since the ground state energy of the free Hamiltonian is embedded in a continuous spectrum. It is also proven that the asymptotic fields exist, the ground state is unique, and spectral gap is closed [19]. In addition, it is shown that the QED Hamiltonian and the total charge operator strongly commute. Then the state space is decomposed with respect to the spectrum of the total charge, and the total charge of the ground state is zero for sufficiently small values of coupling constants [19].

QED is a system of quantum fields interacting with other fields. On the other hand, there is a system of partircles interacting with quantum fields, which includes the non-relativistic QED models. In the last decade the spectral properties of this system have been successfully analyzed, and many results and methods are obtained [14]. In the following sections, we will apply these methods to the analysis of QED Hamiltonian.

2 Dirac fields

Let us denote the creation and annihilation operators of electron by $b_s^*(\mathbf{p})$ and $b_s(\mathbf{p})$, respectively. The creation and annihilation operators of positron are denoted by $d_s^*(\mathbf{p})$ and $d_s(\mathbf{p})$, respectively. These operators satisfy the canonical anti-commutation relation :

$$\{b_s(\mathbf{p}), b_{s'}^*(\mathbf{p}')\} = \{d_s(\mathbf{p}), d_{s'}^*(\mathbf{p}')\} = \delta_{s,s'}\delta(\mathbf{p}-\mathbf{p}'),$$
$$\{b_s(\mathbf{p}), b_{s'}(\mathbf{p}')\} = \{d_s(\mathbf{p}), d_{s'}(\mathbf{p}')\} = \{b_s(\mathbf{p}), d_{s'}(\mathbf{p}')\} = \{b_s(\mathbf{p}), d_{s'}^*(\mathbf{p}')\} = 0,$$

where $\{X,Y\} = XY + YX$. Let us set

$$b_s^\sharp(f) = \int_{\mathbf{R}^3} f(\mathbf{p}) b_s^\sharp(\mathbf{p}) d\mathbf{p}, \qquad d_s^\sharp(g) = \int_{\mathbf{R}^3} g(\mathbf{p}) d_s^\sharp(\mathbf{p}) d\mathbf{p}, \qquad f,g \in L^2(\mathbf{R}^3),$$

where $X^\sharp$ denotes X or X^*. Let Ω_{Dirac} be the vacuum state. The state space of the Dirac field is given by

$$\begin{aligned}\mathfrak{F}_{\mathrm{Dirac}} = L.h.\Big\{ & b_{s_1}^*(f_1)\cdots b_{s_n}^*(f_n) d_{\tau_1}^*(g_1)\cdots d_{\tau_{n'}}^*(g_{n'})\Omega_{\mathrm{Dirac}}, \Omega_{\mathrm{Dirac}} \\ & \Big| \, f_j \in L^2(\mathbf{R}^3), j = 1,\cdots n, \, g_l \in L^2(\mathbf{R}^3), l = 1,\cdots n', \, n,n' \in \mathbf{N}\Big\}^-\end{aligned}$$

where $\mathcal{D}^-$ denotes the closure of $\mathcal{D}$. The one particle energy of the electron with momentum $\mathbf{p}$ is denoted by

$$E_M(\mathbf{p}) = \sqrt{\mathbf{p}^2 + M^2}$$

where $M > 0$ denotes the mass of electron, and we fix it. The free Hamiltonian of the Dirac field is given by

$$H_{\mathrm{Dirac}} = \sum_{s=\pm 1/2} \int_{\mathbf{R}^3} E_M(\mathbf{p}) \left(b_s^*(\mathbf{p}) b_s(\mathbf{p}) + d_s^*(\mathbf{p}) d_s(\mathbf{p}) \right) d\mathbf{p},$$

and the field operator by

$$\psi(\mathbf{x}) = \sum_{s=\pm 1/2} \frac{1}{\sqrt{2\pi}^3} \int_{\mathbf{R}^3} \frac{\chi_{\mathrm{Dirac}}(\mathbf{p})}{\sqrt{E_M(\mathbf{p})}} \left(u_s(\mathbf{p}) b_s(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} + v_s(\mathbf{p}) d_s^*(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}} \right) d\mathbf{p}$$

where u_s and v_s denote the spinors, and χ_{Dirac} the ultraviolet cutoff.

3 Radiation field

Let us consider the radiation field quantized in the Coulomb gauge. The creation and annihilation operators are denoted by $a_r^*(\mathbf{k})$ and $a_r(\mathbf{k})$, respectively. These operator satisfy the canonical commutation relation :

$$\begin{aligned}[a_r(\mathbf{k}), a_{r'}^*(\mathbf{k}')] &= \delta_{r,r'}\delta(\mathbf{k}-\mathbf{k}'), \\ [a_r(\mathbf{k}), a_{r'}(\mathbf{k}')] &= [a_r^*(\mathbf{k}), a_{r'}^*(\mathbf{k}')] = 0,\end{aligned}$$

where $[X,Y] = XY - YX$. Let us set

$$a_r^\sharp(h) = \int_{\mathbf{R}^3} h(\mathbf{k}) a_r^\sharp(\mathbf{k}) d\mathbf{k}, \qquad h \in L^2(\mathbf{R}^3),$$

where $a_r^\sharp$ denotes a_r or a_r^*. Let Ω_{rad} be the vacuum state. The state space of the radiation fields is given by

$$\mathfrak{F}_{\mathrm{rad}} = L.h.\left\{ a_{r_1}^*(h_1)\cdots a_{r_n}^*(h_n)\Omega_{\mathrm{rad}} \, , \, \Omega_{\mathrm{rad}} \, \middle| \, h_l \in L^2(\mathbf{R}^3), \, l = 1,\cdots n, \, n \in \mathbf{N} \right\}^- .$$

The one particle energy of photon with momentum $\mathbf{k}$ is given by

$$\omega(\mathbf{k}) = |\mathbf{k}|.$$

The free Hamiltonian of the radiation field is given by

$$H_{\text{rad}} = \sum_{r=1,2} \int_{\mathbf{R}^3} |\mathbf{k}| a_r^*(\mathbf{k}) a_r(\mathbf{k}) d\mathbf{k},$$

and field operator by

$$\mathbf{A}(\mathbf{x}) = \sum_{r=1,2} \frac{1}{\sqrt{2\pi}^3} \int_{\mathbf{R}^3} \frac{\chi_{\text{rad}}(\mathbf{k}) \mathbf{e}_r(\mathbf{k})}{\sqrt{2\omega(\mathbf{k})}} \left(a_r(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + a_r^*(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right) d\mathbf{k},$$

where $\mathbf{e}_r$ denotes the polarization vector of photon and χ_{rad} the ultraviolet cutoff.

4 Main Results

4.1 Total Hamiltonian

The state space of quantum electrodynamics is given by a boson-fermion Fock space

$$\mathcal{F}_{\text{QED}} = \mathcal{F}_{\text{Dirac}} \otimes \mathcal{F}_{\text{rad}},$$

and the free Hamiltonian by

$$H_0 = H_{\text{Dirac}} \otimes I + I \otimes H_{\text{rad}}.$$

The interactions are given by

$$H_{\text{I}}' = \int_{\mathbf{R}^3} \chi_{\text{I}}(\mathbf{x}) \psi^*(\mathbf{x}) \alpha^j \psi(\mathbf{x}) \otimes A_j(\mathbf{x}) d\mathbf{x},$$

$$H_{\text{II}}' = \int_{\mathbf{R}^3 \times \mathbf{R}^3} \chi_{\text{II}}(\mathbf{x}) \chi_{\text{II}}(\mathbf{y}) \frac{\psi^*(\mathbf{x}) \psi(\mathbf{x}) \psi^*(\mathbf{y}) \psi(\mathbf{y}) \otimes I}{|\mathbf{x}-\mathbf{y}|} d\mathbf{x} d\mathbf{y}$$

where $\alpha^j \in M_4(\mathbf{C})$, $\{\alpha^j, \alpha^l\} = 2\delta_{j,l}$, and χ_{I}, χ_{II} denote the spatial cutoffs. The total Hamiltonian of quantum electrodynamics is given by

$$H_{\text{QED}} = H_0 + \kappa_{\text{I}} H_{\text{I}}' + \kappa_{\text{II}} H_{\text{II}}', \qquad \kappa_{\text{I}}, \kappa_{\text{II}} \in \mathbf{R}.$$

Remark In [5, 7] the spatial cutoffs do not appear explicitly, but a similar function is considered.

By virtue of the ultraviolet cutoffs and the spatial cutoffs, the interactions are relatively bounded with respect to the free Hamiltonian. Then we can prove that H_{QED} is self adjoint by Kato-Rellich theorem.

Lemma 1 ***(Self-adjointness, [5, 7, 19])***
H_{QED} is self adjoint on $\mathcal{D}(H_0)$. Moreover H_{QED} is essentially self-adjoint on any core of H_0 and bounded from below.

4.2 Ground State

Let $\mathcal{O}$ be an operator on a Hilbert space. We denote the spectrum of $\mathcal{O}$ by $\sigma(\mathcal{O})$ and the set of eigenvalues of $\mathcal{O}$ by $\sigma_p(\mathcal{O})$. Let us assume that $\mathcal{O}$ is self-adjoint and bounded from below. Then we call $\mathcal{O}$ has a ground state if the infimum of the spectrum is an eigenvalue, i.e. $E_0(\mathcal{O}) := \inf\sigma(\mathcal{O}) \in \sigma_p(\mathcal{O})$. Let us consider the spectrum of the QED Hamiltonian. It is seen that the spectrum of H_0 is as follows :

$$\sigma(H_0) = [0,\infty), \qquad \sigma_p(H_0) = \{0\}.$$

Thus we see that the free Hamiltonian H_0 has the ground state. But it is not trivial that $H_{\rm QED}$ has the ground state even if H_0 has the ground state, since the ground state energy of H_0 is embedded in the continuous spectrum $[0,\infty)$. It is noted that the analytic perturbation theory can be applied to only discrete eigenvalues [16].
The same difficulty occurs in the analysis of the system of particles interacting with massless quantum fields, which includes the non-relativistic QED models. In the last decade, the spectral properties of the system has been successfully analyzed [14]. It is already proven that ground sates of the non-relativistic QED Hamiltonian exist in [4, 9, 10, 12, 18].

It is seen that low energy bosons cause the absence of the ground states [3, 11]. To neglect the influence of low energy photons, we assume the infrared regularity condition :

$$\int_{\mathbf{R}^3} \frac{|\chi_{\rm rad}(\mathbf{k})|^2}{\omega(\mathbf{k})^3} d\mathbf{k} < \infty.$$

Remark In [5, 7] they assume the conditions similar to the infrared regularity condition.

Theorem 2 ***(Existence of ground state, [5, 7, 19])***
Assume that $|\kappa_I|$ and $|\kappa_{II}|$ are sufficiently small. Then H_{QED} has a ground state under the infrared regularity condition.

4.3 Asymptotic fields

In [17], the existence of the asymptotic field of the massive scalar field is proven by applying the cook method. In [8, 12] they prove the existence of the asymptotic radiation fields of non-relativistic QED models by a similar method used in [17]. In [1], the fermionic scattering thoery is considered, and it is also proven that the asymptotic fields exist by applying the cook method. In [19], it is proven that the asymptotic Dirac and radiation fields exists in a similar way of [12].

Theorem 3 ***(Asymptotic fields, [19])***
(1) Let $\xi \in \mathcal{D}(\omega^{-1/2})$. Then for $\Psi \in \mathcal{D}(H)$, the asymptotic field

$$a^{\sharp}_{r,\pm\infty}(\xi)\Psi := s - \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}(I\otimes a^{\sharp}_r(\xi))e^{itH_0}e^{-itH}\Psi,$$

exists.
(2) Let η, $\zeta \in L^2(\mathbf{R}^3)$. *Then the asymptotic fields*

$$b^{\sharp}_{s,\pm\infty}(\eta) := s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} (b^{\sharp}_{s}(\eta) \otimes I) e^{itH_0} e^{-itH},$$
$$d^{\sharp}_{s,\pm\infty}(\zeta) := s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} (d^{\sharp}_{s}(\zeta) \otimes I) e^{itH_0} e^{-itH}$$

exist.

4.4 Properties of Ground States

In [15], Hiroshima proves the uniqueness of the ground states of the general systems which include the non-relativistic QED models. To prove the uniqueness of the ground state, he use asymptotic fields. We can apply this method to QED, and prove the uniqueness of the ground state.

Proposition 4 ***(Uniqueness of ground state, [19])***
Assume that $|\kappa_I|$ *and* $|\kappa_{II}|$ *are sufficiently small. Then dimker* $(H - E_0(H)) = 1$ *holds where* $E_0(H) = \inf \sigma(H)$.

In [12], the absence of the spectral gap of the non-relativistic QED models is proven. He use the asymptotic fields and wave operators. By using the same methods, we can prove the absence of the spectrum gap of the QED Hamiltonian.

Proposition 5 ***(Absence of spectral gap, [19])***
It holds that $\sigma(H) = [E_0(H), \infty)$.

Let us denote the total charge by

$$Q = \sum_{s=\pm 1/2} \int_{\mathbf{R}^3} \left(b^*_s(\mathbf{p}) b_s(\mathbf{p}) - d^*_s(\mathbf{p}) d_s(\mathbf{p}) \right) d\mathbf{p}.$$

In [19], it is proven that H_{QED} and Q commute strongly. Then the state space $\mathfrak{F}_{\text{QED}}$ is decomposed with respect to the spectrum of Q

$$\mathfrak{F}_{\text{QED}} = \bigoplus_{\alpha \in \mathbf{Z}} \mathfrak{F}_{\alpha},$$

and we see the values of the total charge of the ground state.

Theorem 6 ***(Total charge of ground state, [19])***
Let Ψ_g *be the ground state. Assume that* $|\kappa_I|$ *and* $|\kappa_{II}|$ *are sufficiently small. Then value of the total charge of the ground state is zero, i.e.* $\Psi_g \in \mathfrak{F}_0$.

References

[1] Z.Ammari, Scattering theory for a class of fermionic Pauli-Fierz models, J. Funct. Anal. **208** (2004), 302-359,

[2] A.Arai and M.Hirokawa, On the existence and uniqueness of ground states of a generalized spin-boson model, J. Funct. Anal. **151** (1997), 455-503.

[3] A.Arai, M.Hirokawa and H.Hiroshima, On the absence of eigenvectors of Hamiltonians in a class of massless quantum field models without infrared cutoff, J. Funct. Anal. **168** (1999), 470-497.

[4] V.Bach, J.Fröhlich and I.M.Sigal, Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field, Comm. Math. Phys. **207** (1999), 249-290.

[5] J.Barbaroux, M.Dimassi, and J.Guillt, Quantum electrodynamics of relativistic bound states with cutoffs, J.Hyper. Differ. Equa. **1** (2004), 271-314.

[6] J.Derezínski and C.Gérard, Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonian, Rev. Math. Phys. **11** (1999), 383-450.

[7] M.Dimassi and J.C.Guillot, The quantum electrodynamics of relativistic bound states with cutoffs.I, Appl. Math. Lett. **16** (2003), 551-555.

[8] J. Fröhlich, M. Griesemer and B.Schlein, Asymptotic electromagnetic fields in a mode of quantum mechanical matter interacting with the quantum radiation field, Adv. Math. **164** (2001), 349-398.

[9] C.Gérard, On the existence of ground states for massless Pauli-Fierz Hamiltonians, Ann. H. Poincaré, **1** (2000), 443-459.

[10] M.Griesemer, E.Lieb and M.Loss, Ground states in non-relativistic quantum electrodynamics, Invent. Math. **145** (2001), 557-595.

[11] M.Hirokawa, Infrared catastrophe for Nelson's model -non-existence of ground state and soft-boson divergence-, Publ. RIMS, Kyoto Univ. **42** (2006), 897-922.

[12] F.Hiroshima, Ground states and spectrum of quantum electrodynamics of nonrelativistic particles, Trans. Amer. Math. Soc. **353** (2001), 4497-4528.

[13] F.Hiroshima, Analysis of ground states of atoms interacting with a quantized radiation field, *Topics in the Shrödinger operators* , (H.Araki and H.Ezawa eds.) World Scientific, 2004, 145-273.

[14] F.Hiroshima, Perturbation problems of embedded eigenvalues in field theories. (Japanese) Sugaku 57 (2005), no. 1, 70-92.

[15] F.Hiroshima, Multiplicity of ground states in quantum field models; applications of asymptotic fields, J. Funct. Anal. **224** (2005), 431-470.

[16] P.D.Hislop and I.M.Sigal, Introduction to spectral theory, Springer-Verlag, 1996.

[17] R.Høegh-Krohn, Asymptotic fields in some models of quantum field theory I, J. Math. Phys. **9** (1968), 2075-2080.

[18] H.Spohn, Ground state of quantum particle coupled to a scalar boson field, Lett. Math. Phys. **44** (1998), 9-16.

[19] T.Takaesu, The spectral analysis on quantum electrodynamics with spatial cutoff.I, (arxiv).