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On the Uniqueness of Pairs of a Hamiltonian and a Strong Time Operator in Quantum Mechanics

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Abstract

Let H be a self-adjoint operator (a Hamiltonian) on a complex Hilbert space $\mathcal{H}$. A symmetric operator T on $\mathcal{H}$ is called a *strong time operator* of H if the pair (T, H) obeys the operator equation $e^{itH}Te^{-itH} = T + t$ for all $t \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers and i is the imaginary unit). In this note we review some results on the uniqueness (up to unitary equivalences) of the pairs (T, H).

Keywords: canonical commutation relation, Hamiltonian, strong time operator, weak Weyl relation, weak Weyl representation, Weyl representation, spectrum.

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1 Introduction

A pair (T, H) of a symmetric operator T and a self-adjoint operator H on a complex Hilbert space $\mathcal{H}$ is called a *weak Weyl representation* of the canonical commutation relation (CCR) with one degree of freedom if it obeys the *weak Weyl relation*: For all $t \in \mathbb{R}$ (the set of real numbers), $e^{-itH}D(T) \subset D(T)$ (i is the imaginary unit and $D(T)$ denotes the domain of T) and

$$Te^{-itH}\psi = e^{-itH}(T + t)\psi, \ \forall t \in \mathbb{R}, \forall \psi \in D(T). \tag{1.1}$$

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It is easy to see that the weak Weyl relation is equivalent to the *operator equation*

$$e^{itH}Te^{-itH} = T + t, \quad \forall t \in \mathbb{R}, \tag{1.2}$$

implying that $e^{-itH}D(T) = D(T), \forall t \in \mathbb{R}$.

One can prove that, if (T, H) is a weak Weyl representation of the CCR, then (T, H) obeys the CCR

$$[T, H] = i \tag{1.3}$$

on $D(TH) \cap D(HT)$, where $[X, Y] := XY - YX$. But *the converse is not true.*

In the context of quantum theory where H is the Hamiltonian of a quantum system, T is called a *strong time operator* of H [3, 5].

We remark that a standard time operator (simply a time operator) of H is defined to be a symmetric operator T on $\mathcal{H}$ obeying CCR (1.3) on a subspace $\mathcal{D} \neq \{0\}$ (not necessarily dense) of $\mathcal{H}$ (i.e., $\mathcal{D} \subset D(TH) \cap D(HT)$ and $[T, H]\psi = i\psi, \forall \psi \in \mathcal{D}$) (cf. [1]). Obviously this notion of time operator is weaker than that of strong time operator. General classes of time operators (not strong ones) of a Hamiltonian with discrete eigenvalues have been investigated by Galapon [12], Arai-Matsuzawa [9] and Arai [7].

Weak Weyl representations of the CCR were first discussed by Schmüdgen [19, 20] from a purely operator theoretical point of view and then by Miyamoto [14] in application to a theory of time operator in quantum theory. A generalization of a weak Weyl relation was presented by the present author [2] to cover a wider range of applications to quantum physics including quantum field theory.

Arai-Matsuzawa [8] discovered a general structure for construction of a weak Weyl representation of the CCR from a given weak Weyl representation and established a theorem for the former representation to be a Weyl representation of the CCR. These results were extended by Hiroshima-Kuribayashi-Matsuzawa [13] to a wider class of Hamiltonians.

In the previous paper [6] the author considered the problem on uniqueness (up to unitary equivalences) of weak Weyl representations. In the context of theory of time operators, this is a problem on uniqueness (up to unitary equivalences) of pairs (T, H) with H a Hamiltonian and T a strong time operator of H. This problem has an independent interest in the theory of weak Weyl representations. This note is a review of some results obtained in [6].

2 Preliminaries

We denote by $\mathrm{W}(\mathcal{H})$ the set of all the weak Weyl representations on $\mathcal{H}$:

$$\mathrm{W}(\mathcal{H}) := \{(T, H) | (T, H) \text{ is a weak Weyl representation on } \mathcal{H}\}. \tag{2.1}$$

It is easy to see that, if (T, H) is in $\mathrm{W}(\mathcal{H})$, then so are $(\overline{T}, H)$ and $(-T, -H)$, where $\overline{T}$ denotes the closure of T.

For a linear operator A on a Hilbert space, $\sigma(A)$ (resp. $\rho(A)$) denotes the spectrum (resp. the resolvent set) of A (if A is closable, then $\sigma(A) = \sigma(\overline{A})$). Let $\mathbb{C}$ be the set of complex numbers and

$$\Pi_+ := \{z \in \mathbb{C} | \mathrm{Im}\, z > 0\}, \quad \Pi_- := \{z \in \mathbb{C} | \mathrm{Im}\, z < 0\}. \tag{2.2}$$

In the previous paper [4], we proved the following facts:

Theorem 2.1 [4] *Let $(T, H) \in \mathrm{W}(\mathcal{H})$. Then:*

(i) If H is bounded below, then either $\sigma(T) = \overline{\Pi}_+$ (the closure of Π_+) or $\sigma(T) = \mathbb{C}$.

(ii) If H is bounded above, then either $\sigma(T) = \overline{\Pi}_-$ or $\sigma(T) = \mathbb{C}$.

(iii) If H is bounded, then $\sigma(T) = \mathbb{C}$.

This theorem has to be taken into account in considering the uniqueness problem of weak Weyl representations.

A form of representations of the CCR stronger than weak Weyl representations is known as a *Weyl representation* of the CCR which is a pair (T, H) of *self-adjoint* operators on $\mathcal{H}$ obeying the *Weyl relation*

$$e^{itT} e^{isH} = e^{-its} e^{isH} e^{itT}, \quad \forall t, \forall s \in \mathbb{R}. \tag{2.3}$$

It is well known (the von Neumann uniqueness theorem [15]) that, every Weyl representation on a *separable* Hilbert space is unitarily equivalent to a direct sum of the Schrödinger representation (q, p) on $L^2(\mathbb{R})$, where q is the multiplication operator by the variable $x \in \mathbb{R}$ and $p = -iD_x$ with D_x being the generalized differential operator in x (cf. [3, §3.5], [16, Theorem 4.3.1], [17, Theorem VIII.14]).

It is easy to see that a Weyl representation is a weak Weyl representation (but the converse is not true). Therefore, as far as the Hilbert space under consideration is separable, the non-trivial case for the uniqueness problem of weak Weyl representations is the one where they are *not* Weyl representations. A general class of such weak Weyl representations (T, H) are given in the case where H is semi-bounded (bounded below or bounded above). In this case, T is not essentially self-adjoint [2, Theorem 2.8], implying Theorem 2.1.

Two simple examples in this class are constructed as follows:

Example 2.1 Let $a \in \mathbb{R}$ and consider the Hilbert space $L^2(\mathbb{R}_a^+)$ with $\mathbb{R}_a^+ := (a, \infty)$. Let $q_{a,+}$ be the multiplication operator on $L^2(\mathbb{R}_a^+)$ by the variable $\lambda \in \mathbb{R}_a^+$:

$$D(q_{a,+}) := \left\{ f \in L^2(\mathbb{R}_a^+) | \int_a^\infty \lambda^2 |f(\lambda)|^2 d\lambda < \infty \right\}, \tag{2.4}$$

$$q_{a,+} f := \lambda f, \quad f \in D(q_{a,+}) \tag{2.5}$$

and

$$p_{a,+} := -i \frac{d}{d\lambda} \tag{2.6}$$

with $D(p_{a,+}) = C_0^\infty(\mathbb{R}_a^+)$, the set of infinitely differentiable functions on $\mathbb{R}_a^+$ with bounded support in $\mathbb{R}_a^+$. Then it is easy to see that $q_{a,+}$ is self-adjoint, bounded below with $\sigma(q_{a,+}) = [a, \infty)$ and $p_{a,+}$ is a symmetric operator. Moreover, $(-p_{a,+}, q_{a,+})$ is a weak Weyl representation of the CCR. Hence, as remarked above, $(-\overline{p}_{a,+}, q_{a,+})$ also is a weak Weyl representation.

Note that $p_{a,+}$ is not essentially self-adjoint and

$$\sigma(-p_{a,+}) = \sigma(-\overline{p}_{a,+}) = \overline{\Pi}_+. \tag{2.7}$$

In particular, $\pm\overline{p}_{a,+}$ are maximal symmetric, i.e., they have no non-trivial symmetric extensions (e.g., [18, §X.1, Corollary]).

Example 2.2 Let $b \in \mathbb{R}$ and consider the Hilbert space $L^2(\mathbb{R}_b^-)$ with $\mathbb{R}_b^- := (-\infty, b)$. Let $q_{b,-}$ be the multiplication operator on $L^2(\mathbb{R}_b^-)$ by the variable $\lambda \in \mathbb{R}_b^-$. and

$$p_{b,-} := -i \frac{d}{d\lambda} \tag{2.8}$$

with $D(p_{b,-}) = C_0^\infty(\mathbb{R}_b^-)$. Then $q_{b,-}$ is self-adjoint, bounded above with $\sigma(q_{b,-}) = (-\infty, b]$, $p_{b,-}$ is a symmetric operator, and $(-p_{b,-}, q_{b,-})$ is a weak Weyl representation of the CCR. As in the case of $p_{a,+}$, $p_{b,-}$ is not essentially self-adjoint and

$$\sigma(-p_{b,-}) = \overline{\Pi}_-. \tag{2.9}$$

A relation between $(-p_{a,+}, q_{a,+})$ and $(-p_{b,-}, q_{b,-})$ is given as follows. Let $U_{ab} : L^2(\mathbb{R}_a^+) \to L^2(\mathbb{R}_b^-)$ be a linear operator defined by

$$(U_{ab} f)(\lambda) := f(a + b - \lambda), \quad f \in L^2(\mathbb{R}_a^+), \text{ a.e.} \lambda \in \mathbb{R}_b^-.$$

Then U_{ab} is unitary and

$$U_{ab} q_{a,+} U_{ab}^{-1} = a + b - q_{b,-}, \quad U_{ab} p_{a,+} U_{ab}^{-1} = -p_{b,-}. \tag{2.10}$$

In view of the von Neumann uniqueness theorem for Weyl representations, the pair $(-\overline{p}_{a,+}, q_{a,+})$ (resp. $(-\overline{p}_{b,-}, q_{b,-})$) may be a reference pair in classifying weak Weyl representations (T, H) with H being bounded below (resp. bounded above).

By Theorem 2.1, we can define two subsets of $\mathrm{W}(\mathcal{H})$:

$$\mathrm{W}_+(\mathcal{H}) := \{(T, H) \in \mathrm{W}(\mathcal{H}) | H \text{ is bounded below and } \sigma(T) = \overline{\Pi}_+\}, \quad (2.11)$$
$$\mathrm{W}_-(\mathcal{H}) := \{(T, H) \in \mathrm{W}(\mathcal{H}) | H \text{ is bounded above and } \sigma(T) = \overline{\Pi}_-\}. \quad (2.12)$$

Then, as shown above, $(-p_{a,+}, q_{a,+}) \in \mathrm{W}_+(L^2(\mathbb{R}_a^+))$ and $(-p_{b,-}, q_{b,-}) \in \mathrm{W}_-(L^2(\mathbb{R}_b^-))$.

3 Irreducibility

For a set $\mathcal{A}$ of linear operators on a Hilbert space $\mathcal{H}$, we set

$$\mathcal{A}' := \{B \in \mathsf{B}(\mathcal{H}) | BA \subset AB, \forall A \in \mathcal{A}\},$$

called the *strong commutant* of $\mathcal{A}$ in $\mathcal{H}$, where $\mathsf{B}(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$ with $D(B) = \mathcal{H}$.

We say that $\mathcal{A}$ is *irreducible* if $\mathcal{A}' = \{cI | c \in \mathbb{C}\}$, where I is the identity on $\mathcal{H}$.

Proposition 3.1 *For all $a \in \mathbb{R}$, the set $\{\overline{p}_{a,+}, p_{a,+}^*, q_{a,+}\}$ (Example 2.1) is irreducible.*

To prove this proposition, we need a lemma.

Let $a \in \mathbb{R}$ be fixed. For each $t \geq 0$, we define a linear operator $U_a(t)$ on $L^2(\mathbb{R}_a^+)$ as follows: For each $f \in L^2(\mathbb{R}_a^+)$,

$$(U_a(t)f)(\lambda) := \begin{cases} f(\lambda - t) & \lambda > t + a \\ 0 & a < \lambda \leq t + a \end{cases} \quad (3.1)$$

Then it is easy to see that $\{U_a(t)\}_{t \geq 0}$ is a strongly continuous one-parameter semigroup of isometries on $L^2(\mathbb{R}_+^a)$.

Lemma 3.2 *The generator of $\{U_a(t)\}_{t \geq 0}$ is $-i\overline{p}_{a,+}$:*

$$\frac{dU_a(t)f}{dt} = -i\overline{p}_{a,+}U_a(t)f, \quad \forall f \in D(\overline{p}_{a,+}), t \in \mathbb{R}, \quad (3.2)$$

where the derivative in t is taken in the strong sense.

Proof. Let iA be the generator of $\{U_a(t)\}_{t \geq 0}$:

$$\frac{dU_a(t)f}{dt} = iAU_a(t)f, \quad \forall f \in D(A), t \in \mathbb{R}.$$

Then it follows from the isometry of $U_a(t)$ that A is a closed symmetric operator. It is easy to see that $-p_{a,+} \subset A$ and hence $-\overline{p}_{a,+} \subset A$. As already remarked in Example 2.1, $-\overline{p}_{a,+}$ is maximal symmetric. Hence $A = -\overline{p}_{a,+}$. ∎

Proof of Proposition 3.1

Let $B \in \{\overline{p}_{a,+}, p_{a,+}^*, q_{a,+}\}'$. Then

$$B\overline{p}_{a,+} \subset \overline{p}_{a,+}B, \tag{3.3}$$
$$Bp_{a,+}^* \subset p_{a,+}^*B, \tag{3.4}$$
$$Bq_{a,+} \subset q_{a,+}B. \tag{3.5}$$

As in the case of bounded linear operators on $L^2(\mathbb{R})$ strongly commuting with q (the multiplication operator by the variable $x \in \mathbb{R}$) [3, Lemma 3.13], (3.5) implies that there exists an essentially bounded function F on $\mathbb{R}_a^+$ such that $B = M_F$, the multiplication operator by F.

Let $f \in D(\overline{p}_{a,+})$ and $g(t) := BU_a(t)f$. Then, by Lemma 3.2, g is strongly differentiable in $t \geq 0$ and

$$\frac{dg(t)}{dt} = B(-i\overline{p}_{a,+})U_a(t)f = -i\overline{p}_{a,+}g(t),$$

where we have used (3.3). Note that $g(0) = Bf$. Hence, by the uniqueness of solutions of the initial value problem on differential equation (3.2), we have $g(t) = U_a(t)Bf$. Therefore it follows that $BU_a(t) = U_a(t)B, \forall t \geq 0$. Hence $FU_a(t)f = U_a(t)Ff, \forall f \in L^2(\mathbb{R}_a^+)$, which implies that

$$F(\lambda)f(\lambda - t) = F(\lambda - t)f(\lambda - t), \quad \lambda > t + a.$$

Hence $F(\lambda) = F(\lambda + t)$, a.e.$\lambda > 0, \forall t > 0$. This means that F is equivalent to a constant function. Hence $B = M_F = cI$ with some $c \in \mathbb{C}$. ∎

Proposition 3.3 *For all $b \in \mathbb{R}$, the set $\{\overline{p}_{b,-}, p_{b,-}^*, q_{b,-}\}$ (Example 2.2) is irreducible.*

Proof. Let $B \in \{\overline{p}_{b,-}, p_{b,-}^*, q_{b,-}\}'$. Then, by (2.10), the operator $C := U_{ab}^{-1}BU_{ab}$ is in $\{\overline{p}_{a,+}, p_{a,+}^*, q_{a,+}\}'$. Hence, by Proposition 3.1, $C = cI$ with some constant $c \in \mathbb{C}$. Thus $B = cI$. ∎

4 Uniqueness Theorem

One can prove the following theorem:

Theorem 4.1 *Let $\mathcal{H}$ be separable and $(T, H) \in \mathrm{W}_+(\mathcal{H})$ with $\varepsilon_0 := \inf \sigma(H)$. Suppose that $\{\overline{T}, T^*, H\}$ is irreducible. Then there exists a unitary operator $U : \mathcal{H} \to L^2(\mathbb{R}_{\varepsilon_0}^+)$ such that*

$$U\overline{T}U^{-1} = -\overline{p}_{\varepsilon_0,+}, \quad UHU^{-1} = q_{\varepsilon_0,+}. \tag{4.1}$$

In particular

$$\sigma(H) = [\varepsilon_0, \infty). \tag{4.2}$$

Remark 4.1 It is known that, for every weak Weyl representation $(T, H) \in \mathrm{W}(\mathcal{H})$ ($\mathcal{H}$ is not necessarily separable), H is purely absolutely continuous [14, 19].

We prove Theorem 4.1 in the next section. For the moment, we note a result which immediately follows from Theorem 4.1:

Theorem 4.2 *Let $\mathcal{H}$ be separable and $(T, H) \in \mathrm{W}_-(\mathcal{H})$ with $b := \sup \sigma(H)$. Suppose that $\{\overline{T}, T^*, H\}$ is irreducible. Then there exists a unitary operator $V : \mathcal{H} \to L^2(\mathbb{R}_b^-)$ such that*

$$V\overline{T}V^{-1} = -\overline{p}_{b,-}, \quad VHV^{-1} = q_{b,-}. \tag{4.3}$$

In particular

$$\sigma(H) = (-\infty, b]. \tag{4.4}$$

Proof. As remarked in Section 2, $(-T, -H) \in \mathrm{W}_+(\mathcal{H})$ with $a := \inf \sigma(-H) = -b$ and $\sigma(-T) = \overline{\Pi}_+$. Hence, we can apply Theorem 4.1 to conclude that there exists a unitary operator $U : \mathcal{H} \to L^2(\mathbb{R}_a^+)$ such that

$$U\overline{T}U^{-1} = \overline{p}_{a,+}, \quad UHU^{-1} = -q_{a,+}.$$

By Example 2.2, we have

$$U_{ab}\overline{p}_{a,+}U_{ab}^{-1} = -\overline{p}_{b,-}, \quad U_{ab}q_{a,+}U_{ab}^{-1} = -q_{b,-},$$

where we have used that $a+b = 0$. Hence, putting $V := U_{ab}U$, we obtain the desired result. ∎

Remark 4.2 In view of Theorems 4.1 and 4.2, it would be interesting to know when $\sigma(T) = \overline{\Pi}_+$ (resp. $\overline{\Pi}_-$) for $(T, H) \in \mathrm{W}(\mathcal{H})$ with H bounded below (resp. above). Concerning this problem, we have the following results [5]:

(i) Let $(T, H) \in \mathrm{W}(\mathcal{H})$ and H be bounded below. Suppose that, for some $\beta_0 > 0$, $\mathrm{Ran}(e^{-\beta_0 H}T)$ (the range of $e^{-\beta_0 H}T$) is dense in $\mathcal{H}$. Then $\sigma(T) = \overline{\Pi}_+$.

(ii) Let $(T, H) \in \mathrm{W}(\mathcal{H})$ and H be bounded above. Suppose that, for some $\beta_0 > 0$, $\mathrm{Ran}(e^{\beta_0 H}T)$ is dense in $\mathcal{H}$. Then $\sigma(T) = \overline{\Pi}_-$.

5 Proof of Theorem 4.1

Lemma 5.1 *Let S be a closed symmetric operator on $\mathcal{H}$ such that $\sigma(S) = \overline{\Pi}_+$. Then there exists a unique strongly continuous one-parameter semi-group $\{Z(t)\}_{t\geq 0}$ whose generator is iS. Moreover, each $Z(t)$ is an isometry:*

$$Z(t)^* Z(t) = I, \quad \forall t \geq 0. \tag{5.1}$$

Proof. This fact is probably well known. But, for completeness, we give a proof. By the assumption $\sigma(S) = \overline{\Pi}_+$, we have $\sigma(iS) = \{z \in \mathbb{C} | \mathrm{Re}\, z \leq 0\}$. Therefore the positive real axis $(0, \infty)$ is included in the resolvent set $\rho(iS)$ of iS. Since S is symmetric, it follows that

$$\|(iS - \lambda)^{-1}\| \leq \frac{1}{\lambda}, \quad \lambda > 0.$$

Hence, by the Hille-Yosida theorem, iS generates a strongly continuous one-parameter semi-group $\{Z(t)\}_{t\geq 0}$ of contractions. For all $\psi \in D(iS) = D(S)$, $Z(t)\psi$ is in $D(S)$ and strongly differentiable in $t \geq 0$ with

$$\frac{d}{dt} Z(t)\psi = iSZ(t)\psi = Z(t) iS\psi.$$

This equation and the symmetricity of S imply that $\|Z(t)\psi\|^2 = \|\psi\|^2, \forall t \geq 0$. Hence (5.1) follows. ∎

Lemma 5.2 *Let $(T, H) \in \mathrm{W}_+(\mathcal{H})$. Then there exists a unique strongly continuous one-parameter semi-group $\{U_T(t)\}_{t\geq 0}$ whose generator is $i\overline{T}$. Moreover, each $U_T(t)$ is an isometry and*

$$U_T(t) e^{-isH} = e^{its} e^{-isH} U_T(t), \quad t \geq 0, s \in \mathbb{R}. \tag{5.2}$$

Proof. We can apply Lemma 5.1 to $S = \overline{T}$ to conclude that $i\overline{T}$ generates a strongly continuous one-parameter semi-group $\{U_T(t)\}_{t\geq 0}$ of isometries on $\mathcal{H}$. For all $\psi \in D(\overline{T})$ and all $t \geq 0$, $U_T(t)\psi$ is in $D(\overline{T})$ and strongly differentiable in $t \geq 0$ with

$$\frac{d}{dt} U_T(t)\psi = i\overline{T} U_T(t)\psi = U_T(t) i\overline{T}\psi.$$

Let $s \in \mathbb{R}$ be fixed and $V(t) := e^{its} e^{-isH} U_T(t) e^{isH}$. Then $\{V(t)\}_{t\geq 0}$ is a strongly continuous one-parameter semi-group of isometries. Let $\psi \in D(\overline{T})$. Then $e^{-isH}\psi \in D(\overline{T})$ and

$$\overline{T} e^{-isH}\psi = e^{-isH} \overline{T}\psi + s e^{-isH}\psi.$$

Hence $V(t)\psi$ is in $D(\overline{T})$ and strongly differentiable in t with

$$\frac{d}{dt}V(t)\psi = i\overline{T}V(t)\psi.$$

This implies that $V(t)\psi = U_T(t)\psi, \forall t \in \mathbb{R}$. Since $D(\overline{T})$ is dense, it follows that $V(t) = U_T(t), \forall t \in \mathbb{R}$, implying (5.2). ∎

We recall a result of Bracci and Picasso [10]. Let $\{U(\alpha)\}_{\alpha\geq 0}$ and $\{V(\beta)\}_{\beta\in\mathbb{R}}$ be a strongly continuous one-parameter semi-group and a strongly continuous one-parameter unitary group on $\mathcal{H}$ respectively, satisfying

$$U(\alpha)^*U(\alpha) = I, \quad \alpha \geq 0, \tag{5.3}$$

$$U(\alpha)V(\beta) = e^{i\alpha\beta}V(\beta)U(\alpha), \quad \alpha \geq 0, \beta \in \mathbb{R}. \tag{5.4}$$

Then, by the Stone theorem, there exists a unique self-adjoint operator P on $\mathcal{H}$ such that

$$V(\beta) = e^{-i\beta P}, \quad \beta \in \mathbb{R}. \tag{5.5}$$

Lemma 5.3 [10] *Let $\mathcal{H}$ be separable and P is bounded below with $\nu := \inf \sigma(P)$. Suppose that $\{U(\alpha), U(\alpha)^*, V(\beta) | \alpha \geq 0, \beta \in \mathbb{R}\}$ is irreducible. Then, there exists a unitary operator $Y : \mathcal{H} \longrightarrow L^2(\mathbb{R}_\nu^+)$ such that*

$$YV(\beta)Y^{-1} = e^{-i\beta q_{\nu,+}}, \beta \in \mathbb{R}, \tag{5.6}$$

$$YU(\alpha)Y^{-1} = U_\nu(\alpha), \quad \alpha \geq 0. \tag{5.7}$$

We denote the generator of $\{U(\alpha)\}_{\alpha\geq 0}$ by iQ. It follows that Q is closed and symmetric.

Lemma 5.4 *Under the assumption of Lemma 5.3,*

$$YPY^{-1} = q_{\nu,+}, \tag{5.8}$$

$$YQY^{-1} = -\overline{p}_{\nu,+}. \tag{5.9}$$

In particular

$$\sigma(P) = [\nu, \infty). \tag{5.10}$$

Proof. Lemma 5.3 and (5.6) imply (5.8). Simlarly (5.9) follows from Lemma 5.3, (5.7) and Lemma 3.2. ∎

Lemma 5.5 *Let $(T, H) \in \mathrm{W}(\mathcal{H})$ with $\sigma(T) = \overline{\Pi}_+$. Suppose that $\{\overline{T}, T^*, H\}$ is irreducible. Then $\{U_T(t), U_T(t)^*, e^{-isH} | t \geq 0, s \in \mathbb{R}\}$ is irreducible.*

Proof. Let $B \in \mathsf{B}(\mathcal{H})$ be such that

$$BU_T(t) = U_T(t)B, \tag{5.11}$$
$$BU_T(t)^* = U_T(t)^*B, \tag{5.12}$$
$$Be^{-isH} = e^{-isH}B, \forall t \geq 0, \forall s \in \mathbb{R}. \tag{5.13}$$

Let $\psi \in D(\overline{T})$. Then, by (5.11), we have $BU_T(t)\psi = U_T(t)B\psi, \forall t \geq 0$. By Lemma 5.2, the left hand side is strongly differentiable in t with $d(BU_T(t)\psi)/dt = iB\overline{T}U_T(t)\psi$. Hence so does the right hand side and we obtain that $B\psi \in D(\overline{T})$ and $B\overline{T}\psi = \overline{T}B\psi$. Therefore $B\overline{T} \subset \overline{T}B$. Note that (5.12) implies that $U_T(t)B^* = B^*U_T(t)$. Hence it follows that $B^*\overline{T} \subset \overline{T}B^*$, which implies that $BT^* \subset T^*B$, where we have used the following general facts: for every densely defined closable linear operator A on $\mathcal{H}$ and all $C \in \mathsf{B}(\mathcal{H})$, $(CA)^* = A^*C^*, (AC)^* \supset C^*A^*, (\bar{A})^* = A^*$. Similarly (5.13) implies that $BH \subset HB$. Hence $B \in \{\overline{T}, T^*, H\}'$. Therefore $B = cI$ for some $c \in \mathbb{C}$.

Proof of Theorem 4.1

By Lemmas 5.2 and 5.5, we can apply Lemma 5.3 to the case where $V(\beta) = e^{-i\beta H}, \beta \in \mathbb{R}$ and $U(\alpha) = U_T(\alpha), \alpha \geq 0$. Then the desired results follow from Lemmas 5.3 and 5.4. ∎

Remark 5.1 Recently Bracci and Picasso [11] have obtained an interesting result on the reducibility of the von Neumann algebra generated by $\{U(\alpha), U(\alpha)^*, V(\beta) | \alpha \geq 0, \beta \in \mathbb{R}\}$ obeying (5.3) and (5.4). By employing the result, one can generalize Theorem 4.1 to the case where $\{\overline{T}, T^*, H\}$ is not necessarily irreducible.

6 Application to Construction of a Weyl representation

In the previous paper [8], a general structure was found to construct a Weyl representation from a weak Weyl representation. Here we recall it.

Theorem 6.1 [8, Corollary 2.6] *Let (T, H) be a weak Weyl representation on a Hilbert space $\mathcal{H}$ with T closed. Then the operator*

$$L := \log |H| \tag{6.1}$$

is well-defined, self-adjoint and the operator

$$D := \frac{1}{2}(TH + \overline{HT}) \tag{6.2}$$

is a symmetric operator. Moreover, if D *is essentially self-adjoint, then* $(\overline{D}, L)$ *is a Weyl representation of the CCR and* $\sigma(|H|) = [0, \infty)$.

To apply this theorem, we need a lemma.

Lemma 6.2 [6]*Let* $a \in \mathbb{R}$ *and*

$$d_a := -\frac{1}{2}(p_{a,+}q_{a,+} + \overline{q_{a,+}p_{a,+}}) \tag{6.3}$$

acting in $L^2(\mathbb{R}_a^+)$. *Then* d_a *is essentially self-adjoint if and only if* $a = 0$.

Theorem 6.3 *Let* $\mathcal{H}$ *be separable and* $(T, H) \in \mathrm{W}_+(\mathcal{H})$ *with* $\inf \sigma(H) = 0$ *and* T *closed. Suppose that* $\{T, T^*, H\}$ *is irreducible. Let* L *and* D *be as in (6.1) and (6.2) respectively. Then* D *is essentially self-adjoint and* $(\overline{D}, L)$ *is a Weyl representation of the CCR.*

Proof. Let $\hat{d}_0$ be the operator d_0 with $p_{0,+}$ replaced by $\overline{p}_{0,+}$. Then, by Theorem 4.1, D is unitarily equivalent to $\hat{d}_0$. We have $d_0 \subset \hat{d}_0$. By Lemma 6.2, d_0 is essentially self-adjoint. Hence $\hat{d}_0$ is essentially self-adjoint. Therefore it follows that D is essentially self-adjoint. The second half of the theorem follows from Theorem 6.1. ∎

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