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On Monotone Convolution and Monotone Infinitely Divisibility

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1 Introduction: Non-commutative Probability Theory

Gel'fand-Naimark Theorem is important to understand the essence of non-commutative probability theory. Let Ω be a locally compact Hausdorff space. We denote by $C_0(\Omega)$ the set of continuous functions on Ω which vanish at infinity. The following theorem is a consequence of the Gel'fand-Naimark theorem.

Theorem 1.1. *Let $\mathcal{A}$ be a commutative C^* algebra with a state ϕ. Then there exist a locally compact Hausdorff space Ω and a Radon measure μ such that $\mathcal{A}$ is isomorphic to $C_0(\Omega)$ by a map $\widehat{}: \mathcal{A} \to C_0(\Omega)$, $A \mapsto \widehat{A}$, and such that*

$$\phi(A) = \int_\Omega \widehat{A}(\omega)d\mu(\omega). \tag{1.1}$$

Therefore, a probability space (Ω, μ), where Ω is a locally compact Hausdorff space and μ is a Radon measure, is seen to be a commutative C^* algebra with a state. One can extend the technique of probability theory to a non-commutative algebra equipped with a state. In other words, we can construct a theory in the setting of an algebra equipped with a state, which reduces to the usual probability theory when the algebra is commutative. Such a theory is called non-commutative probability theory. In particular, the fundamental notion of independence in probability theory can be considered also in a non-commutative algebra with a state. An interesting aspect in non-commutative probability theory is that there are various notions of independence, which cannot be realized in commutative algebras, i.e., in the usual probability theory. Today many notions of independence are known. Some examples are shown in Table 1. We give the definition of monotone independence in Section 2. For boolean independence and free independence, the reader is referred to [S-W] and [V-D-N].

There is a central limit theorem for each notion of independence, which we now explain briefly. If $\phi(X_k) = 0, \phi(X_k^2) = 1$ and X_k's are "independent" self-adjoint operators, the probability distribution of

$$\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}$$

converges to some limit distribution as $n \to \infty$. The limit distribution changes as we exchange the notion of independence.

Table 1: Several notions of independence and the limit distributions in central limit theorems

independence	probability distribution	orthogonal polynomials
boson	$\frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx$ on $\mathbb{R}$	Hermite polynomials
free	$\sqrt{2 - x^2} dx$ on $[-\sqrt{2}, \sqrt{2}]$	Chebysheff polynomials of the second kind
boolean	$\frac{1}{2}(\delta_1 + \delta_{-1})$	$1, x$
monotone	$\frac{1}{\pi\sqrt{2-x^2}} dx$ on $[-\sqrt{2}, \sqrt{2}]$	Chebysheff polynomials of the first kind

Another interesting aspect is the deformation of Fock space which is called an interacting Fock space. This aspect is also connected with independence, which we explain in Section 2 in the case of monotone independence. In the usual probability theory, the connection to a Boson Fock space is well known through the Wiener-Itô isomorphism. In the case of free independence, the reader is referred to [K-S] for the Fock structure and a stochastic analysis.

The motivation of the author is to clarify the similarity and dissimilarity among the various notions of independence and to find a way to treat these notions in a more general viewpoint. Then it will be important to study properties of each notion of independence. Monotone independence is, however, not well understood in terms of probability distributions. In this research, we aim to clarify properties of infinitely divisible distributions in monotone probability theory.

2 Monotone Independence

Definition 2.1. Let $\mathcal{A}$ be a C^* algebra and let ϕ be a state.
(1) Let $\{\mathcal{A}_m\}_{m=1}^n$ be a sequence of $*$-subalgebras in $\mathcal{A}$. Then $\{\mathcal{A}_m\}_{m=1}^n$ is said to be *monotone independent* if the following condition holds.

$$\phi(a_1 a_2 \cdots a_n) = \phi(a_k)\phi(a_1 a_2 \cdots \check{a}_k \cdots a_n) \tag{2.1}$$
$$\text{if } a_m \in \mathcal{A}_{i_m} \text{ for all } 1 \le m \le n \text{ and } k \text{ satisfies } i_{k-1} < i_k > i_{k+1}.$$

If $k = 1$ (resp. $k = n$), the above inequality is understood to be $i_1 > i_2$ (resp. $i_{n-1} < i_n$).
(2) Let $\{b_i\}_{i=1}^n$ be a sequence of elements in $\mathcal{A}$. $\{b_i\}_{i=1}^n$ is said to be monotone independent if the $*$-algebras $\mathcal{A}_i$ generated by each b_i without unit form a monotone independent family.

Let $\mathcal{A}$ be a C^* algebra. For a self-adjoint element $a \in \mathcal{A}$, there exists a unique probability distribution μ of a satisfying

$$\int_{\mathbb{R}} f(x)\mu(dx) = \phi(f(a)), \quad f \in C_b(\mathbb{R}),$$

where $f(a) \in \mathcal{A}$ is defined by the functional calculus. The existence of μ is a consequence of the Riesz-Markov-Kakutani theorem. The distribution ν of b is defined similarly. Moreover, if a and b are monotone independent, we can define monotone convolution $\mu \rhd \nu$ of

μ and ν by the distribution of $a + b$, which is a basic object in probability theory. When we consider only convolutions, we can forget the original self-adjoint elements a and b; we can treat only probability distributions.

Basic properties

(1) Monotone convolution is characterized by the reciprocal of Cauchy transform:

$$H_{\mu \triangleright \nu} = H_\mu \circ H_\mu,$$

where

$$H_\mu(z) := \left(\int_{\mathbb{R}} \frac{1}{z - x} d\mu(x) \right)^{-1}.$$

This relation is seen to be the analogue of the relation between convolution and Fourier transform: $\widehat{\mu * \nu} = \widehat{\mu} \cdot \widehat{\nu}$.

(2) $\mu \triangleright \nu$ is linear only in the left factor:

$$(a\mu + b\lambda) \triangleright \nu = a(\mu \triangleright \nu) + b(\lambda \triangleright \nu), \quad a + b = 1, a, b > 0.$$

(3) Monotone convolution is associative and non-commutative.

Definition 2.2. A probability distribution μ is called monotone infinitely divisible if for any $n \in \mathbb{N}$ there exists μ_n such that $\mu = \mu_n \triangleright \cdots \triangleright \mu_n$ (n times).

Theorem 2.3 (Muraki (unpublished), Belinschi06). *There is a one-to-one correspondence among the following four objects:*

(1) *a monotone infinitely divisible distribution μ;*

(2) *a weakly continuous monotone convolution semigroup $\{\mu_t\}$ with $\mu_0 = \delta_0, \mu_1 = \mu$;*

(3) *a composition semigroup of reciprocal Cauchy transforms $\{H_t\}$ ($H_t \circ H_s = H_{t+s}$) with $H_0 = id$, $H_1 = H_\mu$, where $H_t(z)$ is a continuous function of $t \geq 0$ for any $z \in \mathbb{C} \setminus \mathbb{R}$;*

(4) *a vector field on the upper half plane $A(z) = \lim_{t \searrow 0} \frac{H_t(z) - z}{t}$, which has the form $A(z) = -\gamma + \int_{\mathbb{R}} \frac{1 + xz}{x - z} d\tau(x)$, where $\gamma \in \mathbb{R}$ and τ is a positive finite measure. (This is the Lévy-Khintchine formula in monotone probability theory.)*

Example 2.4. We show examples of monotone infinitely divisible distributions.

arcsine law: $d\mu_t(x) = \frac{1}{\pi \sqrt{2t - x^2}} dx$ on $(-\sqrt{2t}, \sqrt{2t})$,
$H_t(z) = \sqrt{z^2 - 2t}$, $A(z) = -\frac{1}{z}$,

delta measure: $d\mu_t = \delta_{at}$, $H_t(z) = z + at$, $A(z) = a$,

Cauchy distribution: $d\mu_t(x) = \frac{bt}{\pi(x^2 + (bt)^2)} dx$, $H_t(z) = z + ibt$.

Now we explain the structure of a Fock space in monotone probability theory. The reader is referred to [Lu], [Mur1] and [Mur2] for details. The monotone Fock space over $L^2(\mathbb{R})$ is defined by

$$\Gamma_m(L^2(\mathbb{R})) := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2(\mathbb{R}_<^n),$$

where $\mathbb{R}_<^n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n; x_1 < x_2 < \cdots < x_n\}$. This structure is important in understanding a Brownian motion. A Brownian motion is defined by the operator

$$B_m(t) := a_m(1_{[0,t)}) + a_m^*(1_{[0,t)}),$$

where $a_m^*(f)$ is defined by

$$a_m^*(f)f_1 \otimes \cdots \otimes f_n := f \otimes f_1 \otimes \cdots \otimes f_n$$

for $f \in L^2(\mathbb{R})$. $a_m(f)$ is the adjoint operator of $a_m^*(f)$. The following fact indicates the connection between monotone independence and the monotone Fock space over $L^2(\mathbb{R})$.

Theorem 2.5. *The Brownian motion on the monotone Fock space has independent increments w.r.t. the vacuum state: for any $0 < t_1 < \cdots < t_n < \infty$,*

$$B_m(t_2) - B_m(t_1), \cdots, B_m(t_n) - B_m(t_{n-1})$$

are monotone independent.

3 Main Results

3.1 Properties of atomic measures

It is expected that the behavior of atomic measures plays an important role in the understanding of a convolution. We show a property of atomic measures which is very different from the usual convolution $*$.

Theorem 3.1. (1) *Let $\nu := \sum_{k=1}^{m} \lambda_k \delta_{a_k}$ be an atomic probability measure such that $\lambda_k > 0$, $\sum \lambda_k = 1$ and $a_1 < a_2 < \cdots < a_m$. For any $b \in \mathbb{R}$, $b \neq 0$, $\delta_b \triangleright \nu$ has distinct m atoms. When we write $\delta_b \triangleright \nu = \sum_{k=1}^{m} \mu_k \delta_{b_k}$ with $b_1 < \cdots < b_m$, the atoms satisfy either $b_1 < a_1 < b_2 < a_2 < \cdots < a_{m-1} < b_m < a_m$ or $a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m$. The coefficients μ_k are given by $\mu_i = \frac{\prod_{k=1}^{m}(b_i - a_k)}{b \prod_{k \neq i}^{m}(b_i - b_k)}$.*
(2) *Moreover, if b and c are distinct real numbers, the $2m$ atoms appearing in $\nu_b = \delta_b \triangleright \nu$ and $\nu_c = \delta_c \triangleright \nu$ are all different.*

Remark 3.2. Theorem 3.1 shows a sharp difference between $\delta_b \triangleright \nu$ and $\delta_b * \nu$: for instance, we can take $b > 0$ large enough so that the atoms $\{b_j\}$ of $\delta_b * \nu$ satisfy $a_1 < a_2 < \cdots < a_m < b_1 < b_2 < \cdots < b_m$, since $b_j = a_j + b$.

Corollary 3.3. *Let μ be an atomic probability measure with distinct m atoms and let ν be an atomic probability measure with distinct n atoms. Then $\mu \triangleright \nu$ consists of exactly distinct mn atoms.*

3.2 Properties of Monotone Infinitely Divisible Distributions

The Lebesgue decomposition (decomposition into an atomic part, a singular continuous part and an absolutely continuous part) of a probability distribution is a basic property. We state the result of the atomic part of a monotone infinitely divisible distribution.

Theorem 3.4. *If a probability measure ν contains more than one atom and if one of the atoms is isolated, then ν is not $\triangleright$-infinitely divisible.*

Thus the number of atoms is restricted strongly in monotone probability theory. This is a property special to monotone probability theory: in probability theory, there are many infinitely divisible distributions which contain more than one atom such as Poisson distribution. In contrast to the above, we show three theorems which are closely similar to the results in probability theory.

Theorem 3.5. *Let $\{\mu_t\}_{t\geq 0}$ be a weakly continuous $\triangleright$-convolution semigroup with $\mu_0 = \delta_0$. Then the following statements are equivalent:*

(1) *there exists $t_0 > 0$ such that $\mathrm{supp}\,\mu_{t_0} \subset [0,\infty)$;*

(2) *$\mathrm{supp}\,\mu_t \subset [0,\infty)$ for all $0 \leq t < \infty$;*

(3) *A is analytic in $\mathbb{C}\setminus[0,\infty)$ and $A < 0$ on $(-\infty,0)$;*

(4) *$\mathrm{supp}\,\tau \subset [0,\infty)$, $\tau(\{0\}) = 0$, $\int_0^\infty \frac{1}{x}d\tau(x) < \infty$ and $\gamma \geq \int_0^\infty \frac{1}{x}d\tau(x)$.*

It is interesting that the characterization (4) is the same as the one in Theorem 24.11 in [Sat]. The following fact is also similar to the result in probability theory. We say a probability distribution μ is symmetric if $\mu(-dx) = \mu(dx)$.

Theorem 3.6. *We assume that the support of each μ_t is compact (this is a time-independent property). Then the following statements are all equivalent.*

(1) *There exists $t_0 > 0$ such that μ_{t_0} is symmetric.*

(2) *μ_t is symmetric for all $t > 0$.*

(3) *$\gamma = 0$ and τ is symmetric.*

Recently, so-called power-law behaviors become more important in physics. The notion of (strictly) stable distributions is very important in connection to power-law behaviors such as in fractals. We can define strictly stable distributions in the non-commutative case, when one notion of independence is given. Therefore, we define strictly stable distributions in monotone probability theory.

Definition 3.7. Let $\{\mu_t\}$ be a weakly continuous monotone convolution semigroup with $\mu_0 = \delta_0$. μ_1 is called strictly $\triangleright$-stable if for any $a > 0$ there exists $b(a) > 0$ such that $\mu_a(dx) = \mu_1(b(a)^{-1}dx)$. (In terms of a process, this is understood to be the relation $X_a \equiv b(a)X_1$.)

Moreover, it can be proved that there exists an index α: $b(a) = a^{\frac{1}{\alpha}}$.

Theorem 3.8. *Define a semigroup of distributions* $\{\mu_t^{(\alpha,b)}\}_{t\geq0}$ *by their reciprocal Cauchy transforms:*

$$H_t^{(\alpha,b)}(z) := (z^\alpha + bt)^{\frac{1}{\alpha}}.$$

Any strictly stable distribution coincides with $\mu_1^{(\alpha,b)}$ *for some* α, b. *Here* α *is the index of* $\mu_1^{(\alpha,b)}$. α *satisfies* $0 < \alpha \leq 2$ *and* b *satisfies* $0 \leq \arg b \leq \alpha\pi$ *if* $0 < \alpha \leq 1$; $(\alpha - 1)\pi \leq \arg b \leq \pi$ *if* $1 < \alpha \leq 2$.

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