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Quantum algorithm for SAT problem with entangled degree

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Abstract

There are some quantum algorithms to solve SAT problem. Ohya, Masuda and Volovich showed the quantum algorithm with chaos amplifier solving it in polynomial time. Ohya and Accardi proposed the quantum algorithm using a stochastic limit.

In this paper, we show a new approach by using an entangled degree. We calculate it with the merginal states of the result state to check whether the SAT condition is held or not.

1 Introduction

Ohya, Masuda and Volovich proposed a quantum algorithm with chaos amplifier for SAT problem[1, 2]. The computational complexity of it is in polynomial of input data[7]. The part of calculation of objective function is written as a product of unitary operators and there are no Black boxes, so called Oracles. Since the probability to obtain the correct result is very small in some cases, we apply a chaos amplifier to make it larger than $1/2$. The whole process of the quantum algorithm can be written in the form of generalized quantum Turing machine which is a mathematical model of quantum computation defined in the paper[8].

In this paper, we propose the new quantum algorithm which calculates an entangled degree of the result state in order to check that it holds the conditions of problem. Entanglement is the one of properties of quantum state that the classical state does not have. To calculate an entangled degree, we can reduce the computational complexity of quantum algorithm. We show the examples of Deutch-Jozsa problem and SAT problem.

2 Outline of Quantum Algorithm

We first explain the usual quantum algorithm that is represented by unitary operators. The computational complexity of them is defined by the number of fundamental quantum gates in it, introduced below.

For the mathematical expression of problem, we construct the quantum algorithm for it following these steps.

1. Define the Hilbert space for computation.
2. Construct the initial state.
3. Construct the unitary operators to solve the problem.
4. Apply it for the initial density operator and obtain the result.
5. Measure the observable with the result state.

In the first step, we define the Hilbert space by the instance of the problem. This Hilbert space should be qubit space for correspondence to digital computation that uses classical bits. Let $\mathcal{H} = \mathbb{C}^2$ be a Hilbert space spanned by $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, a normalized vector $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ on $\mathcal{H}$ is called a qubit.

One can apply Hadamard transformation defined by

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

to create a superposition. For $|0\rangle$ and $|1\rangle$, it works as

$$\begin{aligned} H|0\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \\ H|1\rangle &= \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle . \end{aligned}$$

Here we introduce logical gates on qubit space, which are NOT gate, C-NOT gate and CC-NOT gate. We call these gates fundamental gates. We can also construct AND and OR gate by considering the product of fundamental gates and some imprementations. NOT gate U_{NOT} is defined on one qubit Hilbert space as

$$U_{NOT} = |1\rangle \langle 0| + |0\rangle \langle 1| .$$

C-NOT U_{CN} gate and CC-NOT U_{CCN} are given on two and three qubit Hilbert space as

$$\begin{aligned} U_{CN} &= |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U_{NOT} \\ U_{CCN} &= |0\rangle\langle 0| \otimes I \otimes I + |1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes U_{NOT}, \end{aligned}$$

respectively. The unitary operator to solve the problem is constructed by these fundamental gates.

3 SAT problem

The followings are discussed more precisely in the papers[1, 2, 4, 7]. Let $X = \{x_1, \cdots, x_n, \bar{x}_1, \cdots, \bar{x}_n\}$ be a set of literals and $t : X \to \{0,1\}$ a truth assignment holding $t(x) = 1 - t(\bar{x})$. Let $F(X)$ ne all subsets of literals and an element of it is called a clause. Let $C = \{C_1, \cdots, C_m | C_k \in F(X)\}$ be a set of clauses.

[Problem(SAT problem)] Are there any truth assignments to make $f(C) = 1$ (satisffiable)?

$$f(C) \equiv \wedge_{i=1}^{m} \vee_{x_j \in C_i} t(x_j)$$

SAT problem is one of NP-complete problems. In [4] we discussed the quantum algorithm of the SAT problem. In [1, 2] it is shown that the chaotic quantum algorithm can solve the SAT problem in polynomial time.

4 OMV SAT algorithm

The computational basis of this algorithm is on the Hilbert space $\mathcal{H} = \left(\mathbb{C}^2\right)^{\otimes n+\mu+1}$ where μ is a number of dust qubits. Let

$$|v_{in}\rangle \equiv |0^n, 0^\mu, 0\rangle$$

be an initial state vector. A unitary operator $U_C : \mathcal{H} \to \mathcal{H}$ computes $t(C)$ for truth assignment e_i $(i = 1, \cdots, 2^{n-1})$ as follows

$$\begin{aligned} U_C |v_{in}\rangle &= U_C |0^n, 0^\mu, 0\rangle \\ &= \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |e_i, d^\mu, t(C)\rangle \\ &\equiv |v_{out}\rangle \end{aligned}$$

where $|d^\mu\rangle$ is dust qubits denoted by μ strings of binary symbols.

Theorem 1 *[7]The number μ of dust qubits for algorithm of SAT problem is*

$$\mu \leq 2nm$$

Theorem 2 *[7]For a set of clauses $C = \{C_1, \ldots, C_m\}$, we can construct the unitary operator U_C to calculate the truth value of C as*

$$U_C \equiv \prod_{i=1}^{m-1} U_{AND}(i) \prod_{j=1}^{m} U_{OR}(j) H(n)$$

where

$$H(k) = (H \cdot I)^{\otimes k} I^{\otimes N-k}$$

U_{AND} and U_{OR} is constructed by the product of fundamental gates. We prove this by calculating the indices of qubit and the information of clauses.

The computational complexity of quantum computation depends on the number of unitary operator in the quantum circuit. Let U be the unitary operator, it is written as

$$U = U_n U_{n-1} \cdots U_1$$

where $U_n, \cdots, U_1$ are fundamental gates. The computational complexity $T(U)$ is considered as n.

We need to combine some fundamental gates such as U_{NOT}, U_{C-N} and U_{C-C-N} to construct the quantum circuit in fact. U_{AND} and U_{OR} can be written as a combination of fundamental gates. Here we obtain the computational complexity of SAT algorithm by the number of U_{NOT}, U_{AND} and U_{OR}.

Theorem 3 *For a set of clauses $C = \{C_1, \ldots, C_m\}$ and literal $X = \{x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n\}$, $T(U_C)$ is*

$$\begin{aligned} T(U_C) &= m - 1 + \sum_{k-1}^{m} \left(|C_k| + 2i'_k - 1\right) \\ &\leq 4mn - 1 \end{aligned}$$

4.1 Chaos Amplifier

Here we briefly review how chaos can play a constructive role in computation (see [1, 2] for the details).

Consider the so called logistic map which is given by the equation

$$x_{n+1} = ax_n(1 - x_n) \equiv g_a(x), \quad x_n \in [0, 1].$$

The properties of the map depend on the parameter a. If we take, for example, $a = 3.71$, then the Lyapunov exponent is positive, the trajectory is very sensitive to the initial value and one has the chaotic behavior [2]. It is important to notice that if the initial value $x_0 = 0$, then $x_n = 0$ for all n.

The state $|\psi\rangle$ of the previous subsection is transformed into the density matrix of the form

$$\overline{\rho} = q^2 P_1 + \left(1 - q^2\right) P_0$$

where P_1 and P_0 are projectors to the state vectors $|1\rangle$ and $|0\rangle$. One has to notice that P_1 and P_0 generate an Abelian algebra which can be considered as a classical system. The following theorems is proven in [1, 2, 3].

Theorem 4 *For the logistic map $x_{n+1} = ax_n(1 - x_n)$ with $a \in [0, 4]$ and $x_0 \in [0, 1]$, let x_0 be $\frac{1}{2^n}$ and a set J be $\{0, 1, 2, \ldots, n, \ldots, 2n\}$. If a is 3.71, then there exists an integer k in J satisfying $x_k > \frac{1}{2}$.*

Theorem 5 *Let a and n be the same in above theorem. If there exists k in J such that $x_k > \frac{1}{2}$, then $k > \frac{n-1}{\log_2 3.71 - 1}$.*

Corollary 6 *Let $|t(C)|$ be the cardinality of these assignments, if $x_0 \equiv \frac{r}{2^n}$ with $r = |t(C)|$ and there exists k in J such that $x_k > \frac{1}{2}$, then there exists k satisfying the following inequality if C is SAT.*

$$\left[\frac{n-1-\log_2 r}{\log_2 3.71 - 1}\right] \leq k \leq \left[\frac{5}{4}(n-1)\right].$$

From these theorems, for all k, it holds

$$M_k \begin{cases} = 0 & \text{iff } C \text{ is not SAT} \\ > 0 & \text{iff } C \text{ is SAT} \end{cases}$$

where $M_k = g_a^k(q^2)$.

OMV SAT algorithm is represented in the form of generalized quantum Turing machine(GQTM) and the computational complexity is discussed in [2, 4, 8, 9].

5 Computational complexity of OMV SAT algorithm

Let $T(\Lambda)$ be a computational complexity of Λ. The following theorems are proven in [7].

Theorem 7 *Let $X' = X \cup \bar{X} = \{x_1, \cdots x_n, \bar{x}_1, \cdots, \bar{x}_n\}$ be literals and $C = \{C_1, \cdots, C_m\}$ a set of literals, the computational complexity $T(\Lambda_C)$ is*

$$\begin{aligned} T(\Lambda_C) &= 3\sum_{k=1}^{m}(card(C_k) - 1) + \sum_{k-1}^{m} 2card(C_k \cap \{\bar{x}_1, \ldots, \bar{x}_n\}) + m - 1 + n \\ &\leq (8mn - 2m - 1) \end{aligned}$$

Theorem 8 *For a set of clauses C and n Boolean variables, the computational complexity of the OMV SAT algorithm including the chaos amplifier, $T(\Lambda_{SAT})$ is obtained as follows*

$$T(\Lambda_{SAT}) = T(\Lambda_C) + T(\Lambda_T) + T(\Lambda_{CA}^k) = O(poly(n)),$$

where $poly(n)$ denotes a polynomial of n.

6 Entanglement degree

It is important to measure the degree of entanglement of entangled states. There exist several such measures, for most of such measures it is difficult to compute. Thus Belavkin and Ohya introduced another measure called the degree of entanglement D_{EN}[5].

Let θ be the entangled state with the marginal states ρ and σ.

Definition 9 *The degree of entanglement is defined by*

$$D_{EN}(\theta;\rho,\sigma) \equiv \frac{1}{2}\left(S(\rho)+S(\sigma)-I_\theta(\rho,\sigma)\right)$$

where $S(\rho) = -\mathrm{tr}\rho\log\rho$ *and*

$$I_\theta(\rho,\sigma) \equiv \mathrm{tr}\theta\left(\log\theta - \log\rho\otimes\sigma\right)$$

Definition 10 *A state* θ_1 *has stronger entanglement than* θ_2 *iff*

$$D_{EN}(\theta_1;\rho,\sigma) < D_{EN}(\theta_2;\rho,\sigma)$$

Theorem 11 *For a pure state* θ *with the marginal state* ρ *and* σ,

1. θ *is separable iff* $D_{EN}(\theta;\rho,\sigma)=0$
2. θ *is entangled iff* $D_{EN}(\theta;\rho,\sigma)<0$

7 Quantum algorithm with entangled degree solving SAT problem

Let n be a number of literals in SAT problem, μ a number of qubits to compute $f(C)$, and $\mathcal{H} = (\mathbb{C})^{\otimes n+\mu+1}$ a Hilbert space to solve SAT problem.

We prepare an initial $|\psi_0\rangle \in \mathcal{H}$ state as

$$|\psi_0\rangle = |0^n, 0^\mu, 0\rangle$$

Applying the unitary operator U_C which computes $f(C)$ for all truth assignments, we have the final state as

$$\begin{aligned} U_C|\psi_0\rangle &= \frac{1}{\sqrt{2^n}}\sum_{t=0}^{2^n-1}|e_i, x_i, f_{e_i}(C)\rangle \\ &= |\psi_f\rangle \end{aligned}$$

where $f_{e_i}(C)$ is a value of $f(C)$ for a truth assignment e_i, and x_{e_i} is the qubits that is used for the computation of it.

If C is SAT, there exists the truth assignment e such that

$$f_e(C) = 1$$

Let r be a number of truth assignments making $f_e(C) = 1$. Here, we consider the following density operator

$$\sigma_f = |\psi_f\rangle\langle\psi_f|$$

This is a pure state on the Hilbert space

$$\begin{aligned} \mathcal{H} &= (\mathbb{C})^{\otimes n+\mu+1} \\ &= \mathcal{K}_1 \otimes \mathcal{K}_2 \end{aligned}$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ are two Hilbert spaces whose dimensions are

$$\dim \mathcal{K}_1 = 2^{\mu+\mu}$$

$$\dim \mathcal{K}_2 = 2$$

We define two marginal states of σ_f as

$$\rho_1 = \mathrm{tr}_{\mathcal{K}_2} \sigma_f$$

$$\rho_2 = \mathrm{tr}_{\mathcal{K}_1} \sigma_f$$

Then we compute Entanglement Degree $D_{EN}(\rho; \rho_1, \rho_2)$ with these marginal states. Since the final state σ_f is a pure state, it becomes

$$\begin{aligned} D_{EN}(\rho; \rho_1, \rho_2) &= -\frac{1}{2}(S(\rho_1) + S(\rho_2)) \\ &= S(\rho_1) = S(\rho_2) \end{aligned}$$

Here, we exclude the dust qubits and we obtain

$$\rho_1 = \begin{pmatrix} A_r & 0 \\ 0 & A_{2^n - r} \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} \frac{2^n - r}{2^n} & 0 \\ 0 & \frac{r}{2^n} \end{pmatrix}$$

where

$$A_x \in M(x) = (a_{i,j})$$

$$a_{i,j} = \frac{1}{2^n}$$

ρ_1 is decomposed as

$$\begin{aligned} \rho_1 &= \frac{1}{2^n} \sum_{0 \le i,j < r} |e_i\rangle \langle e_j| + \frac{1}{2^n} \sum_{r \le k,l < 2^n} |e_k\rangle \langle e_l| \\ &= \frac{r}{2^n} |\varphi_r\rangle \langle \varphi_r| + \frac{2^n - r}{2^n} |\varphi_{2^n - r}\rangle \langle \varphi_{2^n - r}| \end{aligned}$$

where

$$|\varphi_r\rangle = \frac{1}{\sqrt{r}} \sum_{0 \le i < r} |e_i\rangle$$

$$|\varphi_{2^n - r}\rangle = \frac{1}{\sqrt{2^n - r}} \sum_{r \le k < 2^n} |e_k\rangle$$

Then we calculate $S(\rho_1)$ and $S(\rho_2)$ as

$$S(\rho_1) = S(\rho_2) = -\frac{2^n - r}{2^n} \log \frac{2^n - r}{2^n} - \frac{r}{2^n} \log \frac{r}{2^n}$$

We obtain

$$D_{EN}(\rho;\rho_1,\rho_2)\begin{cases} =0 & r=0 \\ <0 & 0<r<2^n \\ =0 & r=2^n \end{cases}$$

In the cases of $r=0$ and $r=2^n$, $D_{EN}(\rho;\rho_1,\rho_2)$ is equal to 0, this means that the final state is a separable state. In fact, one can see

$$\sigma_f=\begin{cases} |\eta\rangle\langle\eta|\otimes|0\rangle\langle 0| & r=0 \\ |\eta\rangle\langle\eta|\otimes|1\rangle\langle 1| & r=2^n \end{cases}$$

where

$$|\eta\rangle=\frac{1}{\sqrt{2^n}}\sum_{i=0}^{2^n-1}|e_i\rangle$$

If $D_{EN}(\rho;\rho_1,\rho_2)<0$, we can say that is SAT immediately.

Moreover, even if $D_{EN}(\rho;\rho_1,\rho_2)=0$, we can check satisfiability easily by the following algorithm:

1. Compute for one truth assignment (e.g. $x_1=0$, $x_2=0$, $\cdots$, $x_n=0$)
2. If $f(\mathcal{C})=0$ then $\mathcal{C}$ is not SAT
3. If $f(\mathcal{C})=1$ then $\mathcal{C}$ is SAT

In the case of $D_{EN}(\rho;\rho_1,\rho_2)=0$, there are only two possibilities that are $r=0$ or $r=2^n$. Hence, we can check satisfiability by computing $f(\mathcal{C})$ for one assignment. Therefore, we can represent the flow of quantum algorithm as the following:

1. Define the Hilbert space for computation.
2. Construct the initial state and **define its marginal states.**
3. Construct the unitary operators to solve the problem.
4. Apply it for the initial density operator and obtain the result.
5. **Calculate an entanglement degree with the marginal states of the result state.**

7.1 Example(D-J problem)

In this section we show the quantum algorithm with entangled degree for Deutch-Jozsa problem.

Let N be a positive integer, $h:\{0,\cdots,2N-1\}\to\{0,1\}$ a function holding one of the following conditions:

A $card\{x|h(x)=0\}=0$ or $card\{x|h(x)=1\}=0$ ($h(x)$ is a constant)

B $card\{x|h(x)=0\}=card\{x|h(x)=1\}$ (h is a balanced function) [problem] Decide which condition h holds.

h is also given by a unitary operator U_h explained below. A quantum algorithm of this problem is described the following unitary operators $U_h W U_h$ on the Hilbert space $\left(\mathbb{C}^2\right)^{\otimes 1+[\log 2N]}$

$$U_h \left|x, y\right\rangle = \left|x, y + h\left(x\right) \bmod 2\right\rangle$$

$$W \left|x, y\right\rangle = (-1)^y \left|x, y\right\rangle$$

For an initial state vector

$$\left|\psi_{in}\right\rangle = \frac{1}{\sqrt{2N}} \sum_{x=0}^{2N-1} \left|x, 0\right\rangle$$

apply $U_h W U_h$. Then we have

$$\begin{aligned} U_h W U_h \left|\psi_{in}\right\rangle &= U_h W \frac{1}{\sqrt{2N}} \sum_{x=0}^{2N-1} \left|x, h\left(x\right)\right\rangle \\ &= U_h \frac{1}{\sqrt{2N}} \sum_{x=0}^{2N-1} (-1)^{h(x)} \left|x, h\left(x\right)\right\rangle \\ &= \frac{1}{\sqrt{2N}} \sum_{x=0}^{2N-1} (-1)^{h(x)} \left|x, 0\right\rangle \\ &\equiv \left|\psi_{out}\right\rangle \end{aligned}$$

Therefore, we measure the observable M with $\left|\psi_{out}\right\rangle$:

$$\begin{aligned} M &= \left\langle\psi_{out}\right| P \left|\psi_{out}\right\rangle \\ &= \frac{1}{2N} \left| \sum_{x=0}^{2N-1} (-1)^{h(x)} \right|^2 \end{aligned}$$

where $P = \left|\psi_{in}\right\rangle \left\langle\psi_{in}\right|$. According the result of this measurement, we can judge :

$$\begin{aligned} M = 1 &\Leftrightarrow h \text{ holds A} \\ M = 0 &\Leftrightarrow h \text{ holds B} \end{aligned}$$

Here, we rewrite this algorithm by our new quantum algorithm as

1. Apply U_h to the initial state.
2. Compute D_{EN} of the result of (1) with certain marginal states.
3. Judge the character of h by the information of (2)

Applying U_h to the initial state, we have

$$U_h |\psi_{in}\rangle = \frac{1}{\sqrt{2N}} \sum_{x=0}^{2N-1} |x, h(x)\rangle$$
$$= |\psi'_{out}\rangle \quad (\neq |\psi_{out}\rangle)$$

Here we define two Hilbert spaces

$$\mathcal{K}_1 = \left(\mathbb{C}^2\right)^{\otimes[\log 2N]}$$
$$\mathcal{K}_2 = \mathbb{C}^2$$

and two marginal states

$$\rho_1 = \mathrm{tr}_{\mathcal{K}_2} |\psi'_{out}\rangle \langle\psi'_{out}|$$

$$\rho_2 = \mathrm{tr}_{\mathcal{K}_1} |\psi'_{out}\rangle \langle\psi'_{out}|$$

After the calculation of $D_{EN}\left(|\psi'_{out}\rangle \langle\psi'_{out}| ; \rho_1, \rho_2\right)$, we obtain

$$D_{EN}\left(|\psi'_{out}\rangle \langle\psi'_{out}| ; \rho_1, \rho_2\right) \begin{cases} = 0 & h \text{ holds A} \\ < 0 & h \text{ holds B} \end{cases}$$

If h holds A, then $|\psi'_{out}\rangle \langle\psi'_{out}|$ is separable state on $\mathcal{K}_1 \otimes \mathcal{K}_2$. In fact,

$$|\psi'_{out}\rangle = \frac{1}{\sqrt{2N}} \sum_{x=0}^{2N-1} |x, 0 \text{ or } 1 \text{ for all } x\rangle .$$

Therefore, the computational complexities of above algorithms become as the following table:

	computational complexity
D-J	$T(U_h) + T(W) + T(U_h) + T(M)$
ours	$T(U_h) + T(D_{EN})$

where $T(U)$ is a computational complexity of unitary operator U, $T(M)$ is of measurement M, and $T(D_{EN})$ is of calculation of D_{EN}.

8 Conclusion

We found the third quantum algorithm to solve SAT problem :

1. OMV SAT algorithm : Unitary operations and amplification processes (e.g. chaos amplifier).

2. GQTM : Transition function and transition channels. $\rightarrow$ computational complexity of quantum algorithm.

3. **Quantum circuit and calculation of entanglement degree.**

The computational complexity T_{SAT} of OMV SAT algorithm is

$$T_{SAT} = \mathcal{O}\,(poly\,(n))$$

with halting probability

$$p = \frac{1}{2}$$

However, the computational complexity of the new quantum algorithm T is

$$T = T\,(U_C) + T\,(D_{EN})$$

where $T\,(U_C)$ is the computational complexity of unitary operator U_C calculating f_C and $T\,(D_{EN})$ is that of calculation of entanglement degree.

The entanglement degree is obtained with probability 1, then the halting probability of this algorithm is 1 (c.f. OMV SAT algorithm).

Now we have to construct a physical model which achieves this algorithm. And we apply this to the other problems.

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