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SOME EXAMPLES OF CONDITIONALLY FREE PRODUCT

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INTRODUCTION

Free convolution is a binary operation $\boxplus$ on the class of probability measures on $\mathbb{R}$, which corresponds to the notion of free independence. More precisely, if X_1, X_2 are free random variables in a noncommutative probability space $(\mathcal{A}, \psi)$ (i.e. $\mathcal{A}$ is a unital complex $*$-algebra, ϕ is a state on $\mathcal{A}$), with distributions ν_1, ν_2 respectively, then $\nu_1 \boxplus \nu_2$ is the distribution of $X_1 + X_2$ (for the background on the free probability theory we refer to the books [10, 12]). The free convolution of two measures can only be described indirectly, either analytically, using the Voiculescu R-transform [12, 2, 4] or combinatorially, by free cumulants [10, 9].

Bożejko, Leinert and Speicher [3] introduced notion of conditionally freeness on a noncommutative probability space $\mathcal{A}$, equipped with two states. This leads to *conditionally free convolution* $\boxplus_c$, a binary operation on *pairs* of compactly supported probability measures on $\mathbb{R}$, see [3, 8, 9]. The aim of this paper is to show that in some important cases the conditionally free convolution can be reduced to the free convolution.

1. FREE AND CONDITIONALLY FREE PRODUCT

Let $\mathcal{M}$ (resp. $\mathcal{M}^c$) denote the class of (compactly supported) probability measures on $\mathbb{R}$. Then for $\mu \in \mathcal{M}$ we define the *Cauchy transform*:

$$G_\mu(z) := \int_{\mathbb{R}} \frac{d\mu(x)}{z - x},$$

which is an analytic map from the upper half-plane $\mathbb{C}^+ := \{z \in \mathbb{C} : \Im z > 0\}$ into the lower half-plane $\mathbb{C}^- := \{z \in \mathbb{C} : \Im z < 0\}$, satisfying

$$\lim_{y \to +\infty} iy G_\mu(iy) = 1. \tag{1}$$

Moreover, every analytic function $G : \mathbb{C}^+ \to \mathbb{C}^-$ satisfying (1) is Cauchy transform of a unique probability measure on $\mathbb{R}$, see [1, 5].

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If $\nu \in \mathcal{M}^c$ then $G_\nu(z)$ can be represented as a continued fraction

$$G_\nu(z) = \cfrac{1}{z - u_0 - \cfrac{\alpha_0}{z - u_1 - \cfrac{\alpha_1}{z - u_2 - \cfrac{\alpha_2}{z - u_3 - \cfrac{\alpha_3}{\ddots}}}}}, \tag{2}$$

where the *Jacobi parameters* satisfy: $\alpha_k \geq 0$, $u_k \in \mathbb{R}$ and if $\alpha_m = 0$ for some $m \geq 0$ then $\alpha_n = u_n = 0$ for all $n > m$.

For a pair $\mu, \nu \in \mathcal{M}^c$ we define the free and the conditionally free transform, R_ν and $R_{\mu,\nu}$, as complex functions which satisfy

$$\frac{1}{G_\nu(z)} = z - R_\nu(G_\nu(z)), \tag{3}$$

$$\frac{1}{G_\mu(z)} = z - R_{\mu,\nu}(G_\nu(z)). \tag{4}$$

Then, for $\mu_1, \nu_1, \mu_2, \nu_2 \in \mathcal{M}^c$, the conditionally free convolution

$$(\mu, \nu) = (\mu_1, \nu_1) \boxplus_c (\mu_2, \nu_2) \tag{5}$$

is defined by the equalities

$$R_\nu(z) = R_{\nu_1}(z) + R_{\nu_2}(z), \tag{6}$$

$$R_{\mu,\nu}(z) = R_{\mu_1,\nu_1}(z) + R_{\mu_2,\nu_2}(z). \tag{7}$$

In particular, ν is the free product $\nu_1 \boxplus \nu_2$.

2. A family of transforms

For $a \geq 0$, $u, v \in \mathbb{R}$ we define a transform $T(a, u, v) : \mathcal{M} \to \mathcal{M}$ defining $\mu := T(a, u, v)(\nu)$ by

$$\frac{1}{G_\mu(z)} := z - u - \frac{a}{\frac{1}{G_\nu(z)} - v} = z - u - \frac{aG_\nu(z)}{1 - vG_{\nu(z)}}. \tag{8}$$

Note that the measure μ is well defined, as the reciprocal of the right hand side is a function $\mathbb{C}^+ \to \mathbb{C}^-$ satisfying (1). Moreover, if G_ν admits the expansion (2) as continued fraction then

$$G_\mu(z) = \cfrac{1}{z - u - \cfrac{a}{z - u_0 - v - \cfrac{\alpha_0}{z - u_1 - \cfrac{\alpha_1}{z - u_2 - \cfrac{\alpha_2}{z - u_3 - \cfrac{\alpha_3}{\ddots}}}}}}. \tag{9}$$

Combining (4) with (8) we observe that

$$R_{\mu,\nu}(w) = u + \frac{aw}{1 - vw}. \tag{10}$$

Proposition 2.1. *Assume that* $a_1, a_2 \geq 0$, $u_1, u_2, v \in \mathbb{R}$, $\nu_1, \nu_2 \in \mathcal{M}^c$ *and that* $\mu_1 := T(a_1, u_1, v)(\nu_1)$, $\mu_2 := T(a_2, u_2, v)(\nu_2)$. *Then*

$$(\mu_1, \nu_1) \boxplus_c (\mu_2, \nu_2) = (\mu, \nu_1 \boxplus \nu_2),$$

where

$$\mu = T(a_1 + a_2, u_1 + u_2, v)(\nu_1 \boxplus \nu_2).$$

In particular, if ν *is infinitely divisible with respect to free convolution,* $a \geq 0$, $u, v \in \mathbb{R}$, *then the pair* $(T(a, u, v)(\nu), \nu)$ *is infinitely divisible with respect to the conditionally free convolution.*

Proof. The first statement is a consequence of (6), (7) and (10). Consequently, if $\nu \in \mathcal{M}^c$ is $\boxplus$-infinitely divisible then the family

$$\left(T(ta, tu, v)(\nu^{\boxplus t}), \nu^{\boxplus t}\right),$$

$t > 0$, is a $\boxplus_c$-semigroup of pairs of measures. □

Example. For $a, b > 0$, $u, v \in \mathbb{R}$ denote by $\mu(a, b, u, v)$ the unique measure satisfying

$$G_{\mu(a,b,u,v)}(z) = \cfrac{1}{z - u - \cfrac{a}{z - v - \cfrac{b}{z - v - \cfrac{b}{z - v - \cfrac{b}{\ddots}}}}}$$

(this family of measures was studied in [11]). Then, in view of the results from [6], for $a, b > 0$, $u, v, \alpha, \beta \in \mathbb{R}$, with $a + \alpha, b + \alpha > 0$, we have

$$\mu(a, a + \alpha, u, u + \beta) \boxplus \mu(b, b + \alpha, v, v + \beta) = \mu(a + b, a + b + \alpha, u + v, u + v + \beta).$$

With this notation the limit pairs of measures in the central and Poisson theorems for the conditionally free convolution can be represented as

$$(11) \qquad \left(\mu(a, b, 0, 0), \mu(b, b, 0, 0)\right) = \left(T(a, 0, 0)(\mu(b, b, 0, 0)), \mu(b, b, 0, 0)\right),$$

$$(12) \quad \left(\mu(a, b, a, b + 1), \mu(b, b, b, b + 1)\right) = \left(T(a, a, 1)(\mu(b, b, b, b + 1)), \mu(b, b, b, b + 1)\right),$$

respectively, where $a, b > 0$ are parameters (see [3, 7]). Denoting the former pair (11) by $\overrightarrow{\nu}(a, b)$ and the latter (12) by $\overrightarrow{\pi}(a, b)$, we note that the families $\{\overrightarrow{\nu}(a, b)\}_{a,b>0}$ and $\{\overrightarrow{\pi}(a, b)\}_{a,b>0}$ are both two-parameter semigroups with respect to the conditionally free convolution, i.e. for $a_1, b_1, a_2, b_2 >$ we have:

$$\overrightarrow{\nu}(a_1, b_1) \boxplus_c \overrightarrow{\nu}(a_2, b_2) = \overrightarrow{\nu}(a_1 + a_2, b_1 + b_2),$$

$$\overrightarrow{\pi}(a_1, b_1) \boxplus_c \overrightarrow{\pi}(a_2, b_2) = \overrightarrow{\pi}(a_1 + a_2, b_1 + b_2).$$

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