

Dual pairs の系から見たホワイトノイズ超汎関数空間の特徴.

A characteristic properties of the space of generalized white noise functionals viewed through a system of dual pairs.

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概要

ホワイトノイズ超汎関数空間 $(L^2)^-$ について、各種の dual pair により その特徴をみる。扱う対象は、無限次元空間上の超関数の空間であるため、きわめて複雑な構造をもつが種々の部分空間の間の duality を見ることによりその構造の一側面を伺うことができる。超汎関数空間の定義は $(S)^*$ を採用するのがスマートのように見えるが本報告では Sobolev 空間の性質を使いたいので、定義法の一つである $(L^2)^-$ を利用する。ここでは Fock 空間を基礎にする具体的表現が役立つ。このため、超汎関数の概念や S -変換、 T -変換の意義の再認識が必要となる。特に、真に無限次元の特性がよく現れる 2 次超汎関数の果す重要な役割、また $(L^2)^-$ が種々の双対性を内蔵することに注目したい。

1 Introduction

First, we have a quick review of the Fock space of (ordinary) white noise functionals in classical stochastic analysis:

$$(L^2) = \bigoplus_0^\infty H_n,$$

where (L^2) is the complex Hilbert space involving square integrable functionals of white noise, i.e. $L^2(E^*, \mu)$,

where the measure μ is the probability distribution of white noise $\dot{B}(t), t \in R$, that is the *white noise measure* defined on a space E^* of generalized functions on R^1 , E^* being the dual space of some nuclear space E .

The subspace H_n is the collection of homogeneous chaos in the sense of N. Wiener or that of multiple Wiener integrals in the sense of K. Itô, which is of degree n .

It is well-known that the space H_n is isomorphic to $\widehat{L}^2(R^n)$, the subspace of $L^2(R^n)$ involving symmetric functions, up to the constant $\sqrt{n!}$:

$$H_n \cong \widehat{L}^2(R^n). \quad (1.1)$$

Such an isomorphism can be realized by the so-called S -transform defined by, for $\varphi(x) \in (L^2)$, and for $\xi \in E$,

$$(S\varphi)(\xi) = C(\xi) \int \exp[\langle x, \xi \rangle] \varphi(x) d\mu(x), \quad (1.2)$$

where $C(\xi)$ is the characteristic functional of the white noise measure,

$$C(\xi) = \exp\left[-\frac{1}{2}\|\xi\|^2\right].$$

We now pause to give some interpretation to the S -transform. The expression of the transform looks like an infinite dimensional analogue of the Laplace transform, however it is quite different.

Originally the so-called T -transform was introduced in order to construct a *reproducing kernel Hilbert space* (RKHS) determined by characteristic functional $C(\xi)$. It is of the form, for $\varphi(x) \in (L^2)$

$$(T\varphi)(\xi) = \int \exp[i\langle x, \xi \rangle] \varphi(x) d\mu(x). \quad (1.3)$$

The idea is similar to the case where the author tried to have the factorization of the covariance function of

a Gaussian process (see [2]) to establish the canonical representation theory for Gaussian processes. As a generalization of this method to use a RKHS, and with other reasons, this transform was used in the paper (Hida-Ikeda, the 5th Berkeley Symp. Proc. 1966), where nonlinear functions of white noise are discussed. Then, with some additional ideas, RKHS method appeared again to introduce generalized white noise functionals in 1975 (see [3]).

We see that

$$C(\xi - \eta) = \int_{E^*} e^{i\langle x, \xi - \eta \rangle} d\mu(x), \quad (1.4)$$

The right hand side is written as

$$\int_{E^*} e^{i\langle x, \xi \rangle} \cdot e^{-i\langle x, \eta \rangle} d\mu(x)$$

in a factorization formula.

Based on this formula, we consider functions of the form $\sum a_j e^{-i\langle x, \eta_j \rangle}$ which will span the entire space (L^2). While, the left hand side $\sum_j a_j C(\xi - \eta_j)$ forms a dense subspace of the RKHS. Through this transform, called T -transform, gives a representation of white noise functionals. In addition, the T -transform plays a role of *factorization*, see [15].

Shortly after (around 1980) this T -transform, the S -transform was introduced and developed by Kubo-Takenaka, and further Potthoff-Streit continued development extensively.

Now, S - and T -transform play basic role in white noise analysis in many places and in various manner, e.g. to get RKHS, to have factorization and others.

Remark We do not confuse our transforms with the Bargmann-Segal type transforms or with the Gauss transform. Indeed, essentially different.

We want to take this opportunity to insist strongly that generalized white noise functionals can "not" be reduced to the classical functionals of Brownian motion introduced before.

Generalized white noise functionals [3], [6].

There can be a restriction of this isomorphism by introducing a stronger topology in such a way that

$$\widehat{K}^{(n+1)/2}(R^n) \cong H_n^{(n)}, \quad (1.5)$$

where we use the notation $\widehat{K}^m(R^n)$ to denote the symmetric Sobolev space over R^n of degree m .

Here again and after, the constant $\sqrt{n!}$ is omitted.

Then, we take the dual space of both side of this isomorphism based on symmetric $\widehat{L}^2(R^n)$ and H_n , respectively. We can define $H_n^{(-n)}$ the space of generalized white noise functionals of degree n by the following isomorphism:

$$\widehat{K}^{-(n+1)/2}(R^n) \cong H_n^{(-n)}. \quad (1.6)$$

Finally, with a suitable choice of a positive increasing sequence c_n , we have the *test functional space*

$$(L^2)^+ = \oplus c_n H_n^{(n)} \quad (1.7)$$

and its dual space

$$(L^2)^- = \oplus c_n^{-1} H_n^{(-n)}, \quad (1.8)$$

which is called the space of *generalized white noise functionals*.

In this note we shall discuss various kind of dualities that exist among subspaces of $(L^2)^-$.

We have established in [6], Chpt. 2, the structure of $H_1^{(-1)}$, where we have given the *identity* to the white

noise $\dot{B}(t)$ (or its sample function $x(t)$ with $x \in E^*$). The space $H_1^{(-1)}$ is spanned by the $\dot{B}(t)$'s and each $\dot{B}(t)$ is taken to be the variables of generalized white noise functionals. This fact provides a basic method in what we are going to discuss.

2 Duality in the space $H_1^{(-1)}$

Significant duality can be seen between two Gaussian processes which are in pair living in $H_1^{(-1)}$. There is an interesting pair of multiple Markov Gaussian processes. To fix the idea we shall consider an N -ple Markov Gaussian process $X(t), t \geq 0$, in the restricted sense, which can be dealt with rigorously in the space $H_1^{(-1)}$. It is determined by a differential equation given by

$$L_t X(t) = \dot{B}(t), \quad (2.1)$$

with initial data

$$X(0) = 0, \quad (2.2)$$

where L_t is an N -th ($N \geq 1$) order ordinary differential operator expressed in the form

$$L_t = \sum_{k=0}^N a_k(t) D^{N-k}, \quad D = \frac{d}{dt}. \quad (2.3)$$

We may assume $a_k(t)$'s are sufficiently smooth.

Such a process discussed by J.L. Doob (1944) and in the paper [2] within the framework of general multiple Markov Gaussian process. As for the duality, the paper [17] by Si Si et al has recently discussed in the framework of white noise analysis.

It is known (see [2] Part II) that $X(t)$ has the canonical representation expressed in the form

$$X(t) = \int_0^t R(t, u) \dot{B}(u) du, \quad (2.4)$$

where the kernel $R(t, u)$ is the Riemann function associated with L_t .

It is noted that the expression $\dot{B}(t)$ is no more formal, but it has correct meaning in the space $H_1^{(-1)}$ and analysis concerning the equation (2.1) can be carried on within that space.

We claim that L_t is expressed in the Frobenius formula in such a way that

$$L_t = \frac{1}{v_0(t)} D \frac{1}{v_1(t)} D \cdots D \frac{1}{v_N(t)}. \quad (2.5)$$

Set

$$f_i(t) = v_N(t) \int_0^t v_{N-1}(t_1) dt_1 \int_0^{t_1} v_{N-2}(t_2) dt_2 \cdots \int_0^{t_{i-2}} v_{N-i+1}(t_{i-1}) dt_{i-1}, \quad 1 \leq i \leq N. \quad (2.6)$$

Now define the formal adjoint operator L_u^* :

$$L_u^* = \frac{1}{v_N(u)} D \frac{1}{v_{N-1}(u)} D \cdots D \frac{1}{v_0(u)} \quad (2.7)$$

and set

$$g_i(u) = (-1)^{N-i} v_0(u) \int_0^u v_1(u_1) du_1 \int_0^{u_1} v_2(u_2) du_2 \cdots \int_0^{u_{N-i-1}} v_{N-i}(u_{N-i}) du_{N-i}, \quad 1 \leq i \leq N. \quad (2.8)$$

Obviously we have

$$L_u^* g_i(u) = 0, \quad 1 \leq i \leq N.$$

It can be proved that the Riemann function $R(t, u)$ is expressed in the form of Goursat kernel of order N :

$$R(t, u) = \sum_1^N f_i(t) g_i(u).$$

We are now ready to state the duality of Gaussian Markov processes in the restricted sense. Set

$$R^*(t, u) = R(u, t).$$

Note that a kernel function of canonical representation of a Gaussian process is of Volterra type. However, in the present case, $R(t, u)$ can be defined on the entire space $[0, \infty) \times [0, \infty)$. The same for $R^*(t, u)$.

We restrict the time parameter to the unit interval $[0, 1]$. Define

$$X^*(t) = \int_t^1 R^*(t, u) \dot{B}(u) du. \quad (2.9)$$

The following theorem comes from Si Si, Win Win Htay and Accardi [17].

Theorem 1 The $X^*(t)$ is a backward N -ple Markov Gaussian process in the restricted sense satisfying

$$L_t^* X^*(t) = \dot{B}(t),$$

with the initial data

$$X^*(1) = 0.$$

By this result we may say that $X(t), 0 \leq t \leq 1$, and $X^*(t), 1 \geq t \geq 0$, form a dual pair.

Remark Given an N -ple Markov Gaussian process $X(t)$ in the restricted sense determined by (2.1) and (2.2). Then, the exact expressions of v_i 's, f_i 's and g_i 's are not unique, but N , the degree of Goursat kernel, is uniquely determined.

We shall show that a dual pair can be formed under somewhat weaker assumption than multiple Markov property in the restricted sense. Our forthcoming paper will report the result on this fact.

3 Passage from finite dimensional analysis to infinite dimensional calculus

We shall be concerned with spaces of functionals of white noise $\dot{B}(t), t \in R^1$.

[I] Finite dimensional approximations.

We now come to discuss duality that holds among the spaces of nonlinear functionals of white noise. In fact, we shall consider the space of generalized functionals of the $\dot{B}(t), t \in R^1$. To this end, we take the finite dimensional approximation to Brownian motion $B(t)$ (or approximation to white noise $\dot{B}(t)$) due to P. Lévy. Although there are many methods of approximations to Brownian motion, we claim that Lévy's method is most essential and quite fitting for our purpose to carry on, so to speak, essentially infinite dimensional stochastic calculus.

The relevance of this method is that i) it uses successive approximation method in such a way that the approximation is getting finer and finer as the step proceeds, ii) each step the approximation is uniform in t in a visualized manner, iii) it is easily applied to have white noise functionals approximated (le passage du fini à l'infini), and iv) an approximation of white noise is obtained simply by taking the time-derivative.

Actual method, we have demonstrated in many places, e.g. in [6] Chapt. 2 with fig 2.1. We shall, therefore, explain only the idea quickly.

Construction of a Brownian motion (white noise).

We now show how to construct a Brownian motion $B(t), t \in [0, 1]$. First, let a sequence $\{Y_k, k \geq 1, \}$ of independent identically distributed standard Gaussian

random variables be provided.

Define a sequence of stochastic processes $X_n(t), t \in [0, 1], n = 1, 2, \dots$, successively.

$$X_1(t) = tY_1. \quad (3.1)$$

Let T_n be the set of binary numbers $k/2^{n-1}, k = 0, 1, 2, \dots, 2^{n-1}$, and set $T_0 = \cup_{n \geq 1} T_n$. Assume that $X_j(t) = X_j(t, \omega), j \leq n$, are defined. Then, $X_{n+1}(t)$ is defined in the following manner. At every binary point $t \in T_{n+1} - T_n$ add new random variables Y_k as many as 2^n to $X_n(t)$. On the t -set T_{n+1}^c we have linear interpolation to define $X_{n+1}(t)$.

Then, we have

Theorem 2 i) The sequence $X_n(t), n \geq 1$, is consistent in n , and the uniform L^2 -limit of the $X_n(t)$ exists. The limit is a version of a Brownian motion $B(t)$.

ii) The time derivative $X'_n(t)$ converges to a (version of) white noise $\dot{B}(t)$ which is in $H_1^{(-1)}$.

Realizations of white noise functionals and functional derivatives

By using the approximation (construction) of Brownian motion, white noise functionals can be approximated. The S -transform (1.2) is applied to have U -functionals $U(\xi)$,

We remind the *Volterra form* of a variation of the S -transform $U(\xi)$ of white noise functional φ :

$$\delta U(\xi) = \int U'_\xi(\xi, t) \delta \xi(t), \quad (3.2)$$

where $\delta \xi(t)$ is a continuous analogue of the differential du_j of $u(x_1, x_2, \dots, x_n)$. The functional derivative $U'_\xi(\xi, t)$ is called the *Fréchet derivative* and denoted by $\frac{\delta}{\delta \xi(t)} U$.

Define the partial derivative in $\dot{B}(t)$ by

$$\partial_t = S^{-1} \frac{\delta}{\delta \xi(t)}. \quad (3.3)$$

Formally speaking, ∂_t may be considered as $\frac{\partial}{\partial B(T)}$.

It is noted that this definition of the partial derivative is fitting to our white noise calculus. Part of the reason we shall see later. The adjoint is defined and is expressed as ∂_t^* .

[II] Infinite dimensional rotation group.

Take a suitable nuclear space E and let $O(E)$ be the collection of linear isomorphisms of E which are orthogonal in $L^2(R^1)$. It is topologized by the compact-open topology and we call it *rotation group* of E , or if E is not specified, it is called *infinite dimensional rotation group*.

Let g^* be the adjoint of $g \in O(E)$, Each g^* is a μ measure preserving transformation acting on E^* .

Thus, our white noise analysis has an aspect of the *harmonic analysis* arising from the infinite dimensional rotation group. The harmonic analysis can, in some parts, be approximated by finite dimensional analysis. But, to be very important, there are lots of significant results that are essentially infinite dimensional; in fact, those results can not be well approximated by finite dimensional concepts.

We show an example, that is the Laplacian (indeed, the Lévy Laplacian) Δ_L :

$$\Delta_L = \int \partial_t^2 (dt)^2, \quad (3.4)$$

We shall see some interpretation later in this report.

4 Quadratic functionals of white noise

We are now ready to discuss nonlinear functions (actually functionals) of the $\dot{B}(t)$. We claim that among others the subspace $H_2^{(-2)}$ consisting of *quadratic* generalized white noise functionals is particularly important. There is the isomorphism

$$H_2^{(-2)} \cong \widehat{K}^{-3/2}(R^2).$$

As was established by (1.5). More explicitly, for $\varphi \in H_2^{(-2)}$ we find a function $F(u, v)$ in the space $\widehat{K}^{-3/2}(R^2)$ to have the representation

$$\varphi(\dot{B}) = \int F(u, v) : \dot{B}(u)\dot{B}(v) : dudv, \quad (4.1)$$

where the notation $: \cdot :$ means the Wick product, i.e. renormalized product. (See e.g. [6].) We shall classify those quadratic functionals according to the analytic properties of the kernel. The idea is in line with *le passage du fini à l'infini* proposed by P. Lévy.

We shall, therefore, start with a quadratic form in the elementary theory of linear algebra. A quadratic form $Q(x)$, $x \in R^n$, is expressed as

$$Q(x) = \sum_{j,k} a_{j,k} x_j x_k,$$

It is significant to decompose the $Q(x)$ into two subforms $Q_1(x)$ and $Q_2(x)$:

$$Q(x) = Q_1(x) + Q_2(x),$$

where

$$Q_1(x) = \sum_j a_j x_j^2, \text{ and } Q_2(x) = \sum_{j \neq k} a_{j,k} x_j x_k. \quad (4.2)$$

According to the method to have *le passage à l'infini*, we can consider how and why the above two terms

should be discriminated when we take the limits of them as $n \rightarrow \infty$. Note that the x_j 's are equally weighted variables regardless they are coordinates of finite or infinite dimensional vectors. Here, we shall make some quite elementary observations.

i) Suppose x_i 's are mutually independent random variables and are subject to the standard Gaussian distribution $N(0, 1)$. If both are infinite sum, then for $Q_1(x)$ to be convergent the coefficients a_j 's should be of trace class, but for $Q_2(x)$ it is sufficient that the coefficients $a_{j,k}$ are square summable. In short, the way of convergence is strictly different.

ii) As for analytic properties, any partial sum of $Q_2(x)$ is harmonic, while each partial sum of $Q_1(x)$ is not always so.

iii) Start with a Brownian motion $B(t), t \in [0, 1]$. Consider an approximation to white noise $\dot{B}(t), t \in [0, 1]$ by taking $\frac{\Delta_j B(t)}{\Delta_j}$ in place of x_j (see Theorem 2, ii)). Let $|\Delta_j|$ tend to 0. Then, each term of Q_1 needs a trick of renormalization in order to converge to a member of $H_2^{(-2)}$, while the trick is unnecessary for Q_2 .

iv) The renormalized limit of Q_1 satisfies certain invariance. The collection of such limits accepts an irreducible continuous representation of the group G the collection of the 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a \neq 0, b \in R^1$.

We now come to the expression of generalized quadratic functionals of white noise, having our approximation

applied. We have representations of quadratic functionals $\varphi(\dot{B}) \in H_2^{(-2)}$. It is expressed in the form (4.1) with the kernel F in $\widehat{K}^{-3/2}(R^2)$.

Applying the S -transform, we have the U -functional expressed in the form

$$U(\xi) = \int \int F(u, v) \xi(u) \xi(v) du dv,$$

which is a quadratic form of ξ .

We now recall the entire functionals of the second order due to P. Lévy. He focuses his attention on the *normal* form, which is expressible as

$$U(\xi) = \int g(t) \xi(t)^2 dt + \int \int f(u, v) \xi(u) \xi(v) du dv. \quad (4.3)$$

We assume suitable conditions posed on f and g . Indeed, the sub-space of $H_2^{(-2)}$ involving normal functionals has special meaning as is illustrated below.

The generalized function F , which is in the Sobolev space, should now be chosen such that singularity, if exists, is involved only on the diagonal $u = v$. Namely, we may understand that $g(u)$ is considered as $g(\frac{u+v}{2})\delta(u-v)$, so that F has been decomposed into a singular part g and an ordinary function f .

We are now in a position to realize the observations made in i), ii), iii) and iv) just above.

If permitted to say rather formally, the quadratic form $Q(x)$, which is divided into $Q_1(x)$ and $Q_2(x)$ (see (4.2)), goes to the Lévy's formula for *normal* functionals as the dimension of the vector x tends to infinity. It is worth to be mentioned that Q_1 is magnified when n tends to ∞ .

Write the equation (4.3) as a sum $U(\xi) = Q_1(\xi) +$

$Q_2(\xi)$. We understand that $Q_1(\xi) = \int g(t)\xi(t)^2 dt$ is in the domain of the Laplacian Δ_L given by (3.4). The same for $Q_2(\xi)$. A difference is that for ordinary function f , the functional $Q_2(\xi) = \iint f(u, v)\xi(u)\xi(v) dudv$, is harmonic.

A question arises naturally. Why is a $H_2^{(-2)}$ -functional having off-diagonal singularities of the kernel $F(u, v)$ not so important? The answer is just simple; it is not in the domain of the Laplacian.

Remark. It is natural to ask what is the role of quadratic functional that has singularity is off diagonal. For example

$$\varphi(\dot{B}) = \int g(u)\dot{B}(\alpha(u))\dot{B}(\beta(u))du,$$

where $C = (\alpha(u), \beta(u)), u \in R^1$ is a C^∞ curve that defines a bijection between R^1 to the curve C .

It is easy to see that the second order functional derivative does not exist, so that it is not in the domain of the Laplacian.

With the properties of the Sobolev space of order $-3/2$ (this is a crucial choice) we can now prove

Theorem 3. If an $H_2^{(-3/2)}$ -functional is in the domain of the Lévy Laplacian, then it is a normal functional in the sense of P. Lévy.

Proof. Note that off diagonal singularity is not accepted.

Define a subspace L_2^* of $H_2^{(-2)}$ by

$$L_2^* = \left\{ \int h(u) : \dot{B}(u)^2 : du; h \in K^{-1}(R^1) \right\}.$$

Then, we have

Fact 1) Take the group G given in iv) of this section.

An irreducible continuous representation of the group G is given on the space L_2^* in such a way that for $g \in G$:

$$g : u \rightarrow au + b$$

$$U_g \varphi(\dot{B}) = \varphi(a\dot{B} + b) \sqrt{|a|}.$$

Proof. Suppose φ is expressed in the form

$$\varphi(\dot{B}) = \int h(u) : \dot{B}(u)^2 : du, \quad g \in K^{-1}(R^1).$$

Then

$$U_g \varphi(\dot{B}) = \int h\left(\frac{u-b}{a}\right) |a|^{-1/2} : \dot{B}(u)^2 : du.$$

The kernel function is an image of a $K^{-1}(R^1)$ -continuous mapping of h by g . Irreducibility is implied from the unitary representation of the group G on $L^2(R^1)$.

Fact 2) While a representation of G on $S^{-1}Q_2$ under the same idea is not irreducible.

Remark When we apply the trick "the passage from finite to infinite" to the quadratic form $Q_1(x)$, it is necessary to have it magnified, in addition to subtracting constant, while nothing is necessary for $Q_2(x)$.

5 Duality in the space of quadratic generalized functionals

We can establish an identity of the renormalized square : $\dot{B}(t)^2$: of white noise, as we did in the case of $\dot{B}(t)$ in $H_1^{(-1)}$ (see § 2.6 in [6]).

Having done this, we can now use the subspace L_2^* spanned by quadratic normal functionals of the $\dot{B}(t)$'s,

prepared in the last section. That is the collection of $\varphi(\dot{B}) = \int g(u) : \dot{B}(u)^2 : du$.

It should be reminded that the function g above may be regarded as the restriction of a function f in $\widehat{K}^{-3/2}(R^2)$ down to the diagonal line of R^2 , as was mentioned in the last section. There the trace theorem for Sobolev space is applied.

Our aim is to explain the following theorem that comes from the Si Si's papers [11] and others.

Theorem 4 There exists a subspace L_2 of $H_2^{(2)}$ such that L_2^* is the dual space of L_2 , where the topologies of L_2 comes from that of $H_2^{(2)}$.

Proof. Elementary computations can prove the theorem. But, in reality, there can we see some detailed structure of quadratic generalized white noise functionals. Step by step computations are now in order.

The Fourier transform of $g(\frac{u+v}{2})$ is

$$\frac{1}{2\pi} \iint e^{i(\lambda_1 u + \lambda_2 v)} g\left(\frac{u+v}{2}\right) \delta(u-v) du dv = \sqrt{2\pi} \widehat{g}(\lambda_1 + \lambda_2),$$

where \widehat{g} is the Fourier transform of g of one variable. By the definition of the Sobolev space of order $3/2$ over R^2

$$\frac{1}{2\pi} \iint \frac{|\widehat{g}(\lambda_1 + \lambda_2)|^2}{(1 + \lambda_1^2 + \lambda_2^2)^{3/2}} d\lambda_1 d\lambda_2$$

is finite. This fact implies that $2^{-1/2}g(\frac{u}{\sqrt{2}})$ belongs to the Sobolev space $K^1(R^1)$, in addition its norm is equal to the $K^{-3/2}(R^2)$ -norm of $g(\frac{u+v}{2})\delta(u-v)$ up to an universal constant.

Numerical values are as follows. Let $\|\cdot\|_{n,m}$ be the Sobolev norm of order m over R^n . Then, we can actu-

ally show the following equality

$$\|g\|_{2,3/2}^2 = \frac{c}{2\pi} \|g'\|_{1,1}^2,$$

where $c = \int (1+x^2)^{-3/2} dx$ and $g'(u) = 2^{-1/2} g(\frac{u}{\sqrt{2}})$.

Finally, we come to the stage of determinations of the space L_2 and L_2^* . Remind (see e.g. [6]).

$$H_2^{(2)} = \left\{ \varphi(\dot{B}) = \iint f(u, v) : \dot{B}(u) \dot{B}(v) : dudv, f \in \widehat{K}^{3/2}(R^2) \right\},$$

and introduce an equivalence relation \sim in $H_2^{(2)}$ defined by

$$\iint f_1(u, v) : \dot{B}(u) \dot{B}(v) : dudv \sim \iint f_2(u, v) : \dot{B}(u) \dot{B}(v) : dudv$$

if and only if $f_1(u, u) = f_2(u, u)$.

Set

$$H_2^{(2)} / \sim \equiv L_2.$$

Note. Since $f_i, i = 1, 2$ is in $K^{3/2}$, the relation to the diagonal $u = v$ is a continuous function. Hence, the equivalence relation is defined without any ambiguity.

We now see, what we have computed so far can prove that there is the dual pairing between L_2 and L_2^* . This fact proves the theorem.

This is somewhat a rephrasement, in a formal tone, of Theorem 4. Suppose that $f \in \widehat{K}^{3/2}(R^2)$ and that $g((u+v)/2)\delta(u-v) \in \widehat{K}^{-3/2}(R^2)$ or $g \in K^1(R^1)$. Then, formal computation shows

$$\begin{aligned} & \left\langle \int g(u) : \dot{B}(u)^2 : du, \iint f(u, v) : \dot{B}(u) \dot{B}(v) : dudv \right\rangle \\ &= 2 \int g(u) f(u, u) du. \end{aligned}$$

This equality is derived from

$$E[(\dot{B}(t)^2)^2] = 2 \frac{1}{(dt)^2}.$$

Remark The relationship between $\int : \dot{B}(t)^2 : dt$ and the Lévy Laplacian has been discussed in [11].

6 White noise functionals of higher degree

To fix the idea, we shall discuss dualities in $H_3^{(-3)}$. Let φ be homogeneous functional of degree 3. Its kernel function $F(u_1, u_2, u_3)$ is found in the Sobolev space $\widehat{K}^{-2}(R^3)$. The S -transform $U(\xi) = (S\varphi)(\xi)$ can be expressed in the form

$$U(\xi) = \int \int \int F(u_1, u_2, u_3) \xi(u_1) \xi(u_2) \xi(u_3) du^3.$$

Our method with the idea le passage du fini à l'infini leads us to consider the class of *normal functionals*, namely we are interested in the following forms of degree three.

Type [2,1]

$$\int \int g(u, v) \xi(u)^2 \xi(v) du dv.$$

To have a standard expression, we need to make the kernel g symmetric.

Type [3,0]

$$\int h(u) \xi(u)^3 du.$$

We can define subspaces $L_{2,1}^*$ and $L_{3,0}^*$ of $H_3^{(-3)}$ spanned by generalized functionals of the types (2.1) and (3,0), respectively. Then, we have

Theorem 5 There exist factor spaces $L_{2,1}$ and $L_{3,0}$ of subspaces of H_3 such that $(L_{2,1}, L_{2,1}^*)$ and $(L_{3,0}, L_{3,0}^*)$ are dual pairs, respectively.

Proof can be given by slight modifications of that of the last Theorem.

Now it is clear how to form dualities in the class of entire homogeneous functionals of each degree, by using singularities on the diagonals. The system of dual pairs is one of the characteristics of the space $(L^2)^-$ of generalized white noise functionals.

7 Concluding remarks

We have observed significance of quadratic forms of random elements, through the dualities. From some other viewpoints the same subjects are discussed in [15]. As for the quadratic forms of operators, creation and annihilation operators in white noise analysis have been discussed to some extent in our earlier notes [9]. We can now give further interpretation, in particular to the Lévy Laplacian Δ_L , where the meaning of $(dt)^2$ seems quite natural.

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