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$\mathcal{T}$-transform and S-transform on the space of Hida distributions

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Abstract

The $\mathcal{T}$- and S-transforms play an essential role in the theory of Hida distribution. We will explain the idea behind the establishment of $\mathcal{T}$- and S-transforms.

1 From δ-function to $\mathcal{T}$- and S-transforms

The $\mathcal{T}$- and S-transforms play an essential role in the theory of white nise analysis. In this section we will explain how and why $\mathcal{T}$- and S-transforms were introduced. The idea behind is the **factorization**.

One of Hida's original ideas of white noise theory is as follows, although it is quite naive.

We might say that white noise $\dot{B}(t)$ is a random square root of the δ-function:

$$(\text{random})\sqrt{\delta_t} = \dot{B}(t). \tag{1.1}$$

Ther expression is, of course, formal, but reasonable in a sense that

$$E(\dot{B}(t)\dot{B}(s)) = \delta(t-s),$$

and a correct interpretation will be given in Section 6.

It is noted that $\dot{B}(t)$ is *atomic* as an *idealized elemental* random variable. While, Brownian motion $\{B(t), t \in R^1\}$ is atomic as a stochastic process, where the *causality* is always taken into account.

While, we note that a smeared variable like $\dot{B}(f) = \int f(u)\dot{B}(u)du$ is *not* atomic random variable in white noise space.

In the present report, in particular later sections, the $\mathcal{T}$-transform and S-transform will play dominant roles in both explicitly and implicitly. There, we can see factorization problem for positive definite functions.

2 Factorization due to the Karhunen theory

We refer to the literature [8].

For the moving average representation of a weakly stationary stochastic process $X(t)$ of the form:

$$X(t) = \int_{-\infty}^{t} F(t-u)dZ(u), \tag{2.1}$$

where $Z(u)$ is a process with stationary orthogonal increments such that $E(|dZ(u)|^2) = du$.

There the canonical kernel $F(t,u)$ (see [2]) is obtained by the factorization of the covariance function

$$\gamma(h) = E(X(t+h)X(t)) \tag{2.2}$$

in such a way that the following equality holds:

$$\gamma(h) = \int^{t} F(t+h-u)F(t-u)du. \tag{2.3}$$

The factorization is possible since $\gamma(h)$ has spectral representation with spectral density $f(\lambda)$ such that

$$\int \frac{\log f(\lambda)}{1+\lambda^2} d\lambda > -\infty \tag{2.4}$$

since $X(t)$ is purely non-deterministic. Hence, the Hardy class theory for analytic functions on half space can be applied. Now, we see that

$$c(w) = \sqrt{2\pi i} \exp\left[-\frac{1}{2\pi_i} \int_{-\infty}^{\infty} \frac{\lambda - w}{1+\lambda w} \frac{\log f(\lambda)}{1+\lambda^2} d\lambda \right], w \in \mathbf{C} \tag{2.5}$$

is defined, and it is known that the limit

$$c(\lambda) = \lim_{\mu \to 0} -c(\lambda + i\mu), \mu < 0 \tag{2.6}$$

exists. The Fourier transform

$$\hat{C}(u) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{iu\lambda} c(\lambda) d\lambda \tag{2.7}$$

is in agreement with the canonical kernel F up to a multiplicative constant of absolute value 1.

Remark 1 . *This theory is applied to obtaining the canonical representation of a Gaussian process only when the process is stationary.*

Remark 2 . *There is obtained the factorization F (indeed, the optimal kernel) of the covarianace function (2.3) explicitly. This method is purely analytic. We compare with that in the next section for non-stationary case.*

3 Canonical representation theory for Gaussian processes

Factorization of the covariance function is one of the main tools for the study of Gaussian processes.

Lévy's example:

$$X_1(t) = \int_0^t (2t-u)\dot{B}(u)du, \tag{3.1}$$

$$X_2(t) = \int_0^t (-3t+4u)\dot{B}(u)du. \tag{3.2}$$

We claim that two processes are the same. In fact, the two processes have the same covariance $3ts^2 - \frac{2}{3}s^3$ for $t > s \geq 0$.

Most important viewpoint is that the representation (3.1) of $X_1(t)$ is *canonical*, but (3.2) for $X_2(t)$ is not.

In general, a representation of $X(t)$ given by

$$X(t) = \int^t F(t,u)\dot{B}(u)du \tag{3.3}$$

is said to be *canonical* if the following equality for the conditional expectation holds for every $t > s$:

$$E\Big(X(t) \Big| \mathbf{B}_s(X)\Big) = \int^s F(t,u)\dot{B}(u)du. \tag{3.4}$$

A criterion for the canonical property on the kernel $F(t,u)$ is given in [2].

The canonical property of a representation was proposed by P. Lévy in 1955 at the third Berkeley Symposium on Math. Statistic and Probability. General theory of existence was given in [2] and later by H. Cramér in 1961. Useful applications of this theory are found in the theory of multiple Markov property and in the study of Lévy's Brownina motion. They are given also in [2].

The problem of getting the canonical kernel has close connection with the factorization of the covariance function; indeed getting the *optimal* kernel among the possible factorizations. One of the idea of the factorization is the use of the theory of Reproducing Kernel Hilbert Space (RKHS). Here is a short summary how to obtain RKHS from a positive definite kernel.

Given a positive definite function $\Gamma(t,s), t, s \in T$. Then, there is a linear space $\mathbf{F}_1$ spanned by $\Gamma(\cdot,t), t \in T$, where we have a bilinear form (reproducing property)

$$(f(\cdot), \Gamma(\cdot, s)) = f(s).$$

This equation defines a semi-norm $\|\cdot\|$ in $\mathbf{F}_1$. We consider such a minimal space and define a factor space $\mathbf{F} = \mathbf{F}_1/\|\cdot\|$. Thus obtained $\mathbf{F}$ is a Hilbert space where the reproducing property holds. The kernel $\Gamma(t,s)$ is called the reproducing kernel of $\mathbf{F}$. In view of this $\mathbf{F}$ is often written as $\mathbf{F}(\Gamma)$. We have

$$(\Gamma(\cdot,t), \Gamma(\cdot,s)) = \Gamma(s,t), \tag{3.5}$$

where one can see a square root (in a sense) of the covariance function Γ or its factorization.

Thus $X(t)$, corresponds to $\Gamma(\cdot,t)$ which is obtained by (3.5) (the factorization of the covariance function), is obtained. Then, we are led to have the canonical kernel, although we do not come into details to this direction.

4 Nonlinear case; white noise

Let μ be the white noise measure introduced in the space E^* of generalized functions on R^1. We consider the complex Hilbert space $(L^2) = L^2(E^*, \mu)$ involving nonlinear functionals of the $\dot{B}(t)$ or of $x \in E^*(\mu)$ with finite variance. This space is classical. It is generated by the $e^{ia\langle x,\xi\rangle}, \xi \in E, a \in R^1$. Hence, $X(t)$ in (2.2) is replaced by $e^{i\langle x,\xi\rangle}$, so that the covariance is now

$$C(\xi - \eta) = E(e^{i\langle x,\xi\rangle}e^{-i\langle x,\eta\rangle}), \tag{4.1}$$

where $C(\xi)$ is the characteristic functional of white noise. It is positive definite, so that we can form a RKHS $\mathcal{F}$. The reproducing kernel of $\mathcal{F}$ is $C(\xi-\eta)$ which is the characteristic functional of white noise such that $C(\xi) = \exp[-\frac{1}{2}\|\xi\|^2]$.

Observe now the formula (rephrasement of (4.1)).

$$C(\xi - \eta) = \int e^{i\langle x,\xi\rangle}\overline{e^{i\langle x,\eta\rangle}}d\mu(x), \tag{4.2}$$

so that we have a general formula

$$C(\xi - \sum a_j \eta_j) = \int e^{i\langle x,\xi\rangle} \overline{\Pi_j e^{i\langle x,\eta_j\rangle}} d\mu(x). \tag{4.3}$$

The product which is the factor of the integrand extends to a general white noise functional, say $\varphi(x)$. Then, the integral turns into the following formula

$$(\mathcal{T}\varphi)(\xi) = \int e^{i\langle x,\xi\rangle} \varphi(x) d\mu(x), \tag{4.4}$$

which is the $\mathcal{T}$-transform of $\varphi(x)$. Let it be denoted by $V_\varphi(\xi)$ or simply by $V(\xi)$. It holds that

$$(V(\cdot), C(\cdot - \xi)) = V(\xi), \tag{4.5}$$

where, $(\cdot, \cdot)$ is the inner product in the RKHS $\mathcal{F}$. In particular, we have

$$(C(\cdot - \xi), C(\cdot - \eta)) = C(\eta - \xi). \tag{4.6}$$

Thus the characteristic functional is factorized by the $\mathcal{T}$-transform with the help of the RKHS. If this transform is restricted to H_n, the space of the n-ple Wiener integrals, we are given the integral representation up to $i^n C(\xi)$, i.e.

$$V(\xi) = i^n C(\xi) \int \cdots \int_{R^n} F(u_1, \cdots, u_n) \xi(u_1) \cdots \xi(u_n) du^n, \tag{4.7}$$

where F is a symmetric $L^2(R^n)$ function. It is convenient to write $V(\xi) = i^n C(\xi) U(\xi)$.

The S-transform is

$$(S\varphi)(\xi) = C(\xi) \int e^{\langle x,\xi\rangle} \varphi(x) d\mu(x),$$

by which we can immediately get $U(\xi)$.

Theorem 3 *The following facts hold.*

i) The $\mathcal{T}$-transform factorizes the characteristic functional $C(\xi)$ of white noise.

ii) The system $\{C(\cdot - \xi)\}$ corresponds to the system $\{e^{ia\langle x.\xi\rangle}\}$ which is total in (L^2).

iii) A generalization of the representation of a Gaussian process is given by (4.7).

The proofs of i) and ii) have already be given, As for the proof we need some interpretations which can be seen in [6].

Some more details will be reported in the forthcoming paper.

5 Observations

We now make an important *remark* on $\mathcal{T}$- and S-transforms regarding their meanings and roles, through various observations which are in order.

1) They define maps from the space of generalized white noise functionals to the spaces with reproducing kernels. We are given big advantages, since the image involves good functionals of smooth function of ξ. Those functionals are no more random, and are easy to be analyzed, in general, by appealing to the known theory of functional analysis.

Examples:

$$\dot{B}(t) \quad \rightarrow \quad \xi(t) = (\delta_t, \xi),$$

$$: \dot{B}(t)^n : (\text{renormalized } \dot{B}(t)^2) \quad \rightarrow \quad \xi(t)^n,$$

$$N \exp[c \int \dot{B}(t)^2 dt] \quad \rightarrow \exp[\frac{c}{1-2c} \|\xi\|^2], c \neq \frac{1}{2}.$$

2) They help us to determine factorizations (of covariances and others), which is the main tool of what we discuss in this report.

3) They play a role of determining the integral representations of white noise functionals. With the help of the Sobolev spaces we have been led to introduce spaces of *generalized* white noise functionals. For instance, take the sub-space $H_n^{(-n)}$ of the space of Hida distributions. It is an extension of H_n. For some more details on $H_n^{(-n)}$, we refer to [6] Chapt. 2. We have established

$$H_n^{(-n)} \quad \cong \quad K^{-(n+1)/2}(R^n)(\text{symmetric}),$$

where the notation $K^m(R^n)$ denotes the Sobolev space over R^n of order m.

Note It is very important to recognize the real meaning of the $\mathcal{T}$- and S-transforms including the facts mentioned above. Needless to say, the topology equipped with RKHS, the image of the $\mathcal{T}$- or S-transform, is most convenient for our calculus (see [3]). They should **never** be thought of (simply) as similar transforms to the classical ones. Essentially different from them.

6 Random square root of the Dirac delta function

In this section we give a naive verification that we have proposed in Section 1.

Let us state a proposed and formal assertion.

Proposition 4 *The delta function $\delta(t)$ is positive definite.*

This statement may be proved if we were allowed to use a formal calculus using $\pm\infty$ and formulas like

$$\delta(0) = \infty \tag{6.1}$$
$$\infty + a = \infty \tag{6.2}$$
$$a\infty = \infty, \ (a > 0) \tag{6.3}$$

and so on. Then, we can say δ is positive definite and have a reproducing kernel Hilbert space RKHS $\mathbf{F}(\delta)$. Our final conclusion then follows easily.

We are now ready to give a rigorous interpretation on what we wish to claim.

Explanation by using RKHS.

We start with the δ-function $\delta_t(\cdot) = \delta(\cdot - t)$. It is a generalized function in $K^{-1}(R^1)$ and its Fourier transform is $\frac{e^{it\lambda}}{\sqrt{2\pi}}$. Define a mapping Π:

$$\Pi : \delta_t \quad \longrightarrow \quad e^{it\lambda}.$$

The mapping Π defines a bijection between two systems:

$$\Pi : \Delta = \{\delta_t, t \in R^1\} \quad \longrightarrow \quad \Lambda = \{e^{it\lambda}, t \in R^1\}.$$

The $K^{-1}(R^1)$-norm is introduced to Δ, so is topologized Λ. Their closures are denoted by the same symbols, respectively. And they are isomorphic to each other. The inner product of $e^{it\lambda}$ and $e^{is\lambda}$ is

$$\int \frac{e^{i(t-s)\lambda}}{\pi(1+\lambda^2)} d\lambda = e^{-|t-s|}, \tag{6.4}$$

which is positive definite. Hence, we can form a RKHS $\mathbf{F}_\delta$ with reproducing kernel $e^{-|t-s|}$. We can establish a mapping through Π:

$$\delta(\cdot - t) \quad \longrightarrow \quad e^{-|\cdot - t|} \tag{6.5}$$

The inner product

$$\langle e^{-|\cdot - t|}, e^{-|\cdot - s|} \rangle = e^{-|t-s|}$$

implies

$$\langle \delta(\cdot - t), \delta(\cdot - s) \rangle_\Delta = \delta(t - s). \tag{6.6}$$

Remind the mapping $S(\dot{B}(t)) = \xi(t) = \int \delta_t(u)\xi(u)du$. And hence $\dot{B}(t)$ corresponds to $\delta(t-u)$. Now the right hand side of (6.6) is just the delta function and the left hand side is viewed as the (inner) *product* of $\dot{B}(t)$ and $\dot{B}(s)$.

Theorem 5 *In view of the equation (6.6) the random square root of the delta function is a white noise.*

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