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A Covariant Optimal Quantum Clock

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We present a quantum clock model which preserves its accuracy for a long time and is precise as possible. The former requirement is achieved by the covariant measurement in the sense of Holevo, while for the latter we choose the optimal POVM and the optimal initial state. The best precision we have achieved is $\omega\delta t = \pi/(N+1)$ for an N level system, with ω being the angular frequency of an harmonic oscillator.

I. INTRODUCTION

We cannot measure time directly but we can measure a clock and then infer the time from the position of the hand we read. This feature of measuring time is more acute in quantum mechanics because there is no self-adjoint time operator if the Hamiltonian is bounded below. The time in the Schroedinger equation is only a parameter but not a physical observable. Any physical object can be a clock but its quality may vary. It is reasonable to consider a clock is good if it has a high precision and its accuracy is maintained for a long time. We are going to explore the best quantum clock of the highest possible precision given a finite energy range, which keeps accuracy in an arbitrarily long time passage.

Our model is a circular clock with a single hand, which moves along a circle and ticks at the discrete angles :$2\pi m/N \;\; m = 0, 1, \cdots N-1$ for an N level quantum system. . We can read off the time from the position m of the hand.The ticking time is the $2\pi/N\omega$ so that he precision should be less than this to distinguish two different positions.

II. ACCURACY

Let the position of the clock hand be ϕ and the inferred Schroedinger time be t. To mathematically formulate the accuracy, we demand the covariance of measurement according to Holevo[1]. Suppose that a POVM $M_t(\phi)$ is invariant under a translation $t \to t+\delta, \phi \to \phi + \omega\delta$ for all real t. In the case that this invariance holds for arbitrary displacement δ we call it as the continuous covariance, while if δ has a particular value we call it as the discrete covariance. We are going to work in the N dimensional Hibert space with the basis$|0>, |1> \cdots |N-1>$. The angle stae $|\phi>$ is defined by

$$\sum_{n=0}^{N-1} e^{i\phi n}|n> . \tag{1}$$

A projective measurement of ϕ after time t gives the POVM

$$M_t(\phi) = e^{-iHt}|\phi>< \phi|e^{iHt}. \tag{2}$$

A simple example of continuous covariance is given by $H = \omega N$, where N is the generator of translation of ϕ, which we shall discuss in greater detail later. A counter example is the Wigner clock[2], in which the Hamiltonian is the free Hamiltonian of a particle of the position q. The state evolution is dispersive so that the spread of the wave packet increases like the square root of time, i.e. the accuracy gets worse in time. However, if we replace the Hamiltonian by the one for the harmonic oscillator, the POVM has a discrete covariance. The wave packet spreads and shrinks periodically so that the long time accuracy is maintained. Only in the minimum uncertainty case the wave packet keeps its shape and oscillates back and forth. In this case the precision is of the order of $1/\sqrt{N}$, which is too coarse to measure the ticking time of the order of $1/N$. Recall that our discussion of the clock model is under the requisite that the accuracy is maintained for a long time throughout this report.

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III. PRECISION

Holevo studied the optimal covariant POVM and found that it is of the form $M_t(q) = e^{-iN\omega t}|q>< q|e^{iN\omega t}$. If $|q>$ is an eigen-state of a physical observable, i.e. self-adjoint operator, this is a projective measurement of the physical quantity corresponding to the operator to an eigenvalue q. Note that the optimal POVM has a pure state in its expression instead of a general mixed state ρ. For an initial state $|g>$ the probability distribution $P_t[q, |g>]$ is given by $P_t[q, |g>= Tr[|g>< g|M_t(q)]$. We are going to define the precision by introducing the cost function of Holevo type[1]. By minimizing $Tr(M_t(\phi)\rho)$ with respect to ρ with a fixed initial state $|g>$, he found the above form. Buzek et al.[3] further minimizes the cost function with respect to the initial state $|g>$ for a particular case $|q>=|\phi>$ with ϕ being the coordinate conjugate to N, i.e. the angle of the clock hand. Then let the system unitarily evolve as before and so on.To make the projection measurement of $|\phi>< \phi|$, ϕ has to be a discrete angle $\phi_m = \frac{2\pi m}{N}$, since the operator $\Phi = \sum_{m=0}^{N-1} \phi_m |m>< m|$ is a self-adjoint operator, while the angle itself is not. The discrete angle states $|\phi_m>$ are orthogonal to each other for different m's. The clock ticks from one state to another orthogonal state.

To summarize, we start with the (modified) Buzek state

$$|g>= \sqrt{\frac{2}{N}} \sum \sin(\frac{\pi(n+1)}{N})|n>, \tag{3}$$

which evolves by the Hamiltonian $H = \omega N$ and the measurement is performed with the projection $|\phi>< \phi|$. The distribution function is a function of $\phi - t$ and has a peak at zero with the width $\pi/(N+1)$, which is smaller than the ticking time $2\pi/N$. The width coincides with the Margolus-Levitin limit [4]of time for a large N. The Margolus-Levitin limit is the minimum time to go over to an orthogonal state to the initial state. In this sense, our clock is the best we can make. However, there is no intrinsic reason for the precision limit of the optimal measurement in Buzek's method coincides with the Margolus-Levitin limit for a large N. To proceed further we have to push down the resultant state $|\phi>$ back to $|g>$, e.g., by cooling the clock. Otherwise, the state $|\phi>$ instead of $|g>$ gives the spread of order $\frac{1}{\sqrt{N}}$, implying that we cannot read with certain the hand of the ticking time $2\pi/N$.

IV. SUMMARY AND OPEN QUESTIONS

To summarize this progress report based on the joint work with S.Lloyd and Y.Shikano, we have obtained a covariantly optimal quantum clock which can maintain its accuracy for a long time and the precision is

$$\omega\delta t = \pi/(N+1) \tag{4}$$

for the N level system in the case of the discrete angle. One might consider the case of generalized measurement of the continuous angle ϕ. In this case the limit by uncertainty principle comes from the Mandelstam-Tamm limit[5] , which gives the minimum time to distinguish the distribution functions which evolves in time. In our clock model, it is $1/N$, smaller than the obtained precision $\pi/(N+1)$. This may suggest the possibility that model is not yet optimal in the framework of generalized measurement of the continuous angle. We hope that our analysis can help the actual construction of quantum clock[6] .

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