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# Ikeda's conjecture on the period of the Ikeda lift

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**Abstract**

As an affirmative answer to the Duke-Imamoḡlu conjecture, Ikeda constructed a certain lifting of classical cusp forms on the special linear group $\mathrm{SL}_2$ towards Siegel cusp forms, namely cuspidal automorphic forms on the symplectic group $\mathrm{Sp}_{2n}$ of general even genus $2n$. Afterwards he also proposed a certain conjecture concerning the periods (Petersson norms squared) of such forms. In this paper, we would like to explain a brief sketch of a proof of the conjecture. Details will appear elsewhere.

## 1 Introduction

For each positive integer $n \in \mathbb{Z}$, the symplectic modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$ of genus $2n$ is defined to be

$$\mathrm{Sp}_{2n}(\mathbb{Z}) = \left\{\gamma \in \mathrm{GL}_{2n}(\mathbb{Z}) \,\middle|\, {}^t\gamma J\gamma = J, J = \left(\begin{smallmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{smallmatrix}\right)\right\}.$$

For either an integer or a half-integer $\kappa \in \frac{1}{2}\mathbb{Z}$, we denote the complex vector space consisting of all Siegel cusp forms of weight $\kappa$ with respect to a suitable congruence subgroup $\Gamma$ of $\mathrm{Sp}_{2n}(\mathbb{Z})$ by $S_\kappa(\Gamma)$. Then for each $F, G \in S_\kappa(\Gamma)$, we define the Petersson scalar product $\langle F,\, G\rangle$ by

$$\langle F,\, G\rangle := [\mathrm{Sp}_{2n}(\mathbb{Z}) : \Gamma\cdot\{\pm\mathbf{1}_{2n}\}]^{-1}\int_{\Gamma\backslash\mathfrak{H}_n} F(Z)\overline{G(Z)}\det(\mathrm{Im}(Z))^\kappa\, dZ^*,$$

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where $Z = X + \sqrt{-1}Y \in \mathfrak{H}_n = \{ Z \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid {}^tZ = Z,\ \mathrm{Im}(Z) > 0 \}$ and $dZ^* = \det Y^{-(n+1)} dXdY$ is a finite volume element on $\mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathfrak{H}_n$. As is well-known, this defines a Hermitian scalar product on the space $S_\kappa(\Gamma)$ and hence we can introduce the norm $\|F\|^2 := \langle F, F\rangle$ for each $F \in S_\kappa(\Gamma)$. We note that if $F$ is a Hecke eigenform, that is, a common eigenfunction of all Hecke operators, then the Petersson norm squared $\|F\|^2$ plays an important role within the framework of studying critical values of the standard $L$-function $L(s, F, \mathrm{st})$ attached to $F$ (cf. [1]).

On the other hand, for a couple of positive even integers $n$ and $k$ such that $k > n+1$, let $f \in S_{2k-n}(\mathrm{Sp}_2(\mathbb{Z})) = S_{2k-n}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Then we can consider the lift of $f$ towards the space $S_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$ as follows. Namely, Ikeda ([9]) showed that there exists a Hecke eigenform $F_f \in S_k(\mathrm{Sp}_n(\mathbb{Z}))$ such that

$$L(s, F_f, \mathrm{st}) = \zeta(s) \prod_{i=1}^{n} L(s+k-i, f),$$

where $\zeta(s)$ and $L(s, f)$ are the Riemann zeta function and the Hecke $L$-function associated with $f$, respectively. We note that the above lifting coincides with the Saito-Kurokawa lifting in case $n = 2$, and the existence of the lifting was firstly conjectured by Duke and Imamoḡlu in case $n > 2$ (cf. [2]). More precisely, Ikeda explicitly constructed $F_f$ by Fourier expansions of $f$ and a Hecke eigenform $g \in S_{k-n/2+1/2}(\Gamma_0^{(2)}(4))$ corresponding to $f$ under the Shimura correspondence, where $\Gamma_0^{(2)}(4) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod 4\}$. In this paper, we simply call $F_f$ the Ikeda lift of $f$.

As will be explained precisely in the subsequent part, Ikeda also conjectured in [10] that the ratio $\|F_f\|^2/\|g\|^2$ should be expressed in terms of special values of certain $L$-functions attached to $f$. The purpose of this paper is to explain a proof of the conjecture. We note that $F_f$ could not necessarily be realized as a theta lift except for the case $n = 2$. Thus we cannot use a general method for evaluating Petersson scalar products of theta lifts due to Rallis (cf. [24]). The method we use is to give explicit formulae for several kinds of Dirichlet series of Rankin-Selberg type attached to Siegel modular forms and then to compare their residues.

We note that we can consider an application of the main result to a problem concerning congruences between Ikeda lifts and some genuine Siegel modular forms. This has been announced in [13, 16], and the details will be discussed in [14].

# 2 Main results

Throughout this section, we fix a pair of positive even integers $n$, $k \in \mathbb{Z}$ such that $k > n+1$.

## 2.1 Construction of the Ikeda lift

Let $\mathrm{Sym}_n^*(\mathbb{Z})_+$ be the set of all positive definite half-integral symmetric matrices of size $n$. For each $B \in \mathrm{Sym}_n^*(\mathbb{Z})_+$ and a rational prime $p$, we put

$$b_p(B;\, s) := \sum_{R \in \mathrm{Sym}_n(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_n(\mathbb{Z})} \mathbf{e}(\mathrm{tr}(BR))\, p^{-s\cdot\mu_p(R)},$$

where $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$ for $x \in \mathbb{C}$, and $\mu_p(R) = [\mathbb{Z}_p^n R + \mathbb{Z}_p^n : \mathbb{Z}_p^n]$. As is known by Kitaoka ([18]), we have that there exists a unique polynomial $F_p(B;\, X) \in \mathbb{Z}[X]$ such that

$$b_p(B;\, s) = F_p(B;\, p^{-s}) \times \frac{(1-p^{-s})\prod_{i=1}^{n/2}(1-p^{2i-2s})}{1-\chi_B(p)\, p^{n/2-s}},$$

where $\chi_B : \mathbb{Z} \to \{\pm 1,\, 0\}$ denotes the Kronecker character corresponding to the quadratic field extension $\mathbb{Q}(\sqrt{\mathfrak{D}_B})/\mathbb{Q}$ with $\mathfrak{D}_B := (-1)^{n/2}\det(2B)$. In addition, we can write $\mathfrak{D}_B = \mathfrak{d}_B\, \mathfrak{f}_B^2$ in terms of a fundamental discriminant $\mathfrak{d}_B$, that is, the discriminant of $\mathbb{Q}(\sqrt{\mathfrak{D}_B})/\mathbb{Q}$ and $\mathfrak{f}_B = \sqrt{\mathfrak{D}_B/\mathfrak{d}_B} \in \mathbb{Z}$. Then it is also known that the Laurent polynomial $\widetilde{F}_p(B;\, X) := X^{-\mathrm{ord}_p(\mathfrak{f}_B)} F_p(B;\, p^{-(n+1)/2}X)$ is invariant under $X \mapsto X^{-1}$ (cf. [12]).

On the other hand, let

$$f(\tau) = \sum_{m\geq 1} a_f(m)\mathbf{e}(m\tau) \in S_{2k-n}(\mathrm{SL}_2(\mathbb{Z})) \quad (\tau \in \mathfrak{H}_1)$$

be a Hecke eigenform normalized as $a_f(1) = 1$. Then we can associate $f$ with a Hecke eigenform

$$g(\tau) = \sum_{\substack{m\geq 1,\\ (-1)^{k-n/2}m \equiv 0,1 \pmod 4}} c_g(m)\mathbf{e}(m\tau) \quad (\tau \in \mathfrak{H}_1)$$

in Kohnen's plus space $S^+_{k-n/2+1/2}(\Gamma_0^{(2)}(4))$ of half-integral weight $k - n/2 + 1/2$, that is, a subspace of $S_{k-n/2+1/2}(\Gamma_0^{(2)}(4))$ characterized by the Shimura's Hecke-equivariant isomorphism

$$S^+_{k-(n-1)/2}(\Gamma_0^{(2)}(4)) \xrightarrow{\simeq} S_{2k-n}(\mathrm{SL}_2(\mathbb{Z}))$$

(cf. [20]). Then Ikeda's lifting theorem is stated as follows:

**Theorem I** (cf. [9]). *For each* $B \in \mathrm{Sym}_n^*(\mathbb{Z})_+$, *we put*

$$C_{F_f}(B) := c_g(|\mathfrak{d}_B|)\, \mathfrak{f}_B^{k-n/2-1/2} \prod_{p|\mathfrak{f}_B} \widetilde{F}_p(B;\, \alpha_p),$$

*where* $\alpha_p + \alpha_p^{-1} = p^{-k+n/2+1/2} a_f(p)$. *Then*

$$F_f(Z) = \sum_{B \in \mathrm{Sym}_n^*(\mathbb{Z})_+} C_{F_f}(B)\, \mathbf{e}(\mathrm{tr}(BZ)) \quad (Z \in \mathfrak{H}_n)$$

*belongs to the space* $S_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$, *and forms a Hecke eigenform such that*

$$L(s,\, F_f,\, \mathrm{st}) = \zeta(s) \prod_{i=1}^{n} L(s+k-i,\, f).$$

We do not consider Eisenstein series here. However, one can formally look at the Ikeda lift as an analogy to the association between Siegel Eisenstein series $E_k^{(2n)}$ of weight $k$ with respect to $\mathrm{Sp}_{2n}(\mathbb{Z})$ and Eisenstein series $E_{2k-n}^{(2)}$ of weight $2k-n$ with respect to $\mathrm{SL}_2(\mathbb{Z})$. Namely, we have

$$L(s,\, E_k^{(2n)},\, \mathrm{st}) = \zeta(s) \prod_{i=1}^{n} L(s+k-i,\, E_{2k-n}^{(2)}).$$

## 2.2 Ikeda's conjecture and the main theorem

In order to state Ikeda's conjecture precisely, we introduce some notations of $L$-functions as follows. For a given normalized Hecke eigenform $f \in S_{2k-n}(\mathrm{SL}_2(\mathbb{Z}))$ as in the previous section, we put

$$\begin{cases} \widetilde{\xi}(s) := \Gamma_{\mathbb{C}}(s)\, \zeta(s), \\ \Lambda(s,\, f) := \Gamma_{\mathbb{C}}(s)\, L(s,\, f), \\ \widetilde{\Lambda}(s,\, f,\, \mathrm{ad}) := \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s+2k-n-1)\, L(s,\, f,\, \mathrm{ad}), \end{cases}$$

where $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$ and $L(s,\, f,\, \mathrm{ad})$ denotes the adjoint $L$-function of $f$ defined by

$$L(s,\, f,\, \mathrm{ad}) = \prod_p \{(1-p^{-s})(1-\alpha_p^2 p^{-s})(1-\alpha_p^{-2}p^{-s})\}^{-1}.$$

As is well-known, we have $\widetilde{\xi}(2i) = |B_{2i}|/2i \in \mathbb{Q}^\times$ for each positive $i \in \mathbb{Z}$, where $B_{2i}$ is the $2i$-th Bernoulli number. It is also known that $\widetilde{\Lambda}(2i\, -$

1, $f$, Ad$)/\|f\|^2$ is an algebraic number for each $1 \leq i < k - n/2$. In particular, we have $\widetilde{\Lambda}(1, f, \mathrm{Ad}) = 2^{2k-n}\|f\|^2$ (cf. [26]). Then Ikeda proposed the following:

**Conjecture I** (cf. [10]). *Under the same situation as in Theorem I, there exists $\alpha(n,k) \in \mathbb{Z}$ such that*

$$\frac{\|F_f\|^2}{\|g\|^2} = 2^{\alpha(n,k)}\Lambda(k, f)\,\widetilde{\xi}(n)\prod_{i=1}^{n/2-1}\widetilde{\xi}(2i)\,\widetilde{\Lambda}(2i+1, f, \mathrm{ad}).$$

When $n = 2$, it has been already known by Kohnen and Skoruppa that the above conjecture holds true (cf. [21], see also [23]). Then the main theorem in this paper is stated as follows.

**Theorem 2.1.** *Conjecture I holds true for any positive even $n$.*

In the subsequent sections, we will explain a proof of Theorem 2.1 by using a three step-wise approach.

# 3 Rankin-Selberg method for the Fourier-Jacobi expansion of the Ikeda lift

For the moment, let us review the theory of Fourier-Jacobi expansions of Siegel modular forms of genus $2n \geq 4$ and its application towards the evaluation of Petersson norm squared.

For each positive $k \in \mathbb{Z}$, let $F \in S_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$ possess the Fourier expansion

$$F(Z) = \sum_{B \in \mathrm{Sym}_n^*(\mathbb{Z})_+} C_F(B)\,\mathbf{e}(\mathrm{tr}(BZ)) \quad (Z \in \mathfrak{H}_n).$$

Then by decomposing each point $Z \in \mathfrak{H}_n$ into the form

$$\begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix} \quad ((\tau, z) \in \mathfrak{H}_{n-1} \times \mathbb{C}^{n-1},\ \tau' \in \mathfrak{H}_1),$$

we obtain the Fourier-Jacobi expansion

$$F(\begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix}) = \sum_{m=1}^{\infty} \phi_m(\tau, z)\mathbf{e}(m\tau'),$$

where

$$\phi_m(\tau,\, z) := \sum_{\substack{(T,r)\in \mathrm{Sym}^*_{n-1}(\mathbb{Z})\times\mathbb{Z}^{n-1}\\ 4mT-{}^trr>0}} C_F(\begin{pmatrix} m & r/2 \\ {}^tr/2 & T \end{pmatrix})\mathbf{e}(\mathrm{tr}(T\tau + {}^trz)).$$

We note that for each $m$, the function $\phi_m$ belongs to the complex vector space $J^{\mathrm{cusp}}_{k,m}(\mathrm{Sp}_{2n-2}(\mathbb{Z})^J)$ consisting of all holomorphic Jacobi cusp forms of weight $k$ and index $m$ with respect to the Jacobi modular group $\mathrm{Sp}_{2n-2}(\mathbb{Z})^J := \mathrm{Sp}_{2n-2}(\mathbb{Z}) \ltimes (\mathbb{Z}^{2n-2} \times \mathbb{Z})$ of genus $2n-2$ (cf. [28]). Then we define the Dirichlet series $D(s,\, F)$ attached to $F$ by

$$D(s,\, F) := \zeta(2s-2k+2n)\sum_{m=1}^{\infty} \|\phi_m\|^2 m^{-s},$$

where $\|\phi_m\|^2$ denotes the Petersson norm squared of $\phi_m \in J^{\mathrm{cusp}}_{k,m}(\mathrm{Sp}_{2n-2}(\mathbb{Z})^J)$ introduced to be

$$\|\phi_m\|^2 := \int_{\mathrm{Sp}_{2n-2}(\mathbb{Z})^J\backslash\mathfrak{H}_{n-1}\times\mathbb{C}^{n-1}} |\phi_m(\tau,\, z)|^2 \det(\mathrm{Im}(\tau))^k \times \exp(-4\pi m\,\mathrm{Im}(z)\mathrm{Im}(\tau)^t\mathrm{Im}(z))\, d\tau^* dz.$$

We easily see that the Dirichlet series $D(s,\, F)$ converges absolutely for $\mathrm{Re}(s) > k$. Moreover, Yamazaki showed the following:

**Theorem II** (cf. [27], see also [22]). *The function*

$$D^*(s,\, F) := \pi^{k-n-1}(2\pi)^{1-2s}\Gamma(s)\, D(s,\, F)$$

*has a meromorphic conticuation to the whole $s$-plane, and has simple poles at $s = k,\, k-n$ with the residue $\|F\|^2$. Furthermore, it satisfies the functional equation*

$$D^*(2k-n-s,\, F) = D^*(s,\, F).$$

Then, as the first main ingredient of the proof of Theorem 2.1, we have the following:

**Theorem 3.1** (cf. [15]). *Let $n, k$ be as in §2. If $f \in S_{2k-n}(\mathrm{SL}_2(\mathbb{Z}))$ is a normalized Hecke eigenform, then*

$$D(s,\, F_f) = \|\phi_{f,1}\|^2\, \zeta(s-k+1)\zeta(s-k+n)L(s,\, f),$$

*where $\phi_{f,1}$ denotes the first coefficient of the Fourier-Jacobi expansion of $F_f$.*

Moreover, by comparing residues at $s=k$ on both sides, we also obtain

**Corollary 3.1.** *Under the same situation as above, we have*

$$\frac{\|F_f\|^2}{\|\phi_{f,1}\|^2} = 2^{-k+n-1}\Lambda(k,\, f)\widetilde{\xi}(n). \tag{1}$$

When $n = 2$, the above two results have been obtained by Kohnen and Skoruppa ([21]).

# 4 The Eichler-Zagier-Ibukiyama isomorphism

Based on the result in the previous section, let us review in this section that there exists a natural correspondence between holomorphic Jacobi forms of integral weight and index 1 and Siegel modular forms of half-integral weight, and explain the coincidence of Petersson norms squared up to scalar.

We put $\Gamma_0^{(2n-2)}(4) := \left\{ \gamma \in \mathrm{Sp}_{2n-2}(\mathbb{Z}) \mid \gamma \equiv \left( \begin{smallmatrix} * & * \\ 0_{n-1} & * \end{smallmatrix} \right) \pmod 4 \right\}$. Then for each $k \in \mathbb{Z}$, we introduce the generalized Kohnen's plus space by

$$\begin{aligned} &S^+_{k-1/2}(\Gamma_0^{(2n-2)}(4)) \\ &:= \left\{ F(Z) \in S_{k-1/2}(\Gamma_0^{(2n-2)}(4)) \;\middle|\; \begin{array}{c} C_F(A) = 0 \text{ unless } A \equiv (-1)^{k+1}\, {}^t r r \\ \pmod{4\mathrm{Sym}^*_{n-1}(\mathbb{Z})} \text{ for some } r \in \mathbb{Z}^{n-1} \end{array} \right\}. \end{aligned}$$

As is mentioned before, for each positive even $k \in \mathbb{Z}$, we have

$$S^+_{k-1/2}(\Gamma_0^{(2)}(4)) \xrightarrow{\simeq} S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})).$$

Moreover, Eichler and Zagier ([3]) showed that there exists an isomorphism

$$J^{\mathrm{cusp}}_{k,1}(\mathrm{SL}_2(\mathbb{Z})^J) \xrightarrow{\simeq} S^+_{k-1/2}(\Gamma_0^{(1)}(4)),$$

which is compatible with actions of all Hecke operators up to $p=2$. As a generalization of the isomorphism, Ibukiyama showed the following:

**Theorem III** (cf. [4]). *If $n \geq 2$, then for each positive even $k \in \mathbb{Z}$, there exists an isomorphism*

$$\sigma : J^{\mathrm{cusp}}_{k,1}(\mathrm{Sp}_{2n-2}(\mathbb{Z})^J) \xrightarrow{\simeq} S^+_{k-1/2}(\Gamma_0^{(2n-2)}(4)),$$

*which is compatible with actions of Hecke operators up to $p=2$.*

In addition, Eichler and Zagier ([3]) also showed that the isomorphism $\sigma$ is compatible with Petersson norms squared. As its generalization to higher genus, we obtain the following:

**Theorem 4.1.** *Under the same assumtion as in Theorem III, for each* $\phi \in J_{k,1}^{\mathrm{cusp}}(\mathrm{Sp}_{2n-2}(\mathbb{Z})^J)$*, we have*

$$\|\phi\|^2 = 2^{2(k-1)(n-1)-1}\|\sigma(\phi)\|^2. \tag{2}$$

*Proof.* The proof proceeds in a similar way to that of Theorem 5.4 in [3]. □

Thus by combining Corollary 3.1 and Theorem 4.1, we can show Theorem 2.1 in case $n = 2$. Indeed, for a given normalized Hecke eigenform $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$, we denote by $g \in S_{k-1/2}^{+}(\Gamma_0^{(2)}(4))$ and $\phi_{f,1} \in J_{k,1}^{\mathrm{cusp}}(\mathrm{SL}_2(\mathbb{Z})^J)$ a Hecke eigenform corresponding to $f$ under Shimura's isomorphism and the first coefficient of the Fourier-Jacobi expansion of the Saito-Kurokawa lift $F_f \in S_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$ of $f$, respectively. Then we have $\sigma(\phi_{f,1}) = g$, and hence by combining the equations (1) and (2), we obtain

$$\frac{\|F_f\|^2}{\|g\|^2} = \frac{\|F_f\|^2}{\|\phi_{f,1}\|^2}\cdot\frac{\|\phi_{f,1}\|^2}{\|g\|^2} = 2^{k-2}\Lambda(k,\, f)\,\widetilde{\xi}(2),$$

and this proves the assertion. □

# 5 Rankin-Selberg method for Siegel modular forms of half-integral weight

In this section, we derive an explicit formulae for certain Dirichlet series attached to Siegel modular forms of half-integral weight and apply it to evaluate Petersson norms squared of such forms.

For each positive even $k \in \mathbb{Z}$, we consider

$$F(Z) = \sum_{A \in \mathrm{Sym}_{n-1}^*(\mathbb{Z})_+} C_F(A)\mathbf{e}(\mathrm{tr}(AZ)) \in S_{k-1/2}(\Gamma_0^{(2n-2)}(4)).$$

Then we define the Dirichlet series $R(s,\, F)$ attached to $F$ by

$$R(s,\, F) := \sum_{A \in \mathrm{Sym}_{n-1}^*(\mathbb{Z})_+/\mathrm{SL}_{n-1}(\mathbb{Z})} \frac{|C_F(A)|^2}{e(A)\det A^s},$$

where $e(A) = \#\{X \in \mathrm{SL}_{n-1}(\mathbb{Z}) \mid {}^tXAX = A\}$. This kind of Dirichlet series has been studied by Shimura ([25]) and Kalinin ([11]) in case of integral weight. Then by using a similar method, we easily see the following:

**Proposition 5.1.** *We put* $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\xi(s) = \Gamma_{\mathbb{R}}(s)\zeta(s)$ *and*

$$R^*(s, F) := \gamma_{n-1}(s)\xi(2s-2k+n+1)\prod_{i=1}^{n/2-1}\xi(4s-4k-2i+2n+2)R(s, F),$$

*where* $\gamma_{n-1}(s) = 2^{1-2s(n-1)}\prod_{j=1}^{n-1}\Gamma_{\mathbb{R}}(2s-j+1)$. *Then the function* $R^*(s, F)$ *has a meromorphic continuation to the whole s-plane and has a simple pole at* $s = k-1/2$ *with the residue* $\prod_{i=1}^{n/2-1}\xi(2i+1)\,\|F\|^2$.

Then we have an explicit formula for the Dirichlet series $R(s, \sigma(\phi_{f,1}))$ as follows:

**Theorem 5.2** (cf. [17]). *Under the same situation as in Theorem 3.1, we put* $\lambda_n = \frac{1}{2}\prod_{i=1}^{n/2-1}\widetilde{\xi}(2i)$. *Then we have*

$$R(s,\sigma(\phi_{f,1})) = \frac{\lambda_n}{2^{(n-1)(s+1/2)}}\zeta(2s+n-2k+1)^{-1}\prod_{i=1}^{n/2-1}\zeta(4s+2n-4k+2-2i)^{-1}$$
$$\times\{R(s-n/2+1,g)\,\zeta(2s-2k+3)$$
$$\times\prod_{j=1}^{n/2-1}L(2s-2k+2j+2,\ f,\ \mathrm{ad})\,\zeta(2s-2k+2j+2)$$
$$+(-1)^{n(n-2)/8}R(s,g)\,\zeta(2s-2k+n+1)$$
$$\times\prod_{j=1}^{n/2-1}L(2s-2k+2j+1,\ f,\ \mathrm{ad})\,\zeta(2s-2k+2j+1)\}.$$

Moreover, by comparing residues at $s = k-1/2$, we also obtain

**Corollary 5.2.** *Under the same situation as above, we have*

$$\frac{\|\sigma(\phi_{f,1})\|^2}{\|g\|^2} = 2^{\beta(n,k)}\prod_{i=1}^{n/2-1}\widetilde{\xi}(2i)\,\widetilde{\Lambda}(2i+1,\ f,\ \mathrm{ad}), \tag{3}$$

*where* $\beta(n, k) = -3k(n-2) + n(n-3)/2 + 1$.

Therefore, by combining the three equations (1), (2) and (3), we can show Theorem 2.1. Indeed, we have

$$\begin{aligned}\frac{\|F_f\|^2}{\|g\|^2} &= \frac{\|F_f\|^2}{\|\phi_{f,1}\|^2}\cdot\frac{\|\phi_1\|^2}{\|\sigma(\phi_{f,1})\|^2}\cdot\frac{\|\sigma(\phi_{f,1})\|^2}{\|g\|^2}\\ &= 2^{-k+n-1}\Lambda(k,\, f)\,\widetilde{\xi}(n)\cdot 2^{2(k-1)(n-1)-1}\cdot 2^{\beta(n,k)}\prod_{i=1}^{n/2-1}\widetilde{\xi}(2i)\,\widetilde{\Lambda}(2i+1,\, f,\, \mathrm{ad})\\ &= 2^{-(n-3)(k-n/2)-n+1}\Lambda(k,\, f)\,\widetilde{\xi}(n)\prod_{i=1}^{n/2-1}\widetilde{\xi}(2i)\,\widetilde{\Lambda}(2i+1,\, f,\, \mathrm{ad}),\end{aligned}$$

and this proves the assertion. □

# 6 Proof of Theorem 5.2

The rest of the paper is devoted to a sketch of a proof of Theorem 5.2. Details will appear in [17]. For each positive $m \in \mathbb{Z}$, we simply write $\mathcal{S}_{m,p} = \mathrm{Sym}_m^*(\mathbb{Z}_p)$ and $\mathcal{S}_{m,p}^\times = \mathcal{S}_{m,p} \cap \mathrm{GL}_m(\mathbb{Q}_p)$. In particular, if $m$ is odd, then we put

$$\mathcal{S}_{m,p}^{(1)} := \left\{A \in \mathcal{S}_{m,p} \;\middle|\; A + {}^t rr \in 4\mathcal{S}_{m,p} \text{ for some } r \in \mathbb{Z}_p^m\right\}.$$

For each $A \in \mathcal{S}_{n-1,p}^{(1)}$, we put

$$\widetilde{F}_p^{(1)}(A;\, X) := \widetilde{F}_p(\begin{pmatrix} 1 & r/2 \\ {}^t r/2 & (A + {}^t rr)/4 \end{pmatrix};\, X),$$

where $r = r_A \in \mathbb{Z}_p^{n-1}$ such that $A + {}^t rr \in 4\mathcal{S}_{n-1,p}$. For each $A \in \mathcal{S}_{m,p}^\times$ and $e \geq 0$, we put

$$\mathcal{A}_e(A,\, A) = \{X \in \mathrm{Mat}_{n-1\times n-1}(\mathbb{Z}_p)/p^e\mathrm{Mat}_{n-1\times n-1}(\mathbb{Z}_p) \mid {}^t XAX - A \in p^e\mathcal{S}_{m,p}\}$$

and

$$\alpha_p(A,\, A) := \frac{1}{2}\lim_{e\to\infty} p^{e\{-m^2+m(m+1)/2\}}\#\mathcal{A}_e(A,\, A).$$

For each $\mathfrak{d} \in \mathbb{Z}_p$ and a $\mathrm{GL}_{n-1}(\mathbb{Z}_p)$-invariant function $\omega_p$ on $\mathcal{S}_{n-1,p}^\times$, we put

$$H_p^{(n-1)}(\mathfrak{d},\, \omega_p;\, X,\, t) := \sum_{l=0}^{\infty}\sum_{A\in\mathcal{A}_p(\mathfrak{d},\, l)} \omega_p(A)\,\frac{|\widetilde{F}_p^{(1)}(A;\, X)|^2}{\alpha_p(A,\, A)}\, t^{\mathrm{ord}_p(\det A)},$$

where $\mathcal{A}_p(\mathfrak{d}, l) = \{A \in \mathcal{S}_{n-1,p}^{(1)} \mid \det A = \mathfrak{d}\, p^{2l+(n-2)\delta_{2,p}}\}/\mathrm{GL}_{n-1}(\mathbb{Z}_p)$. As for $\omega_p : \mathcal{S}_{n-1,p}^{\times}/\mathrm{GL}_{n-1}(\mathbb{Z}_p) \to \{\pm 1, 0\}$, we consider either the constant function $\iota_p$ on $\mathcal{S}_{n-1,p}^{\times}$ taking the value 1 or the function $\varepsilon_p$ assigning the Hasse invariant of $A$ for $A \in \mathcal{S}_{n-1,p}^{\times}$ (cf. [19]). Then by using the same method to Ibukiyama and Saito ([8]), similarly to [5, 6], we have

**Theorem 6.1.** *We have*

$$R(s, \sigma(\phi_{f,1})) = \kappa_{n-1} \sum_{\mathfrak{d}} |c_g(|\mathfrak{d}|)|^2 \, |\mathfrak{d}|^{-k+n/2+1/2}$$

$$\times \left\{ \prod_p H_p^{(n-1)}(\mathfrak{d}, \iota_p; \alpha_p, p^{-s+k-1/2}) + \prod_p H_p^{(n-1)}(\mathfrak{d}, \varepsilon_p; \alpha_p, p^{-s+k-1/2}) \right\},$$

*where the summation is taken over all fundamental discriminant $\mathfrak{d} \in \mathbb{Z}$ such that $(-1)^{n/2}\mathfrak{d} > 0$ and we put $\kappa_{n-1} = 2^{(n-2)(n-1)/2-\delta_{n,2}} \pi^{-n(n-1)/4} \prod_{i=1}^{n-1} \Gamma(i/2)$.*

Moreover, we obtain the following explicit formulae for the power series $H_p^{(n-1)}(\mathfrak{d}, \omega_p; X, t)$:

**Theorem 6.2.** *Let $\mathfrak{d} \in \mathbb{Z}$ be a fundamental discriminant and $\xi = (\frac{\mathfrak{d}}{p})$, where $(\frac{\mathfrak{d}}{*})$ denotes the Kronecker symbol associated with $\mathfrak{d}$.*

(1) *For $\omega_p = \iota_p$, we have*

$$\begin{aligned} &H_p^{(n-1)}(\mathfrak{d}, \iota_p; X, t) \\ &= \frac{(2^{-(n-1)(n-2)/2} t^{n-2})^{\delta_{2,p}}}{\prod_{i=1}^{n/2-1}(1-p^{-2i})} (p^{-1}t)^{\mathrm{ord}_p(\mathfrak{d})} (1-p^{-n}t^2) \prod_{i=1}^{n/2-1} (1-p^{-2n+2i}t^4) \\ &\quad \times \frac{(1+p^{-2}t^2)(1+\xi^2 p^{-3} t^2) - 2\xi p^{-5/2}(X+X^{-1})t^2}{(1-p^{-2}X^2t^2)(1-p^{-2}X^{-2}t^2)(1-p^{-2}t^2)^2} \\ &\quad \times \frac{1}{\prod_{i=1}^{n/2-1}(1-p^{-2i-1}X^2t^2)(1-p^{-2i-1}X^{-2}t^2)(1-p^{-2i-1}t^2)^2}. \end{aligned}$$

(2) *For $\omega_p = \varepsilon_p$, we have*

$$\begin{aligned} &H_p^{(n-1)}(\mathfrak{d}, \varepsilon_p; X, t) = ((-1)^{n(n-2)/8} 2^{-(n-1)(n-2)/2} t^{n-2})^{\delta_{2,p}} \\ &\quad \times \frac{((-1)^{n/2}, (-1)^{n/2}\mathfrak{d})_p}{\prod_{i=1}^{n/2-1}(1-p^{-2i})} (p^{-n/2}t)^{\mathrm{ord}_p(\mathfrak{d})} (1-p^{-n}t^2) \prod_{i=1}^{n/2-1} (1-p^{-2n+2i}t^4) \\ &\quad \times \frac{(1+p^{-n}t^2)(1+\xi^2 p^{-n-1}t^2) - 2\xi p^{-1/2-n}(X+X^{-1})t^2}{(1-p^{-n}X^2t^2)(1-p^{-n}X^{-2}t^2)(1-p^{-n}t^2)^2} \\ &\quad \times \frac{1}{\prod_{i=1}^{n/2-1}(1-p^{-2i}X^2t^2)(1-p^{-2i}X^{-2}t^2)(1-p^{-2i}t^2)^2}, \end{aligned}$$

*where* $(*,*)_p$ *denotes the Hilbert symbol over* $\mathbb{Q}_p$.

On the other hand, by using the same argument as in Theorem 6.1, we obtain the following:

**Proposition 6.3.** *Let* $f$ *and* $g$ *be a couple of Hecke eigenforms as in § 2. Then we have*

$$R(s,g) = L(2s-2k+n+1, f, \mathrm{ad}) \sum_{\mathfrak{d}} |c_g(|\mathfrak{d}|)|^2 |\mathfrak{d}|^{-s}$$

$$\times \prod_p \{(1+p^{-2s+2k-n-1})(1+\left(\frac{\mathfrak{d}}{p}\right)^2 p^{-2s+2k-n-2} - 2\left(\frac{\mathfrak{d}}{p}\right) a_f(p)\, p^{-2s+2k-n-3/2})\},$$

*where the summation is taken over all fundamental discriminant* $\mathfrak{d} \in \mathbb{Z}$ *such that* $(-1)^{n/2}\mathfrak{d} > 0$.

By combining Theorems 6.1, 6.2 and Proposition 6.3, we can prove Theorem 5.2. □

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