

Spherical functions on $U(n, n)/(U(n) \times U(n))$ and hermitian Siegel series

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§0 Introduction

Let k' be an unramified quadratic extension over a non-archimedean local field k of characteristic 0. We fix a prime element π of k , and the additive value $v_\pi(\cdot)$ and the normalized absolute value $|\cdot|$ on k^\times , where $|\pi|^{-1} = q$ is the cardinality of the residue class field of k . We consider hermitian matrices with respect to the involution $*$ on k' which is identity on k , and set

$$\mathcal{H}_m = \{A \in M_m(k') \mid A^* = A\}, \quad \mathcal{H}_m^{nd} = \mathcal{H}_m \cap GL_m(k'), \quad (0.1)$$

where, for a matrix $A = (a_{ij}) \in M_{mn}(k')$, we denote by A^* the matrix $(a_{ji}^*) \in M_{nm}(k')$.

For $T \in \mathcal{H}_n^{nd}$, we define the spaces

$$\mathfrak{X}_T = \{x \in M_{2n,n}(k') \mid x^* H_n x = T\}, \quad X_T = \mathfrak{X}_T / U(T),$$

where $H_n = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \in \mathcal{H}_{2n}$ and $U(T) = \{g \in GL_n(k') \mid g^* T g = T\}$. We consider spherical functions on X_T , which is isomorphic to $U(n, n)/(U(T) \times U(T))$ over k , where $U(n, n) = U(H_n)$ (cf. Lemma 1.1). We consider the following integral

$$\omega_T(\bar{x}; s) = \int_K |f_T(kx)|^{s+\varepsilon} dk, \quad (\bar{x} \in X_T, s \in \mathbb{C}^n). \quad (0.2)$$

Here dk is the normalized Haar measure on $K = U(n, n) \cap GL_{2n}(\mathcal{O}_{k'})$,

$$\varepsilon = (-1, \dots, -1, -\frac{1}{2}) + \left(\frac{\pi\sqrt{-1}}{\log q}, \dots, \frac{\pi\sqrt{-1}}{\log q}\right) \in \mathbb{C}^n,$$

$$|f_T(x)|^s = \prod_{i=1}^n |d_i(x_2 T^{-1} x_2^*)|^{s_i},$$

where x_2 is the lower half n by n block of $x \in \mathfrak{X}_T$ and $d_i(y)$ is the determinant of the upper left i by i block of y . The right hand side of (0.2) is absolutely convergent

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if $\operatorname{Re}(s_i) \geq 1$ ($1 \leq i \leq n-1$) and $\operatorname{Re}(s_n) \geq \frac{1}{2}$, continued to a rational function of q^{s_1}, \dots, q^{s_n} , and becomes a common eigen function with respect to the action of Hecke algebra $\mathcal{H}(G, K)$ with $G = U(n, n)$; thus we have a spherical function on X_T . It is convenient to introduce the new variable z which is related to s by

$$s_i = -z_i + z_{i+1} \quad (1 \leq i \leq n-1), \quad s_n = -z_n, \quad (0.3)$$

and we write $\omega_T(\bar{x}; z) = \omega_T(\bar{x}; s)$. We denote by W the Weyl group of G with respect to the maximal k -split torus in G , which is isomorphic to $S_n \times (C_2)^n$, S_n acts on z_i by permutation of indices. We denote by Σ^+ the set of positive roots of G with respect to the Borel group, and regard it a subset of \mathbb{Z}^n and write $\langle \alpha, z \rangle = \sum_{i=1}^n \alpha_i z_i$ for $\alpha \in \Sigma^+$ (for details, see §2.2).

Our main results in §1 and §2 are the following.

Theorem 1(i) *For any $T \in \mathcal{H}_n^{nd}$, the function*

$$\prod_{1 \leq i < j \leq n} \frac{(1 + q^{z_i - z_j})}{(1 - q^{z_i - z_j - 1})} \times \omega_T(\bar{x}; z)$$

is holomorphic for all z in \mathbb{C}^n and S_n -invariant, and the function

$$|2|^{-z_1 - z_2 - \dots - z_n} \prod_{1 \leq i < j \leq n} \frac{(1 + q^{z_i - z_j})(1 + q^{z_i + z_j})}{(1 - q^{z_i - z_j - 1})(1 - q^{z_i + z_j - 1})} \times \omega_T(\bar{x}; z)$$

is also holomorphic for all z in \mathbb{C}^n and W -invariant. In particular the latter is an element in $\mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W$.

(ii) *For any $T \in \mathcal{H}_n^{nd}$ and $\sigma \in W$, the following functional equation holds*

$$\omega_T(x; z) = \Gamma_\sigma(z) \cdot \omega_T(x; \sigma(z)), \quad (0.4)$$

where

$$\Gamma_\sigma(z) = \prod_{\substack{\alpha \in \Sigma^+ \\ \sigma(\alpha) < 0}} f_\alpha(\langle \alpha, z \rangle), \quad f_\alpha(t) = \begin{cases} \frac{1 - q^{t-1}}{q^t - q^{-1}} & \text{if } \alpha \text{ is short} \\ |2|^t & \text{if } \alpha \text{ is long} \end{cases}.$$

In §3, we give an explicit expression for $\omega_T(x_T; s)$.

As an application, we consider the hermitian Siegel series in §4. For each $T \in \mathcal{H}_n$, the hermitian Siegel series $b_\pi(T; s)$ is defined by

$$b_\pi(T; s) = \int_{\mathcal{H}_n(k')} \nu_\pi(R)^{-s} \psi(\operatorname{tr}(TR)) dR, \quad (0.5)$$

where ψ is an additive character on k of conductor \mathcal{O}_k , $\operatorname{tr}(\)$ is the trace of matrix and $\nu_\pi(R)$ is the "denominator" of R , which is certain non-negative powers of q (cf. (4.1)) As for Siegel series (for symmetric matrices), F. Sato and the author have given

a new integral expression and related it to a spherical function on the symmetric space $O(2n)/(O(n) \times O(n))$ (cf. [HS]). In the present paper we develop the similar argument for hermitian Siegel series. Since we know well about the functional equations of spherical functions $\omega_T(\bar{x}; s)$ with respect to W as above, we can bring out the functional equation of $b_\pi(T; s)$ as an application; thus we obtain an integral expression of $b_\pi(T; s)$ and its functional equation.

Theorem 2(i) *If $\text{Re}(s) > 2n$, one has*

$$b_\pi(T; s) = \zeta_n(k'; \frac{s}{2})^{-1} \cdot \int_{\mathfrak{X}_T(\mathcal{O}_{k'})} |N_{k'/k}(\det x_2)|^{\frac{s}{2}-n} |\Theta_T|(x), \tag{0.6}$$

where $\mathfrak{X}_T(\mathcal{O}_{k'}) = \mathfrak{X}_T \cap M_{2n,n}(\mathcal{O}_{k'})$, $\zeta_n(k'; \cdot)$ is the zeta function of the matrix algebra $M_n(k')$, and $|\Theta_T|(x)$ is a certain normalized measure on \mathfrak{X}_T .

(ii) *For any $T \in \mathcal{H}_n^{nd}$, one has*

$$\frac{b_\pi(T; s)}{\prod_{i=0}^{n-1} (1 - (-1)^i q^{-s+i})} = \chi_\pi(\det T)^{n-1} |\det(T/2)|^{s-n} \times \frac{b_\pi(T; 2n - s)}{\prod_{i=0}^{n-1} (1 - (-1)^i q^{-(2n-s)+i})},$$

where χ_π is the character on k^\times determined by

$$\chi_\pi(a) = (-1)^{v_\pi(a)} = |a|^{\frac{\pi\sqrt{-1}}{\log q}}, \quad a \in k^\times.$$

We note here that the above functional equation is related to an element of the Weyl group of $U(n, n)$, which was not the case for symmetric case when n is odd. The existence of functional equation of $b_\pi(T; s)$ was known in an abstract form as functional equations of Whittaker functions of a p -adic group by Karel [Kr](cf. also Kudla-Sweet [KS], Ikeda [Ik]).

§1

We follow the notations in the introduction. For $A \in \mathcal{H}_m$ and $X \in M_{mn}(k')$, we write

$$A[X] = X^*AX = X^* \cdot A \in \mathcal{H}_n,$$

then our spaces are given for each $T \in \mathcal{H}_n^{nd}$ by

$$\begin{aligned} \mathfrak{X}_T &= \{x \in M_{2n,n}(k') \mid H_n[x] = T\}, & X_T &= \mathfrak{X}_T/U(T), \\ x_T &= \begin{pmatrix} \frac{1}{2}T \\ 1_n \end{pmatrix} \in \mathfrak{X}_T. \end{aligned} \tag{1.1}$$

The group $G = U(n, n)$ acts on \mathfrak{X}_T , as well as on X_T , through left multiplication, which is transitive by Witt's theorem for hermitian matrices (cf. [Sch], Ch.7, §9). Our first observation is the following.

Lemma 1.1 *The stabilizer subgroup of $G = U(n, n)$ at $x_T U(T) \in X_T$ is given as*

$$\left\{ \tilde{T}^{-1} \begin{pmatrix} h_1^* & 0 \\ 0 & h_2^* \end{pmatrix} \tilde{T} \mid h_1, h_2 \in U(T) \right\}, \quad \tilde{T} = \begin{pmatrix} 1_n & \frac{1}{2}T \\ 1_n & -\frac{1}{2}T \end{pmatrix} \in GL_{2n}(k').$$

In particular, the space X_T is isomorphic to $G / (U(T) \times U(T))$.

We fix the Borel subgroup B of G as

$$B = \left\{ \begin{pmatrix} b & 0 \\ 0 & b^{*-1} \end{pmatrix} \begin{pmatrix} 1_n & a \\ 0 & 1_n \end{pmatrix} \mid \begin{array}{l} b \text{ is upper triangular of size } n, \\ a + a^* = 0 \end{array} \right\}, \quad (1.2)$$

and introduce the B -relative invariants on \mathfrak{X}_T

$$f_{T,i}(x) = d_i(x_2 T^{-1} x_2^*) \quad 1 \leq i \leq n, \quad (1.3)$$

associated with k -rational characters ψ_i of B by

$$f_{T,i}(bx) = \psi_i(b) f_{T,i}(x), \quad \psi_i(b) = N(d_i(b))^{-1}, \quad (1.4)$$

where x_2 is the lower half n by n block of $x \in \mathfrak{X}_T$, $d_i(y)$ is the determinant of upper left i by i block of y and $N = N_{k'/k}$. Since $f_{T,i}(xh) = f_{T,i}(x)$ for any $h \in U(T)$, we understand $f_{T,i}(x)$ as B -relative invariants on X_T , $1 \leq i \leq n$.

Remark 1.2 It is possible to realize above objects as the sets of k -rational points of algebraic sets defined over k and develop the arguments, but we take down to earth way for simplicity of notations. We only note here that X_T is isomorphic to $U(n, n) / (U(n) \times U(n))$ over the algebraic closure \bar{k} of k and $\{x \in X_T \mid f_{T,i}(x) \neq 0, 1 \leq i \leq n\}$ is a Zariski open B -orbit over \bar{k} , where $U(n) = U(1_n)$.

Hereafter, we write an element $\bar{x} = xU(T)$ in X_T by its representative x in \mathfrak{X}_T for simplicity of notations. We set $|0| = 0$ for the absolute value on k^\times for convenience.

The modulus character δ on B (which is characterized by $d_i(bb') = \delta(b')^{-1} d_i(b)$ for the left invariant measure $d_i(b)$ on B) is given by

$$\delta^{\frac{1}{2}}(b) = \prod_{i=1}^{n-1} |\psi_i(b)|^{-1} \times |\psi_n(b)|^{-\frac{1}{2}}.$$

Now we introduce the spherical function $\omega(x; s)$ on $X_T = \mathfrak{X}_T / U(T)$

$$\omega_T(x; s) = \omega_T^{(n)}(x; s) = \int_K |f_T(kx)|^{s+\varepsilon} dk, \quad (1.5)$$

where dk is the normalized Haar measure on $K = G \cap GL_{2n}(\mathcal{O}_{k'})$, $s \in \mathbb{C}^n$

$$\varepsilon = (-1, \dots, -1, -\frac{1}{2}) + \left(\frac{\pi\sqrt{-1}}{\log q}, \dots, \frac{\pi\sqrt{-1}}{\log q} \right) \in \mathbb{C}^n,$$

$$f_T(x) = \prod_{i=1}^n f_{T,i}(x), \quad |f_T(x)|^s = \prod_{i=1}^n |f_{T,i}(x)|^{s_i}.$$

The right hand side of (1.5) is absolutely convergent if $\operatorname{Re}(s_i) \geq 1$ ($1 \leq i \leq n - 1$) and $\operatorname{Re}(s_n) \geq \frac{1}{2}$, continued to a rational function of q^{s_1}, \dots, q^{s_n} , and becomes a common eigenfunction with respect to the action of the Hecke algebra $\mathcal{H}(G, K)$ (cf. [H2], §1).

Since we see

$$\omega_{T[h]}(x; s) = \omega_T(xh^{-1}; s), \quad h \in GL_n(k'), \quad x \in \mathfrak{X}_{T[h]}, \tag{1.6}$$

it suffices to consider only for diagonal T 's for the study of functional properties of $\omega_T(x; s)$ (e.g., Theorem 1 in the introduction),

We write $\omega_T(x; z) = \omega_T(x; s)$ for the new variable z introduced by (1.7). The Weyl group W of G relative to the maximal k -split torus in B acts on rational characters of B as usual (i.e., $\sigma(\psi)(b) = \psi(n_\sigma^{-1}bn_\sigma)$ by taking a representative n_σ of σ), so W acts on $z \in \mathbb{C}^n$ and on $s \in \mathbb{C}^n$ as well. We will determine the functional equations of $\omega_T(x; s)$ with respect to this Weyl group action. The group W is isomorphic to $S_n \times C_2^n$, S_n acts on z by permutation of indices and W is generated by S_n and $\tau : (z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-1}, -z_n)$.

By using a result on spherical functions on the space of hermitian forms((cf. [H1]-§2 or [H3]-§4.2)), we obtain the following.

Theorem 1.3 *For any $T \in \mathcal{H}_n^{nd}$, the function*

$$\prod_{1 \leq i < j \leq n} \frac{q^{z_i} + q^{z_j}}{q^{z_j} - q^{z_i-1}} \times \omega_T(x; z)$$

is holomorphic for any z in \mathbb{C}^n and S_n -invariant. In particular it is an element in $\mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^{S_n}$.

Remark 1.4 For the transposition $\tau_i = (i \ i + 1) \in W$, $1 \leq i \leq n - 1$, the following functional equation holds by Theorem 1.3

$$\omega_T(x; z) = \frac{1 - q^{z_i - z_{i+1} - 1}}{q^{z_i - z_{i+1}} - q^{-1}} \times \omega_T(x; \tau_i(z)), \quad 1 \leq i \leq n - 1. \tag{1.7}$$

On the other hand, one can obtain (1.8) directly in the similar way to the case of τ in § 3, then Theorem 1.3 follows from (1.8).

§2

2.1. We fix a unit $\epsilon \in \mathcal{O}_k^\times$ for which $k' = k(\sqrt{\epsilon})$ and $\epsilon \in 1 + 4\mathcal{O}_k^\times$ if k is dyadic (cf. [Om]-63.3 and 63.4).

Theorem 2.1 *For any $T \in \mathcal{H}_n^{nd}$, the spherical function satisfies the following functional equation:*

$$\omega_T(x; z) = |2|^{2z_n} \omega_T(x; \tau(z)).$$

The case $n = 1$ is easy; we calculate spherical functions explicitly for representatives of K_1 -orbits in \mathfrak{X}_T , where $K_1 = U(H_1) \cap GL_2(\mathcal{O}_{k'})$, and obtain the functional equation. For $n \geq 2$ we take a representative w_τ of $\tau \in W$ by

$$w_\tau = \left(\begin{array}{c|c} 1_{n-1} & \\ \hline & 1 \\ \hline & 1_{n-1} \\ & 0 \end{array} \right) \in G,$$

and take the parabolic subgroup $P = P_\tau$ attached to τ (cf. [Bo], 21.11)

$$\begin{aligned} P &= B \cup Bw_\tau B \\ &= \left\{ \left(\begin{array}{c|c} q & \\ \hline a & b \\ \hline c & q^{*-1}d \end{array} \right) \left(\begin{array}{c|c} 1_{n-1} & \alpha \\ \hline & 1 \\ \hline & 1_{n-1} \\ & -\alpha^* \end{array} \right) \left(\begin{array}{c|c} 1_n & B \quad \beta \\ \hline & -\beta^* \quad 0 \\ \hline & 1_n \end{array} \right) \in G \right\} \\ &\quad \left. \begin{array}{l} q \text{ is upper triangular in } GL_{n-1}(k'), \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1, 1), \alpha, \beta \in M_{n-1,1}(k'), \\ B \in M_{n-1}(k'), B + B^* = 0 \end{array} \right\}, \end{aligned} \quad (2.1)$$

where each empty place in the above expression means zero-entry. Hereafter we fix a diagonal $T \in \mathcal{H}_n^{nd}$, and write $f_i(x) = f_{T,i}(x)$ by abbreviating the suffix T . The B -relative invariants $f_i(x)$ become P -relative invariants associated with ψ_i except $i = n$. We consider the following action of $\tilde{P} = P \times GL_1$ on $\tilde{X}_T = X_T \times V$ with $V = M_{21}(k')$:

$$(p, r) \cdot (x, v) = (px, \rho(p)vr^{-1}),$$

where $\rho(p) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for the decomposition of $p \in P$ as in (2.1). For $(x, v) \in \tilde{X}_T$, set

$$g(x, v) = \det \left[\left(\begin{array}{c|c} 1_{n-1} & \\ \hline & tv \end{array} \right) \begin{pmatrix} x_2 \\ -y \end{pmatrix} \cdot T^{-1} \right],$$

where x_2 is the lower half n by n block of x (the same before) and y is the n -th row of x .

Then we obtain

Lemma 2.2 $g(x, v)$ is a relative \tilde{P} -invariant on \tilde{X}_T associated with character

$$\tilde{\psi}(p, r) = N(d_{n-1}(p))^{-1}N(r)^{-1} = \psi_{n-1}(p)N(r)^{-1}, \quad (p, r) \in \tilde{P} = P \times GL_1,$$

satisfies

$$g(x, v_0) = f_n(x), \quad v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and is expressed as

$$g(x, v) = D(x)[v],$$

with some hermitian matrix

$$D(x) = \begin{pmatrix} a & b + c\sqrt{\epsilon} \\ b - c\sqrt{\epsilon} & d \end{pmatrix}, \quad a, c, d \in k, \quad b = -\frac{1}{2}f_{n-1}(x), \quad ad = b^2 - c^2\epsilon. \quad (2.2)$$

In order to prove Theorem 2.2, we need the functional equation of the following function

$$\zeta_{K_1}(A; s) = \int_{K_1} |d_1(h \cdot A)|^{s-\frac{1}{2}} dh, \quad (A \in \mathcal{H}_2, s \in \mathbb{C}),$$

where dh is the normalized Haar measure on K_1 .

Lemma 2.3 *Let $x \in X_T$ such that $f_T(x) \neq 0$ and $D(x)$ be given by (2.4). Then one has*

$$\zeta_{K_1}(D(x), s) = |2|^{-2s} |f_{n-1}(x)|^{2s} \zeta_{K_1}(D(x), -s).$$

Now Theorem 2.2 is proved as follows. By the embedding

$$K_1 \longrightarrow K = K_n, \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \tilde{h} = \left(\begin{array}{c|c} 1_{n-1} & \\ \hline a & b \\ \hline c & 1_{n-1} \\ & d \end{array} \right),$$

we have

$$\begin{aligned} \omega_T(x; s) &= \int_{K_1} dh \int_K |f(\tilde{h}kx)|^{s+\varepsilon} dk \\ &= \int_K \chi_\pi \left(\prod_{i < n} f_i(kx) \right) \prod_{i < n} |f_i(kx)|^{s_i-1} \left(\int_{K_1} \chi_\pi(f_n(\tilde{h}kx)) |f_n(\tilde{h}kx)|^{s_n-\frac{1}{2}} dh \right) dk. \end{aligned}$$

By definition of $f_n(x)$ and $g(x, v)$ and Lemma 2.3, we see

$$f_n(\tilde{h}x) = g(x, \begin{pmatrix} d \\ -c \end{pmatrix}) = D(x) \left[\begin{pmatrix} d \\ -c \end{pmatrix} \right] = d_1(h^{*-1} \cdot D(x)), \quad (h \in K_1),$$

hence we have

$$\omega_T(x; s) = \int_K \chi_\pi \left(\prod_{i < n} f_i(kx) \right) \prod_{i < n} |f_i(kx)|^{s_i-1} \zeta_{K_1}(D(kx); s_n + \frac{\pi\sqrt{-1}}{\log q}) dk.$$

Then the functional equation of $\omega_T(x; s)$ follows from Lemma 2.4. ■

2.2. We denote by Σ the set of roots of G with respect to the k -split torus of G contained in B and by Σ^+ the set of positive roots with respect to B . We may understand

$$\Sigma^+ = \{e_i - e_j, e_i + e_j \mid 1 \leq i < j \leq n\} \cup \{2e_i \mid 1 \leq i \leq n\},$$

where $e_i \in \mathbb{Z}^n$ whose j -th component is given by the Kronecker delta δ_{ij} , and the set

$$\Sigma_0 = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{2e_n\}$$

forms the set of simple roots. We denote by Δ the subset of W consisting of the reflections associated to elements in Σ_0 . Then $\Delta = \{\tau_i \mid 1 \leq i \leq n-1\} \cup \{\tau\}$ generates W . We write $\alpha < 0$ if $\alpha \in \Sigma$ is negative. We see the pairing \langle, \rangle on $\Sigma \times \mathbb{C}^n$ given by

$$\langle \alpha, z \rangle = \sum_{i=1}^n \alpha_i z_i, \quad (\alpha \in \Sigma, z \in \mathbb{C}^n).$$

is W -invariant. Then we obtain

Theorem 2.4 For $T \in \mathcal{H}_n^{\text{nd}}$ and $\sigma \in W$, the spherical function $\omega_T(x; z)$ satisfies the following functional equation

$$\omega_T(x; z) = \Gamma_\sigma(z) \cdot \omega_T(x; \sigma(z)), \quad (2.3)$$

where

$$\Gamma_\sigma(z) = \prod_{\substack{\alpha \in \Sigma^+ \\ \sigma(\alpha) < 0}} f_\alpha(\langle \alpha, z \rangle),$$

$$f_\alpha(t) = \begin{cases} |2|^t & \text{if } \alpha = 2e_i \text{ for some } i \\ \frac{1 - q^{t-1}}{q^t - q^{-1}} & \text{otherwise,} \end{cases}$$

in particular, the Gamma factor $\Gamma_\sigma(z)$ does not depend on T nor x .

Proof. For an element of Δ , we know the Gamma factor by (1.8) and Theorem 2.2. In general, assume that $\sigma \in W$ has the shortest expression

$$\sigma = \sigma_\ell \cdots \sigma_1,$$

with $\sigma_i \in \Delta$ associated by some $\alpha_i \in \Sigma_0$. Since the Gamma factors satisfy cocycle relations and $\langle \cdot, \cdot \rangle$ is W -invariant, we have

$$\begin{aligned} \Gamma_\sigma(z) &= \Gamma_{\sigma_\ell}(\sigma_{\ell-1} \cdots \sigma_1(z)) \cdots \Gamma_{\sigma_2}(\sigma_1(z)) \cdot \Gamma_{\sigma_1}(z) \\ &= f_{\alpha_\ell}(\langle \alpha_\ell, \sigma_{\ell-1} \cdots \sigma_1(z) \rangle) \cdots f_{\alpha_2}(\langle \alpha_2, \sigma_1(z) \rangle) \cdot f_{\alpha_1}(\langle \alpha_1, z \rangle) \\ &= f_{\alpha_\ell}(\langle \sigma_1 \cdots \sigma_{\ell-1}(\alpha_\ell), z \rangle) \cdots f_{\alpha_2}(\langle \sigma_1(\alpha_2), z \rangle) f_{\alpha_1}(\langle \alpha_1, z \rangle). \end{aligned}$$

Hence $\Gamma_\sigma(z)$ has the required form, since we have

$$\{\alpha \in \Sigma^+ \mid \sigma(\alpha) < 0\} = \{\sigma_1 \cdots \sigma_{k-1}(\alpha_k) \mid 1 \leq k \leq \ell\}.$$

■

Corollary 2.5 Set $\rho \in W$ by

$$\rho(z_1, \dots, z_n) = (-z_n, -z_{n-1}, \dots, -z_1). \quad (2.4)$$

Then

$$\Gamma_\rho(z) = |2|^{2(z_1 + \cdots + z_n)} \prod_{1 \leq i < j \leq n} \frac{1 - q^{z_i + z_j - 1}}{q^{z_i + z_j} - q^{-1}}. \quad (2.5)$$

Remark 2.6 The above ρ gives the functional equation of the hermitian Siegel series (cf. §4), and it is interesting that such ρ corresponds to the unique automorphism of the extended Dynkin diagram of the root system of type (C_n) , which was pointed out by Y. Komori.

By Theorem 1.3 and Theorem 2.5, we obtain the following.

Theorem 2.7 *Set*

$$F(z) = \prod_{\alpha \in \Sigma^+} g_\alpha(z),$$

where, for $\alpha \in \Sigma$,

$$g_\alpha(z) = \begin{cases} |2|^{-\frac{(\alpha, z)}{2}} & \text{if } \alpha = \pm 2e_i \text{ for some } i \\ \frac{1 + q^{(\alpha, z)}}{1 - q^{(\alpha, z)-1}} & \text{otherwise} \end{cases}.$$

Then, for any $T \in \mathcal{H}_n^{nd}$, the function $F(z)\omega_T(x; z)$ is holomorphic for all z in \mathbb{C}^n and W -invariant. In particular it is an element in $\mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W$.

Proof. Take any $\sigma \in \Delta$ associated by $\alpha \in \Sigma_0$. Then $F(z)\omega_T(x; z)$ is σ -invariant, since $g_\alpha(\sigma z) = g_{\sigma\alpha}(z) = g_{-\alpha}(z)$ and $\Gamma_\sigma(z) = g_{-\alpha}(z)/g_\alpha(z)$. Thus, $F(z)\omega_T(x; z)$ is W -invariant, since Δ generates W . Set

$$F_1(z) = \prod_{1 \leq i < j \leq n} \frac{1 + q^{z_i - z_j}}{1 - q^{z_i - z_j - 1}}, \quad F_2(z) = |2|^{-z_1 - \dots - z_n} \prod_{1 \leq i < j \leq n} \frac{1 + q^{z_i + z_j}}{1 - q^{z_i + z_j - 1}}.$$

Then $F(z) = F_1(z)F_2(z)$ and $F_1(z)\omega_T(x; z)$ is holomorphic in $z \in \mathbb{C}^n$ and S_n -invariant by Theorem 1.3. Hence $F(z)\omega_T(x; z)$ is holomorphic in $z \in \mathbb{C}^n$, since it is W -invariant and holomorphic for certain region e.g., $\{z \in \mathbb{C}^n \mid \text{Re}(z_i) \leq 0\}$. ■

§3

3.1. In this section we give an explicit formula of $\omega_T(x; s)$ at x_T by using the general formula of Proposition 1.9 in [H2] (or Theorem 2.6 in [H4]). In order to apply it, we have to check several conditions ((A1) – (A4) in [H4]-§1), and it is obvious our (B, X_T) satisfies them except (A3), which is the same as (C) below.

Proposition 3.1 *The following condition (C) is satisfied.*

(C) : For $y \in X_T$ such that $f_T(y) = 0$, there exists a character $\psi \in \langle \psi_i \mid 1 \leq i \leq n \rangle$ whose restriction to the identity component of the stabilizer of B at y is not trivial.

Theorem 3.2 *Let $T = \text{Diag}(\pi^{\lambda_1}, \dots, \pi^{\lambda_n})$ with $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \geq v_\pi(2)$. Then*

$$\omega_T(x_T; z) = \frac{(-1)^{\sum_i \lambda_i(n-i+1)} q^{\sum_i \lambda_i(n-i+\frac{1}{2})} (1 - q^{-2})^n}{\prod_{i=1}^{2n} (1 - (-1)q^{-i})} \sum_{\sigma \in W} \gamma(\sigma(z)) \Gamma_\sigma(z) q^{\langle \lambda, \sigma(z) \rangle}, \quad (3.1)$$

where $\langle \lambda, z \rangle = \sum_{i=1}^n \lambda_i z_i$, $\Gamma_\sigma(z)$ is defined in Theorem 2.5, and

$$\gamma(z) = \prod_{1 \leq i < j \leq n} \frac{(1 - q^{2z_i - 2z_j - 2})(1 - q^{2z_i + 2z_j - 2})}{(1 - q^{2z_i - 2z_j})(1 - q^{2z_i + 2z_j})} \cdot \prod_{i=1}^n \frac{1 - q^{2z_i - 1}}{1 - q^{2z_i}}.$$

We admit Proposition 3.1 for the moment and prove Theorem 3.2.

The set $X_T^{op} = \{x \in X_T \mid f_T(x) \neq 0\}$ becomes a disjoint union of B -orbits as follows.

$$X_T^{op} = \bigsqcup_{u \in \mathcal{U}} X_{T,u}, \quad \mathcal{U} = (\mathbb{Z}/2\mathbb{Z})^{n-1},$$

$$X_{T,u} = \{x \in X_T \mid v_\pi(f_{T,i}(x)) \equiv u_1 + \cdots + u_i \pmod{2}, \quad 1 \leq i \leq n-1\}.$$

We set

$$\omega_{T,u}(x; s) = \int_K |f_T(kx)|_u^{s+\varepsilon} dk,$$

where

$$|f_T(y)|_u^{s+\varepsilon} = \begin{cases} |f_T(y)|^{s+\varepsilon} & \text{if } y \in X_{T,u}, \\ 0 & \text{otherwise.} \end{cases}$$

For a character $\chi = (\chi_1, \dots, \chi_{n-1})$ of \mathcal{U} , we set

$$L_T(x; \chi; z) = \int_K \chi(f_T(kx)) |f_T(kx)|^{s+\varepsilon} dk = \sum_{u \in \mathcal{U}} \chi(u) \omega_{T,u}(x; z),$$

where $\chi(u) = \prod_{i=1}^{n-1} \chi_i(u_1 + \cdots + u_i)$. Adjusting z according to χ , by adding $\frac{\pi\sqrt{-1}}{\log q}$ to z_i if necessary, we may write

$$L_T(x; \chi; z) = \omega_T(x; z_\chi).$$

Then, by the functional equations of $\omega_T(x; z)$ (Theorem 2.5), we have

$$L_T(x; \chi; z) = \Gamma_\sigma(z_\chi) L_T(x; \sigma(\chi); \sigma(z)), \quad \sigma \in W \quad (3.2)$$

by taking suitable character $\sigma(\chi)$ of \mathcal{U} . If χ is the trivial character $\mathbf{1}$, then (3.2) coincides with the original functional equation of $\omega_T(x; z)$. We obtain

$$(\omega_{T,u}(x_T; z))_u = (A^{-1} \cdot G(\sigma, z) \cdot \sigma A) (\omega_{T,u}(x_T; \sigma(z)))_u,$$

where

$$A = (\chi(u))_{\chi,u}, \quad \sigma A = (\sigma(\chi)(u))_{\chi,u} \in GL_{2^n}(\mathbb{Z}),$$

and $G(\sigma, z)$ is the diagonal matrix of size 2^n whose (χ, χ) -component is $\Gamma_\sigma(z_\chi)$. For T given as in Theorem 3.2, we obtain

$$\begin{aligned} \int_U |f_T(ux_T)|^{s+\varepsilon} du &= |f_T(x_T)|^{s+\varepsilon} \\ &= (-1)^{\sum_i \lambda_i(n-i+1)} q^{\sum_i \lambda_i(n-i+\frac{1}{2})} q^{\langle \lambda, z \rangle}, \end{aligned}$$

where U is the Iwahori subgroup of K compatible with B and du is the normalized Haar measure on U . Setting

$$\delta_u(x_T, z) = \begin{cases} (-1)^{\sum_i \lambda_i(n-i+1)} q^{\sum_i \lambda_i(n-i+\frac{1}{2})} q^{\langle \lambda, z \rangle} & \text{if } x_T U(T) \in X_{T,u} \\ 0 & \text{otherwise,} \end{cases}$$

we have, by Proposition 1.9 in [H2] (or its generalization Theorem 2.6 in [H4]),

$$(\omega_{T,u}(x_T; z))_u = \frac{1}{Q} \sum_{\sigma \in W} \gamma(\sigma(z)) (A^{-1} \cdot G(\sigma, z) \cdot \sigma A) (\delta_u(x_T, \sigma(z)))_u,$$

where

$$Q = \sum_{\sigma \in W} [U\sigma U : U]^{-1} = \prod_{i=1}^{2n} (1 - (-1)^i q^{-i}) / (1 - q^{-2})^n.$$

Hence we obtain

$$\begin{aligned} \omega_T(x_T; z) &= \sum_{u \in \mathcal{U}} \mathbf{1}(u) \omega_u(x_T; z) \\ &= \frac{(-1)^{\sum_i \lambda_i(n-i+1)} q^{\sum_i \lambda_i(n-i+\frac{1}{2})}}{Q} \sum_{\sigma \in W} \gamma(\sigma(z)) \Gamma_\sigma(z) q^{\langle \lambda, \sigma(z) \rangle}. \end{aligned}$$

■

3.2. In order to prove Proposition 3.1, we consider the action of $G \times U(T)$ on \mathfrak{X}_T by $(g, h) \circ x = gxh^{-1}$. Then, the stabilizer B_y of B at $yU(T) \in X_T$ coincides with the image $B_{(y)}$ of the projection to B of the stabilizer $(B \times U(T))_y$ at $y \in \mathfrak{X}_T$ to B . Hence the condition (C) is equivalent to the following:

(C') : For $y \in \mathfrak{X}_T$ such that $f_T(y) = 0$ there exists $\psi \in \langle \psi_i \mid 1 \leq i \leq n \rangle$ whose restriction to the identity component of $B_{(y)}$ is not trivial.

It is sufficient to prove the condition (C) (equivalently, (C')) over the algebraic closure \bar{k} , since, for a connected linear algebraic group \mathbb{H} , $\mathbb{H}(k)$ is dense in $\mathbb{H}(\bar{k})$. In the rest of this section, we consider algebraic sets over \bar{k} , extend the involution $*$ on k' to \bar{k} and denote it by $-$, and write $\bar{x} = (\bar{x}_{ij})$ for any matrix $x = (x_{ij})$. Since \mathfrak{X}_T is isomorphic to $\mathfrak{X}_{T[h]}$ by $x \mapsto xh$ and $B_{(x)} = B_{(xh)}$ for $h \in GL_n$, we may assume that $T = 1_n$. Then, our situation is the following:

$$\begin{aligned} \mathfrak{X} &= \mathfrak{X}_{1_n} = \{x \in M_{2n,n} \mid H_n[x] = 1_n\}, \\ (U(n, n) \times U(n)) \times \mathfrak{X} &\longrightarrow \mathfrak{X}, \quad ((g, h), x) \longmapsto (g, h) \circ x = gxh^{-1}. \end{aligned}$$

We consider the set

$$\tilde{\mathfrak{X}} = \{(x, y) \in M_{2n,n} \oplus M_{2n,n} \mid {}^t y H_n x = 1_n\}$$

together with $GL_{2n} \times GL_n$ -action defined by

$$(g, h) \star (x, y) = (gxh^{-1}, \dot{g}y{}^t h), \quad \dot{g} = H_n {}^t g^{-1} H_n, \quad (3.3)$$

and take the Borel subgroup P of GL_{2n} by

$$P = \left\{ \begin{pmatrix} p_1 & r \\ 0 & p_2 \end{pmatrix} \in GL_{2n} \mid p_1, {}^t p_2 \in B_n, r \in M_n \right\},$$

where B_n is the Borel subgroup of GL_n consisting of upper triangular matrices.

Then, the embedding $\iota : \mathfrak{X} \longmapsto \tilde{\mathfrak{X}}$, $x \longmapsto (x, \bar{x})$ is compatible with the actions, i.e., we have the commutative diagram

$$\begin{array}{ccc} (U(n, n) \times U(n)) \times \mathfrak{X} & \xrightarrow{\circ} & \mathfrak{X} \\ \downarrow id & & \downarrow \iota \\ (GL_{2n} \times GL_n) \times \tilde{\mathfrak{X}} & \xrightarrow{*} & \tilde{\mathfrak{X}} \end{array}$$

For $(x, y) \in \tilde{\mathfrak{X}}$ and $p \in P$, set

$$\tilde{f}_i(x, y) = d_i(x_2 {}^t y_2), \quad \tilde{\psi}_i(p) = \prod_{1 \leq j \leq i} p_j^{-1} p_{n+j}, \quad (1 \leq i \leq n),$$

where x_2 (resp. y_2) is the lower half n by n block of x (resp. y), and p_j is the j -th diagonal entry of p . Then for each i , we see

$$\begin{aligned} \tilde{f}_i((p, r) \star (x, y)) &= \tilde{\psi}_i(p) \tilde{f}_i(x, y), & (p, r) \in P \times GL_n, \\ \tilde{f}_i(x, \bar{x}) &= f_i(x), & (x \in \mathfrak{X}), \quad \tilde{\psi}_i|_B = \psi_i. \end{aligned}$$

We set

$$\mathcal{S} = \left\{ (x, y) \in \tilde{\mathfrak{X}} \mid \prod_{i=1}^n \tilde{f}_i(x, y) = 0, \quad (P \times GL_n) \star (x, y) \cap \mathfrak{X} \neq \emptyset \right\}.$$

For $\alpha = (x, y) \in \tilde{\mathfrak{X}}$, we denote by H_α the stabilizer of $P \times GL_n$ at α , and by P_α its image of the projection to P . In order to prove the condition (C), it is sufficient to show the following:

(\tilde{C}) : For each $\alpha \in \mathcal{S}$, there exists some $\psi \in \langle \tilde{\psi}_i \mid 1 \leq i \leq n \rangle$ whose restriction to the identity component of P_α is not trivial.

We show the condition (\tilde{C}) by taking suitable representatives by $P \times GL_n$ -action.

(i) Assume $\alpha = (x, y) \in \mathcal{S}$ satisfies $\det(x_2) \neq 0$. Then, in the $P \times GL_n$ -orbit containing α , there is $\beta = \left(\begin{pmatrix} 0 \\ 1_n \end{pmatrix}, \begin{pmatrix} 1_n \\ h \end{pmatrix} \right)$ with some hermitian matrix h , further we may assume

$$h = 1_r \perp \langle 0 \rangle \perp h_1 \quad \text{or} \quad h = 1_r \perp h_2,$$

where $0 \leq r \leq n-1$, and for h_2 , there is some i , ($1 < i \leq n-r$) such that each entry in the first row and column or in the i -th row and column is 0 except at $(1, i)$ or $(i, 1)$ which are 1.

Then H_β contains the following elements, according to the above type of h ,

$$\left(\left(\frac{\delta_{r+1}(a)}{1_n} \right), 1_n \right) \quad \text{or} \quad \left(\left(\frac{\delta_{r+1}(a)}{\delta_{r+i}(a)} \right), \delta_{r+i}(a) \right),$$

where $\delta_j(a)$ is the diagonal matrix in GL_n whose diagonal entries are 1 except the j -th which is $a \in GL_1$. Hence we see $\tilde{\psi}_{r+1} \neq 1$ on the identity component of P_β .

(ii) The case $\alpha = (x, y) \in \mathcal{S}$ with $\det(y_2) \neq 0$ is reduced to the case $\det(x_2) \neq 0$, since $\beta = (y, x) \in \mathcal{S}$ and $H_\beta = \{(\dot{p}, {}^t r^{-1}) \mid (p, r) \in H_\alpha\}$ and $\tilde{\psi}_i(\dot{p}) = \tilde{\psi}_i(p)^{-1}$.

(iii) Assume $\alpha = (x', y') \in \mathcal{S}$ satisfies $\det x'_2 = \det y'_2 = 0$. Then, in the $P \times GL_n$ -orbit containing α , there is some $\beta = (x, y)$ of the following type: for some integers r_i, e_j satisfying

$$\begin{aligned} 1 \leq r_1 < r_2 < \dots < r_\ell \leq n \quad (1 \leq \ell < n), \\ 1 \leq e_1 < e_2 < \dots < e_k \leq n \quad (k = n - \ell), \end{aligned}$$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ with $x_i, y_i \in M_n$ is given by

- x_1 : 1 at $(r_i, k + i)$ -entry for $1 \leq i \leq \ell$ and 0 at any other entry;
- x_2 : 1 at (e_i, i) -entry for $1 \leq i \leq k$ and 0 at any other entry;
- y_1 : the e_i -th row is the same as in x_2 for $1 \leq i \leq k$, and the j -th column is 0 if $j > k$;
- y_2 : the r_i -th row is the same as in x_1 for $1 \leq i \leq \ell$, and for each i , any (i, j) -entry is 0 for $j > k$ if some (i, j') -entry is non-zero entry with $j' \leq k$.

Let $D(a)$ be the diagonal matrix in GL_n whose i -th diagonal entry is $a \in GL_1$ (resp. 1) if every (i, j) -entry of y_2 is 0 for $j \leq k$ (resp. otherwise), where the r_i -th diagonal entry of $D(a)$ is a by this choice. Then H_β contains

$$\left(\left(\frac{D(a)}{\quad} \middle| \frac{\quad}{1_n} \right), \left(\frac{1_k}{\quad} \middle| \frac{\quad}{a1_\ell} \right) \right),$$

and $\tilde{\psi}_{r_i} \neq 1$ on the identity component of P_β , $1 \leq i \leq \ell$. ■

§4

We recall the hermitian Siegel series, and give its integral representation and functional equation. Let ψ be an additive character of k of conductor \mathcal{O}_k . For $T \in \mathcal{H}_n(k')$, the hermitian Siegel series $b_\pi(T; s)$ is defined by

$$b_\pi(T; s) = \int_{\mathcal{H}_n(k')} \nu_\pi(R)^{-s} \psi(\text{tr}(TR)) dR, \tag{4.1}$$

where $\text{tr}(\)$ is the trace of matrix and $\nu_\pi(R)$ is defined as follows: if the elementary divisors of R with negative π -powers are $\pi^{-e_1}, \dots, \pi^{-e_r}$, then $\nu_\pi(R) = q^{e_1 + \dots + e_r}$, and $\nu_\pi(R) = 1$ otherwise (cf. [Sh]-§13).

In the following we assume that T is nondegenerate, since the properties of $b_\pi(T; s)$ can be reduced to the nondegenerate case. We recall the set \mathfrak{X}_T for $T \in \mathcal{H}_n^{nd}(k')$

$$\mathfrak{X}_T = \mathfrak{X}_T(k') = \{ x \in M_{2n,n}(k') \mid H_n[x] = T \},$$

which is the fibre space $g^{-1}(T)$ for the polynomial map $g : M_{2n,n}(k') \longrightarrow \mathcal{H}_n(k')$, $g(x) = H_n[x]$ defined over k . We may take the measure $|\Theta_T|$ on \mathfrak{X}_T induced by a k -rational differential form ω on $M_{2n,n}(k')$ satisfying $\omega \wedge g^*(dT) = dx$ where dT is the canonical

gauge form on $\mathcal{H}_n(k')$, dx is the canonical gauge form on $M_{2n,n}(k')$. Then the following identity holds (cf. [Ym], [HS]-§2):

$$\begin{aligned} & \int_{\mathfrak{X}_T(k')} \phi(x) |\Theta_T|(x) \\ &= \lim_{\epsilon \rightarrow \infty} \int_{\mathcal{H}_n(\pi^{-\epsilon})} \psi(-\text{tr}(Ty)) \int_{M_{2n,n}(k')} \phi(x) \psi(\text{tr}(H_n[x]y)) dx dy, \end{aligned}$$

where $\phi \in \mathcal{S}(M_{2n,n}(k'))$, a locally constant compactly supported function on $M_{2n,n}(k')$ and $\mathcal{H}_n(\pi^{-\epsilon}) = \mathcal{H}_n(k') \cap M_n(\pi^{-\epsilon} \mathcal{O}_{k'})$.

The following lemma can be proved in the similar line to the case of symmetric matrices (cf. [HS]-§2).

Lemma 4.1 *If $\text{Re}(s) > n$, one has*

$$\begin{aligned} & \int_{\mathfrak{X}_T(\mathcal{O}_{k'})} |N_{k'/k}(\det x_2)|^{s-n} |\Theta_T|(x) \\ &= \lim_{\epsilon \rightarrow \infty} \int_{\mathcal{H}_n(\pi^{-\epsilon} \mathcal{O}_{k'})} \psi(-\text{tr}(Ty)) dy \int_{M_{2n,n}(\mathcal{O}_{k'})} |N_{k'/k}(\det x_2)|^{s-n} \psi(\text{tr}(H_n[x]y)) dx. \end{aligned} \quad (4.2)$$

Let us recall the zeta function of the matrix algebra $M_n(k')$ and its explicit formula:

$$\begin{aligned} \zeta(k'; s) &= \int_{M_n(\mathcal{O}_{k'})} |\det x|_{k'}^{s-n} dx = \int_{M_n(\mathcal{O}_{k'})} |N_{k'/k}(\det x)|^{s-n} dx \\ &= \prod_{i=1}^n \frac{1 - q^{-2i}}{1 - q^{-2(s-i+1)}}. \end{aligned}$$

Then we obtain the following integral expression of hermitian Siegel series, which can be proved in a similar line to the case of Siegel series.

Theorem 4.2 *If $\text{Re}(s) > 2n$, we have*

$$b_\pi(T; s) = \zeta_n(k'; \frac{s}{2})^{-1} \times \int_{\mathfrak{X}_T(\mathcal{O}_{k'})} |N_{k'/k}(\det x_2)|^{\frac{s}{2}-n} |\Theta_T|(x).$$

We introduce the spherical function on X_T with respect to the Siegel parabolic subgroup $P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \mid a, b, d \in M_n(k') \right\}$ by

$$\tilde{\omega}_T(x; s) = \int_K |N_{k'/k}(\det(kx)_2)|^{s-n} dk.$$

Then we have

$$\tilde{\omega}_T(x; s) = |\det T|^{s-n} \omega_T(x; 1 - \frac{\pi\sqrt{-1}}{\log q}, \dots, 1 - \frac{\pi\sqrt{-1}}{\log q}, s - n + \frac{1}{2} - \frac{\pi\sqrt{-1}}{\log q}), \quad (4.3)$$

which is holomorphic for $s \in \mathbb{C}$ by Theorem 1.3. Next proposition shows the relation between hermitian Siegel series and spherical functions.

Proposition 4.3 Denote the K -orbit decomposition of $\mathfrak{X}_T(\mathcal{O}_{k'})$ as

$$\mathfrak{X}_T(\mathcal{O}_{k'}) = \sqcup_{i=1}^r Kx_i.$$

Then one has

$$b_\pi(T; s) = \zeta_n(k'; \frac{s}{2})^{-1} \cdot \sum_{i=1}^r c_i \tilde{\omega}_T(x_i; \frac{s}{2}), \quad c_i = \int_{Kx_i} |\Theta_T|(y).$$

By Proposition 4.3 and Corollary 2.6, we obtain the following functional equation of hermitian Siegel series.

Theorem 4.4

$$\frac{b_\pi(T; s)}{\prod_{i=0}^{n-1} (1 - (-1)^i q^{-s+i})} = \chi_\pi(\det T)^{n-1} |\det(T/2)|^{s-n} \times \frac{b_\pi(T; 2n-s)}{\prod_{i=0}^{n-1} (1 - (-1)^i q^{-(2n-s)+i})}.$$

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