On the principal series representation of $SU(2, 2)$

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1 Introduction

Let $G$ denote the the special unitary group $SU(2, 2)$. In the paper, we will deal with the principal series representations of $G$ which are parabolically induced by the minimal parabolic subgroup $P_{\text{min}}$ with Langlands decomposition $P_{\text{min}} = MAN$,

$$\pi_{\sigma, \nu} = \text{Ind}^{G}_{P_{\text{min}}}(\sigma \otimes e^{\nu+\rho} \otimes 1_{N}),$$

where $\rho$ is the half sum associated to the root system of the pair $(G, A)$, $\nu$ is a complex valued real linear form on $a = \text{Lie}(A)$, $\sigma$ is a unitary character of $M$.

Let $\eta$ be a continuous unitary character of $N$. We then have the Jacquet functional $J_{\sigma, \nu}$ on the space of differentiable functions of $L^{2}_{\mathbb{R}}(K)$, the representation space of $\pi_{\sigma, \nu}$, such that $J_{\sigma, \nu}(\pi_{\sigma, \nu}(n)f) = \eta(n)J_{\sigma, \nu}(f)$ for any $n \in N$. The functional defines an intertwiner $J$ from $\pi_{\sigma, \nu}|_{K}$ to $\mathcal{A}_{\eta}(N \backslash G)$ by sending any $v \in \pi_{\sigma, \nu}|_{K}$ to the function $J_{\nu}(g) := J_{\sigma, \nu}(\pi_{\sigma, \nu}(g)v), \ (g \in G).$ Here the subspace of all $K$-finite vectors of $\pi_{\sigma, \nu}$ is denoted by $\pi_{\sigma, \nu}|_{K}$ and $\mathcal{A}_{\eta}(N \backslash G)$ is the subspace of $C^\infty(G)$ consisting of all moderate growth functions $f(g)$ such that $f(n g) = \eta(n) f(g)$ for $n \in N$ and $g \in G$. In fact, $J$ is an intertwiner of $K$ and $g$-equivariant, and hence the study of the image of $J$ (the Whittaker model) leads us to the problem of the investigations of the $(g, K)$-module structure and the functions $J_{\nu}(g)$ for certain $K$-types of $\pi_{\sigma, \nu}$.

The main goal of this paper is to describe the above mentioned objects in terms of parameters of the principal series representation $\pi_{\sigma, \nu}$ explicitly. Note that our results are quite similar to that of Ishii [4] and Oda [5], for both $Sp(2, \mathbb{R})$ and $SU(2, 2)$ have the same restricted root system.

We also consider a matrix representations of the Knapp-Stein intertwiner operator which have been motivated by a result of Goodman-Wallach [2].

2 Preliminaries

Let $K$ be the compact group $S(U(2) \times U(2))$. Then $K$ is the maximal compact subgroup of $G$ fixed by the Cartan involution $\theta$ for $G$ given by

$$\theta(g) = {}^tg^{-1}, \ g \in G.$$ 

We fix the following basis for the 7 dimensional Lie algebra $\mathfrak{k}_{\mathbb{C}}$, the complexification of $\mathfrak{k} = \text{Lie}(K)$:

$$h^{1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad h^{2} = \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}, \quad I_{2, 2} = \begin{pmatrix} 12 & 0 \\ 0 & -12 \end{pmatrix},$$

$$e^{1}_{\pm} = \begin{pmatrix} e_{\pm} & 0 \\ 0 & 0 \end{pmatrix}, \quad e^{2}_{\pm} = \begin{pmatrix} 0 & 0 \\ 0 & e_{\pm} \end{pmatrix},$$

where $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ e_{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $e_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.
For every $K$-module $V$, it is clear that $I_{2,2} \in \mathfrak{k}_C$ commutes with the action of $K$ on $V$. If $V$ is irreducible, then by Schur's lemma, the operator is a scalar of the identity map.

**Lemma 2.1** Let $m_1, m_2$ be positive integers and $l$ be an integer. If $m_1 + m_2 + l$ is an even integer, then there is an irreducible $K$-module $(\pi_{m_1, m_2}; l, V_{m_1, m_2})$ with a basis $\{f_{pq} \mid 0 \leq p \leq m_1, 0 \leq q \leq m_2\}$ of $V_{m_1, m_2}$ such that $I_{2,2}f_{pq} = lf_{pq}$ and

\[
\begin{align*}
    h^1(f_{pq}) &= (2p - m_1)f_{pq}, & e^1_{+}(f_{pq}) &= (m_1 - p)f_{p+1,q}, & e^1_{-}(f_{pq}) &= pf_{p-1,q}, \\
    h^2(f_{pq}) &= (2q - m_2)f_{pq}, & e^2_{+}(f_{pq}) &= (m_2 - q)f_{pq+1}, & e^2_{-}(f_{pq}) &= qf_{pq-1}.
\end{align*}
\]

It follows from the fact that $SU(2) \times SU(2) \times \mathbb{C}^{(1)}$ is a twofold covering of $K$ with the projection given by

\[
    pr(g_1, g_2; u) = \text{diag}(ug_1, u^{-1}g_2), \quad g_1, g_2 \in SU(2), \quad u \in \mathbb{C}^{(1)}.
\]

### 3 $K$-finite vectors in the principal series

In this section, for each simple $K$-module $\tau \in \hat{K}$, we associate a matrix function $S^{(\tau)}(k), \ k \in K$, whose entries give a basis for the $\tau$-isotypic component of $\pi_{\sigma,\nu}$. The main feature of this basis is that the both $\mathfrak{g}$ and $K$-actions on $\pi_{\sigma,\nu} |_K$ have simple expressions in terms of parameters of given representation. For more details about this theme, we refer to [5] which is our main reference.

**Proposition 3.1** Let $H(\tau)$ be the $\tau$-isotypic component of $L^2(K)$, and put $\dim(\tau) = n$. There exists a unique square matrix function $S^{(\tau)}(k), \ k \in K$, of size $n$ with entries in $H(\tau)$,

\[
    S^{(\tau)}(k) = \begin{bmatrix} f_{11}(k) & \cdots & f_{n1}(k) \\ \vdots & \ddots & \vdots \\ f_{1n}(k) & \cdots & f_{nn}(k) \end{bmatrix} = \{f_{ij}(k)\}_{1 \leq i, j \leq n},
\]

satisfying the following two conditions:

1. $S^{(\tau)}(1_K) = \text{diag}(1, \ldots, 1) \in M_n(\mathbb{C}),$

2. For each $\alpha (1 \leq \alpha \leq n)$, the set $\{f_{\alpha 1}(k), \ldots, f_{\alpha n}(k)\}$ is a basis for $\tau$ as in Lemma 2.1. Moreover, we have

\[
    H(\tau) = \bigoplus_{\alpha} W_{\alpha},
\]

where $W_{\alpha}$ denotes the space spanned by $f_{\alpha 1}(k), \ldots, f_{\alpha n}(k)$.

**Proof.** The existence of the matrix function is similar to that of [5]. We consider the uniqueness. Assume that there exist two matrices $F^{(\tau)}(k) = \{f_{ij}(k)\}$ and $G^{(\tau)}(k) = \{g_{ij}(k)\}$ as required. Denote by $F_{\alpha}$ the isomorphism between $\tau$ and the space spanned by $\{f_{\alpha j}(k), \ldots, f_{\alpha n}(k)\}$. Similarly, we define $G_{\alpha}$ for the $\alpha$-th column of $G^{(\tau)}(k)$. As a result, we obtain two ordered bases $\{F_{\alpha}\}_\alpha$ and $\{G_{\alpha}\}_\alpha$ for the $n$-dimensional vector space $\text{Hom}_K(\tau, H(\tau))$. Then we have the $n$ by $n$ matrix $A = \{a_{\alpha\beta}\}$, the change of coordinate matrix, such that

\[
    F_{\alpha} = \sum_{\beta} a_{\alpha\beta} G_{\beta}.
\]

For a basis $\{f_\gamma\}$ of $\tau$, one obtains

\[
    f_{\alpha \gamma}(k) = F_{\alpha}(f_\gamma) = \sum_{\beta} a_{\alpha\beta} G_{\beta}(f_\gamma) = \sum_{\beta} a_{\alpha\beta} f_{\beta \gamma}(k).
\]
Evaluation at the point $1_K$ shows that
\[ a_{\alpha\gamma} = \delta_{\alpha\gamma}. \]
If $v \neq 0 \in W_\alpha \cap W_\beta$, then $Kv = W_\alpha = W_\beta$. Schur's lemma and second condition imply that $\alpha = \beta$. Assume there is a matrix $S(\tau)(k)$ as required, we then have the direct sum decomposition of $H(\tau)$. \[ \Box \]

For each $\tau_m = \tau[m_1,m_2;l] \in \hat{K}$, define a finite set $I(\tau_m)$ to be the collection of indices $\alpha$ such that $W_\alpha$ occurs in $\pi_{\sigma,\nu} |_K$ as a $K$-module. Thus, the cardinality of $I(\tau_m)$ is the $K$-multiplicity of $\tau_m$ in $\pi_{\sigma,\nu}$. Let $s$ be the integer parameter corresponding to $\sigma \in \hat{M}$. By setting $n = (m_1 + m_2 + s)/2$, one can see that $p + q = n$ if $\alpha \in I(\tau_m)$ with $\alpha = (m_2 + 1)p + q + 1, (q \leq m_2)$. We identify the index $\alpha$ with the pair $(p, q)$ defined by $\alpha$.

We define a matrix function $S^{(\tau_m)}(k)_{\alpha} \in \hat{K}$ attached to the $\tau$-isotypic component of $\pi_{\sigma,\nu}$ by eliminating all the $\alpha$-th columns of $S^{(\tau_m)}(k)$ when $p + q \neq n$ and change the $\alpha$-th columns by 0 if $\alpha \notin I(\tau_m)$ and $p + q = n$.

4 The $(\mathfrak{g}, K)$-module structure on $\pi_{\sigma,\nu}$

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g} = \text{Lie}(G)$ corresponding to $\theta$. In this section, we explicitly describe $\mathfrak{p}_C$-action on the space

\[ \pi_{\sigma,\nu} \mid_K \cong \bigoplus_{\tau_m \in \hat{K}} \bigoplus_{\alpha \in I(\tau_m)} W_\alpha. \]

Since the adjoint representation of $K$ on $\mathfrak{p}_C$ splits into two irreducible components, the antiholomorphic part $\mathfrak{p}_-$ and the holomorphic part $\mathfrak{p}_+$, it is enough to investigate the $\mathfrak{p}_+$-action for our purpose. Let $E_{ij}$ be the matrix unit of $M_4(\mathbb{R})$ with 1 in the $(i, j)$-entry and zero elsewhere. Then the set $\{E_{ij} \mid i = 1, 2, j = 3, 4\}$ forms a basis for $\mathfrak{p}_+$. For a fixed pair $(\epsilon_1, \epsilon_2)$, $\epsilon_j \in \{\pm 1\}$ with $j = 1, 2$, we define $c_j^T$ by

\[ c_j^T = \frac{t}{m_j + 1} \quad (0 \leq t \leq m_j + \epsilon_j). \]

Let $(\tau_m, V_m)$ be an irreducible representation of $K$ with parametrization $m = [m_1, m_2; l]$. By the well known Clebsch-Gordan theorem, the irreducible components in the $K$-module $\mathfrak{p}_+ \otimes \tau_m$ are precisely the $K$-representations

\[ T = \{ \tau[m_1 + \epsilon_1, m_2 + \epsilon_2; l + 2] \mid \epsilon_1, \epsilon_2 \in \{\pm 1\} \}, \]

and we will denote these by $\tau_{[\pm, \pm; \pm]}$ or $\tau_{[\epsilon_1, \epsilon_2; \pm]}$.

For each $K$-isomorphism between $\tau_m$ and $W_\alpha$ in Proposition 3.1, we have the following surjective homomorphism $\mathfrak{p}_+ \otimes \tau_m \to \mathfrak{p}_+ W_\alpha$ of $K$-modules. Therefore, we obtain an injection

\[ \mathfrak{p}_+ H_{\sigma,\nu}(\tau_m) \hookrightarrow \bigoplus_{\tau_{m'} \in T} H_{\sigma,\nu}(\tau_{m'}) \]

which implies the following theorem. Here $H_{\sigma,\nu}(\tau_m)$ stands for the $\tau_m$-isotypic component of $\pi_{\sigma,\nu}$.

**Theorem 4.1** Let $\tau_{[\epsilon_1, \epsilon_2; \pm]}$ be a simple $K$-submodule of the $K$-module $\mathfrak{p}_+ \otimes \tau_m$ for a given simple $K$-module $\tau_m$ and the $K$-module $(\text{Ad}, \mathfrak{p}_+)$. Then we have that

\[ C_{[\epsilon_1, \epsilon_2; \pm]} S^{(\tau_m)}(k) = S^{(\tau_{[\epsilon_1, \epsilon_2; \pm]})}(k) \Gamma_{[\epsilon_1, \epsilon_2; \pm]}, \]

where the product of matrices of the left hand side is the differential operation. Here, $r = (s + l)/2$ and
1. $\Gamma_{[-,-,+]} = \{a_{ij}\}_{0 \leq i \leq n-1, 0 \leq j \leq n}$ is a matrix whose all non zero entries are given by

\[ a_{t-1,t} = \frac{1}{2}(\nu + 1 + m_1 + r - 2t) \quad \text{if} \ (t, n-t) \in I(\tau_m), \ (t-1, n-t) \in I(\tau_{m'}) \]
\[ a_{t,t} = -\frac{1}{2}(\nu_1 - 1 - m_2 + r - 2t) \quad \text{if} \ (t, n-t) \in I(\tau_m), \ (t, n-t-1) \in I(\tau_{m'}) \]

and $C_{[-,-,+]} = \{C_{ij}\}$ is a matrix of size $(m_1 m_2) \times (m_1 + 1)(m_2 + 1)$ with entries given by

\[ C_{m_2p+q+1,(m_2+1)p+q+1} = -E_{24} \]
\[ C_{m_2p+q+1,(m_2+1)p+q+2} = -E_{13} \]
\[ C_{m_2p+q+1,(m_2+1)(p+1)+q+1} = E_{23} \]
\[ C_{m_2p+q+1,(m_2+1)(p+1)+q+2} = E_{24} \]

for each $0 \leq p \leq m_1 - 1$ and $0 \leq q \leq m_2 - 1$, but all other entries are 0.

2. $\Gamma_{[+,+,+]} = \{a_{ij}\}_{0 \leq i \leq n+1, 0 \leq j \leq n}$ is a matrix whose all non zero entries are given by

\[ a_{t,t} = \frac{1}{2}(\nu + 1 + m_1 + r - 2t)(1-c_t^{1})c_{\nu-t}^{2} \quad \text{if} \ (t, n-t) \in I(\tau_m), \ (t, n-t+1) \in I(\tau_{m'}) \]
\[ a_{t+1,t} = \frac{1}{2}(\nu_1 + 3 + 2m_1 - m_2 + r - 2t)c_{t+1}^{1}(c_{\nu-t}^{2}-1) \quad \text{if} \ (t, n-t) \in I(\tau_m), \ (t+1, n-t) \in I(\tau_{m'}) \]

and $C_{[+,+,+]} = \{C_{ij}\}$ is a matrix of size $(m_1+2)(m_2+2) \times (m_1+1)(m_2+1)$ with entries given by

\[ C_{(m_2+2)p+q+1,(m_2+1)p+q+1} = -(1-c_p^{1})(1-c_q^{2})E_{23} \]
\[ C_{(m_2+2)p+q+1,(m_2+1)p+q} = -c_p^{1}c_q^{2}E_{24} \]
\[ C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q+1} = -(1-c_q^{2})E_{13} \]
\[ C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q} = c_q^{2}E_{14} \]

for each $0 \leq p \leq m_1+1$ and $0 \leq q \leq m_2+1$, but all other entries are 0.

3. $\Gamma_{[+,+,-]} = \{a_{ij}\}_{0 \leq i \leq n, 0 \leq j \leq n}$ is a square matrix whose all non zero entries are given by

\[ a_{t-1,t} = \frac{1}{2}(\nu + 1 + m_1 + r - 2t)c_{\nu-t+1}^{2} \quad \text{if} \ (t, n-t) \in I(\tau_m), \ (t-1, n-t+1) \in I(\tau_{m'}) \]
\[ a_{t,t} = \frac{1}{2}(\nu_1 + 1 + m_2 + r - 2t)c_{t+1}^{1} \quad \text{if} \ (t, n-t) \in I(\tau_m), \ (t+1, n-t) \in I(\tau_{m'}) \]

and $C_{[+,+,-]} = \{C_{ij}\}$ is a matrix of size $m_1(m_2+2)(m_2+2) \times (m_1+1)(m_2+1)$ with entries given by

\[ C_{(m_2+2)p+q+1,(m_2+1)p+q+1} = -(1-c_p^{1})(1-c_q^{2})E_{23} \]
\[ C_{(m_2+2)p+q+1,(m_2+1)p+q} = -c_p^{1}c_q^{2}E_{24} \]
\[ C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q+1} = -(1-c_q^{2})E_{13} \]
\[ C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q} = c_q^{2}E_{14} \]

for $0 \leq p \leq m_1+1$ and $0 \leq q \leq m_2-1$, but all other entries are 0.

4. $\Gamma_{[-,+-,+]} = \{a_{ij}\}_{0 \leq i \leq n, 0 \leq j \leq n}$ is a square matrix whose all non zero entries are given by

\[ a_{t,t} = \frac{1}{2}(\nu + 1 + m_1 + r - 2t)(1-c_t^{1}) \quad \text{if} \ (t, n-t) \in I(\tau_m), \ (t, n-t) \in I(\tau_{m'}) \]
\[ a_{t+1,t} = \frac{1}{2}(\nu_1 + 1 + 2m_1 - m_2 + r - 2t)c_{t+1}^{1} \quad \text{if} \ (t, n-t) \in I(\tau_m), \ (t+1, n-t-1) \in I(\tau_{m'}) \]
and $C_{[+,--;+]} = \{C_{ij}\}$ is a matrix of size $(m_1 + 2)m_2 \times (m_1 + 1)(m_2 + 1)$ with entries given by

$$
C_{m_2p+q+1,(m_2+1)p+q+1} = (1 - c_p^{1})E_{24},
$$
$$
C_{m_2p+q+1,(m_2+1)p+q+2} = (1 - c_p^{1})E_{23},
$$
$$
C_{m_2p+q+1,(m_2+1)(p-1)+q+1} = c_p^{1}E_{14},
$$
$$
C_{m_2p+q+1,(m_2+1)(p-1)+q+2} = c_p^{1}E_{13},
$$

for $0 \leq p \leq m_1 + 1$ and $0 \leq q \leq m_2 - 1$, but all other entries are 0.

4.0.1 The Knapp-Stein operator

In this subsection, we consider a matrix representation of the Knapp-Stein operator with respect to the basis for $\pi_{\sigma,\nu}|_K$. This is motivated by Theorem 6.7 in the paper of Goodman-Wallach [2].

Let us recall the Knapp-Stein intertwining operator $A_{\sigma,\nu}^{s}$ from the space of all $C^\infty$-vectors of $\pi_{\sigma,\nu}$ to that of $\pi_{s(\sigma),s(\nu)}$ defined by

$$(A_{\sigma,\nu}^{s}f)(k) = \int_{\overline{N}_{\theta}}a(n_{\partial}s^{*}k)^{\nu+\rho}f(k(n_{\epsilon}s^{*}k))dn_{s}, \quad (f \in \pi_{\sigma,\nu}^\infty).$$

Here $s^{*} \in K$ such that $s := Ad(s^{*}) \in W(A), \overline{N}_{s} = N \cap s^{*}Ns^{*-1}$ and $s(\sigma)$ is a character of $M$ given by $s(\sigma)(m) = \sigma(s^{*}ms^{*-1}), m \in M$. Since it is a linear map from $\pi_{\sigma,\nu}$ to $\pi_{s(\sigma),s(\nu)}$ satisfying

$$A_{\sigma,\nu}^{s}\pi_{\sigma,\nu}(x)f = \pi_{s(\sigma),s(\nu)}(x)A_{\sigma,\nu}^{s}f,$$

$x \in G$ (or $U(\mathfrak{g}))$, we have a linear map

$$A^{s}(\tau) : \text{Hom}_{K}(\tau, \pi_{\sigma,\nu}|_K) \rightarrow \text{Hom}_{K}(\tau, \pi_{s(\sigma),s(\nu)}|_K).$$

for any $\tau \in \hat{K}$.

Let $[\alpha_{i}]$ be the $K$-isomorphism from $\tau$ to $W_{\alpha_{i}}$ for $\alpha_{i} \in I(\tau)$. We equip the space $\text{Hom}_{K}(\tau, \pi_{\sigma,\nu}|_K)$ with the basis consisting of the $K$-homomorphisms $[\alpha_{i}]$. Similarly, we choose a basis for the space $\text{Hom}_{K}(\tau, \pi_{s(\sigma),s(\nu)}|_K)$. Then we want to compute all entries $a_{ij}$ of the matrix $A^{s}(\tau) = (a_{ij})$ such that

$$A^{s}(\tau)[\alpha_{i}] = \sum_{\alpha_{j} \in I} a_{ij} \cdot [\alpha_{j}^s]$$

where $I = \{\alpha_{i}^{s} \mid W_{\alpha_{i}} \rightarrow \pi_{s(\sigma),s(\nu)}|_K\}$. For each basis vector $f_{pq}$ of $\tau$ as in Lemma 2.1, we have that

$$(A^{s}(\tau)[\alpha_{i}]) (f_{pq}) = \sum_{\alpha_{j} \in I} a_{ij} \cdot [\alpha_{j}^s] (f_{pq}) = \sum_{\alpha_{j} \in I} a_{ij} \cdot f_{\alpha_{j}^s, pq}^{(r)}(k).$$

On the other hand, by definition of the map $A^{s}(\tau)$, one has

$$(A^{s}(\tau)[\alpha_{i}]) (f_{pq}) = (A_{\sigma,\nu}^{s}f_{\alpha_{i}, pq}^{(r)}) (k), \quad \alpha_{i} \in I(\pi_{\sigma,\nu}, \tau).$$

Thus we have the following formula for the coefficients $a_{ij}$ of the matrix $A^{s}(\tau)$ for each $\tau \in \hat{K}$.

**Lemma 4.2** Let $\alpha_{i} \in I(\pi_{\sigma,\nu}, \tau)$ and $\alpha_{j}^{s} \in I(\pi_{s(\sigma),s(\nu)}, \tau)$. Then the $(i,j)$-th coefficient of $A^{s}(\tau)$

$$a_{ij} = (A_{\sigma,\nu}^{s}f_{\alpha_{i}, \alpha_{j}^{s}}^{(m)})(1_{4}).$$

**Example 4.3**
Let $s$ be a generator of $W(A)$ whose image is the matrix $\text{diag}(1, -1)$ under the representation of $W(A)$ on $a^*$. Then we choose the corresponding $s^* \in K$ as the matrix $\text{diag}(1, -i, 1, i)$ and hence

$$
\overline{N}_s = \exp(g_{-2\lambda_2}) = \left\{ n_s(t) = \kappa^{-1} \begin{pmatrix} 1 & t & 1 \\ 1 & 1 & t \\ t & 1 & 1 \end{pmatrix} \kappa : t \in \mathbb{R} \right\}.
$$

Since $n_s \in \overline{N}_s$, one has $^t n_s I_{2,2} n_s = I_{2,2}$ and hence $n_s s^* = I_{2,2}^t n_s^{-1} I_{2,2} s^*$. Thus, we have the following.

Assume $n_s = n_s(t) \in \overline{N}_s$. Let $n' \in N$, $a(n_s s^*) \in A$ and $k(n_s s^*) \in K$ be so that $n_s s^* = n'a(n_s s^*)k(n_s s^*)$.

Then

$$
a(n_s s^*)^{\nu + \rho} = (1 + t^2)^{-\frac{d+1}{2}},
$$

$$
k(n_s s^*) = \text{diag}(1, -iu, -1, -iu^{-1})
$$

where $u = ((1 - it)/(1 + it))^\frac{1}{2}$.

For a fixed $\tau_m \in \hat{K}$, therefore

$$f_{\gamma_i, \beta_j}(k(n_s s^*)) = 0 \text{ when } \gamma_i \neq \beta_j$$

If $\tau = \tau_{[m_1, m_2, l]}$ then we have

$$A^*(\tau) = 2\pi 2^{-\nu_2} \Gamma(\nu_2) \text{ diag} \left[ \frac{(-1)^{\frac{1}{2}(m_1 + m_2)/2} - p + \frac{1}{2} + d + \frac{1}{2} m_2 + r}{\Gamma(\frac{1}{2} \nu_2 + \frac{1}{2} + d)} \Gamma(\frac{1}{2} \nu_2 + \frac{1}{2} - d) \right]_p$$

where $d = \frac{1}{2}(m_1 - 2p)$ for $(p, (m_1 + m_2)/2 - p) \in I(\pi_{\sigma, \nu}, \tau_m)$.

### 5 Whittaker functions

The main focus of this section is on the integral expressions of Whittaker functions on $G$ related to certain principal series. The results of the section 4.1 lead us to the study of Whittaker functions related to some $K$-types. For this purpose, we focus our investigation on the principal series representations which contain one dimensional $K$-types and apply the method used in [4] to evaluate such Whittaker functions. More precisely, in this setting, the character $\sigma$ of $M$ factors through a character $\chi$ of $\mu_2$. Let $(\pi_{\chi, \nu}, L_\chi^2(K))$ denote the principal representation series corresponding to such character $\sigma$.

For an integer $u$, define a function $f_u(k)$ on $K$ by $f_u(k) := \det(k_{21})^u$, $k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$.

**Lemma 5.1** Let $f_u(k)$ be as above. Then $\tau_{[0, 0, 2u]} \cong \mathbb{C} f_u(k)$ as $K$-modules. Moreover, if $\chi(-1) = (-1)^u$ then $f_u(k) \in L_\chi^2(K)$ and $[\pi_{\chi, \nu} : \tau_{[0, 0, 2u]}] = 1$.

#### 5.1 The Jacquet integral.

Let $J_{\chi, \nu}$ be the Jacquet functional on the subspace of differentiable functions of $L_\chi^2(K)$ given by

$$J_{\chi, \nu}(f) = \int_{N} \eta(n)^{-1} a(s^* n)^{\nu + \rho} f(k(s^* n)) dn$$

for a differentiable function $f$ in $L_\chi^2(K)$ and the longest element $s \in W(A)$. Here $W(A)$ is the Weyl group defined as the quotient of $M^* = N_K(a)$, the normalizer of $a$ in $K$, by $M$ and $s^*$ is an element of $M^*$ mapping to the longest element $s \in W(A)$.
Then one has \( J_{X,\nu}(\pi(n)f) = \eta(n)J_{X,\nu}(f) \) and hence
\[
J \in \text{Hom}_{(g,K)}(\pi_{X,\nu}|_{K}, \mathcal{A}_{\eta}(N\backslash G)),
\]
where \( J \) associates \( v \in \pi_{X,\nu}|_{K} \) to the function \( J_{v}(g) := J_{X,\nu}(\pi_{X,\nu}(g)v) \).

In our case, we can choose \( I_{2,2} \in K \) for \( s^{*} \in K \).

### 5.1.1 The first modification

Let \( E_{ij} \) be the usual matrix with 1 in \((i, j)\)-entry and zero elsewhere. Put
\[
E_{0} = \kappa^{-1}(E_{12} - E_{43})\kappa, \quad E_{1} = i\kappa^{-1}(E_{12} + E_{43})\kappa, \quad E_{2} = \kappa^{-1}E_{24}\kappa
\]
\[
F_{0} = \kappa^{-1}(E_{14} + E_{23})\kappa, \quad F_{1} = i\kappa^{-1}(E_{14} - E_{23})\kappa, \quad F_{2} = \kappa^{-1}E_{13^{K}}\kappa
\]
by setting \( i = \sqrt{-1} \) and
\[
\kappa = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -i & 0 & i & 0 \\ 0 & -i & 0 & i \end{pmatrix}.
\]

Then the corresponding root spaces of positive roots in \( \Phi(g, a) \) are given by
\[
g_{\lambda_{1}-\lambda_{2}} = E_{0} \cdot \mathbb{R} \oplus E_{1} \cdot \mathbb{R}, \quad g_{2\lambda_{2}} = E_{2} \cdot \mathbb{R},
\]
\[
g_{\lambda_{1}+\lambda_{2}} = F_{0} \cdot \mathbb{R} \oplus F_{1} \cdot \mathbb{R}, \quad g_{2\lambda_{1}} = F_{2} \cdot \mathbb{R}.
\]

Let \( n \) be a subalgebra defined by \( n = \sum_{\alpha \in \Phi_{+}} g_{\alpha} \). We now describe elements of a maximal nilpotent subgroup \( N \) of \( G \) given by \( N = \exp(n) \).

The Killing form \( B(X, Y) = \text{tr}(\text{ad}X \cdot \text{ad}Y) \), \((X, Y \in g)\) and Cartan involution \( \theta \) of \( g \) induce an inner product \( \langle \cdot, \cdot \rangle \) of \( g \) via
\[
\langle X, Y \rangle = -B(X, Y^{\theta}), \quad (X, Y \in g).
\]

Then one has that \( \langle g_{\alpha}, g_{\beta} \rangle = 0 \) if \( \alpha \neq \beta \), because of the involution \( \theta \). Moreover, one can see that the set \( \{E_{i}, F_{i} | i = 0, 1, 2\} \) is an \( \langle \cdot, \cdot \rangle \)-orthogonal basis for \( n \) such that a each element \( n = n(n_{0}, n_{1}, n_{2}, n_{3}) \) in the maximal unipotent group \( N = \exp(n) \) is expressed in the form:
\[
\kappa^{-1} \begin{pmatrix} 1 & n_{0} & 1 \\ 1 & 1 & n_{1} \\ -n_{0} & 1 & 1 \end{pmatrix} \kappa
\]
for \( n_{1}, n_{3} \in \mathbb{R}, \ n_{0}, n_{2} \in \mathbb{C} \).

**Lemma 5.2** We have

1. Set \( N_{1} = \begin{pmatrix} n_{1} & n_{2} \\ \overline{n}_{2} & n_{3} \end{pmatrix} \) for \( n = n(n_{0}, n_{1}, n_{2}, n_{3}) \in N \). Then
\[
f_{u}(k(I_{2,2}n)) = \left( \frac{\text{det}(1 - \sqrt{-1}N_{1})/\text{det}(1 + \sqrt{-1}N_{1})}{1} \right)^{\frac{q}{2}}.
\]
2. Let \( \eta \) be a character of \( N \) determined by a real number \( c_2 \) and \( c = c_0 + \sqrt{-1}c_1 \in \mathbb{C} \). Then

\[
\eta(\alpha a^{-1}) = \exp(2\sqrt{-1} \left( \frac{a_1}{a_2} \text{Re}(\bar{c}n_0) + c_2a_3^2n_3 \right)),
\]

where \( a_i = \exp(t_i) \), \( (i = 1, 2) \) for \( a = (t_1, t_2) \in A \).

3. For \( \nu = (\nu_1, \nu_2) \in \text{Hom}_R(a, \mathbb{C}) \), we have that \( a(I_{2,2}n)^{\nu+\rho} = \Delta_1^{-\frac{n_1^2}{2} + \frac{n_2^2}{2}} \Delta_2^{-\frac{n_2^2}{2}} \) where

\[
\Delta_1 = 1 + n_1^2 + n_2n_2 + (n_0n_2 + n_0n_2)(n_1 + n_3) + n_0n_0(1 + n_2n_2 + n_3^2),
\]

\[
\Delta_2 = 1 + n_1^2 + 2n_2n_2 + n_3^2 + (n_1n_3 - n_2n_2)^2 \quad \text{for} \quad n = (n_0, n_1, n_2, n_3) \in N.
\]

For future convenience, we choose a new coordinate for \( A \) by

\[
y = (y_1, y_2) = \left( \frac{a_1}{a_2}, a_3^2 \right).
\]

Since \( f \to J_f(g) \) is the Whittaker realization of \( \pi_{\chi, \nu} \), \( J_{\nu}(a) \) is the radial part of a Whittaker function on \( G \) belonging to \( \pi_\nu \). Thus, in the new coordinate system, we can summarize that the radial part of Whittaker function associated with the \( K \)-type \( \tau_u \) can be written in the form

\[
y^{-\nu_1+3}y_2^{-\nu_2+2} \int_N \Delta_1^{-\frac{n_1^2}{2} + \frac{n_2^2}{2}} \Delta_2^{-\frac{n_2^2}{2}} \times \exp(-2\sqrt{-1}(\text{Re}(\bar{c}n_0) + c_2y_2n_3))f_u(k(I_{2,2}n))dn,
\]

where \( dn \) is a multiplicative Haar measure on \( N \). Now we shall give a normalization of Haar measure of \( N \). Since the exponential map of \( n \) onto \( N \) is an analytic isomorphism, there exists a unique Haar measure \( dn \) on \( N \) that corresponds to Lebesgue measure on \( n \).

**Lemma 5.3** The radial part of the moderate growth Whittaker function \( W_{(\nu_1, \nu_2)}(y_1, y_2; u) \) (up to constant) associated with the \( K \)-type \( \tau_u \) can be written in the form

\[
y^{-\nu_1+3}y_2^{-\nu_2+2} \int_{\mathbb{R}^4} \Delta_1^{-\frac{n_1^2}{2} + \frac{n_2^2}{2}} \Delta_2^{-\frac{n_2^2}{2}} \times \exp(-2\sqrt{-1}(c_0y_1 + c_1t_0y_1 - n_3y_2))f_u(k(I_{2,2}n))dn,
\]

with respect to \( dz_0dt_0dn_1dz_2dt_2dn_3 \). Here \( n_i = z_i + \sqrt{-1}t_i \) \( (i = 0, 2) \).

In fact, it suffices to consider the cases \( u = 0 \) and \( 1 \) for our purposes. Let \( K_{\mu}(z) \) be the Bessel function.

**Theorem 5.4** Let \( \pi_{\chi, \nu} \) be irreducible and \( \eta \) be a nondegenerated unitary character \( N \). Then we have the following assertions on the \( A \)-radial part of the primary Whittaker function \( W_{(\nu_1, \nu_2)}(y_1, y_2; u) \).

If \( \chi \) is trivial then the Whittaker function \( W_{(\nu_1, \nu_2)}(y_1, y_2; 0) \) is identified with \( y_1^3y_2^2 \) times

\[
\int_0^\infty \int_0^\infty T_{\nu_1, \nu_2}(y_1, y_2, t_1, t_2) \frac{dt_1}{t_1} \frac{dt_2}{t_2}.
\]

If \( \chi \) is non-trivial then the Whittaker function \( \tilde{W}_{(\nu_1, \nu_2)}(y_1, y_2; 1) \) is identified with \( y_1^3y_2^2/4 \) times

\[
\int_0^\infty \int_0^\infty T_{\nu_1, \nu_2}(y_1, y_2, t_1, t_2) (\sqrt{t_1/t_2} - 1/\sqrt{t_1t_2}) \frac{dt_1}{t_1} \frac{dt_2}{t_2}
\]

where \( T_{\nu_1, \nu_2}(y_1, y_2, 1, 0) \) is the function

\[
K_{\frac{\nu_1-1}{2}}(2\sqrt{t_2/t_1})K_{\frac{\nu_2-1}{2}}(2\sqrt{t_1t_2}) \exp \left( -c_2y_2t_1 - \frac{|c_2|y_2}{t_1} - \frac{t_2}{|c_2|y_2} - (c_0^2 + c_1^2)|c_2|y_2^2/t_2 \right)
\]

Note here that, we need the following formula to reduce the number of integral symbols corresponding to the root spaces \( t_{\lambda_1 - \lambda_2} \) and \( t_{\lambda_1 + \lambda_2} \):

**Formula 5.5** Let \( a, c \in \mathbb{R}^*_+ \) and \( b, a, \beta \in \mathbb{R} \) such that \( a^2 + \beta^2 = 1 \). Then we have

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \exp(-c(x^2 + y^2) - a(ax + \beta y)^2 + 2\sqrt{-1}b(ax + \beta y))dxdy = \frac{\pi}{(c^2 + ac)^{1/2}} \exp\left(-\frac{b^2}{a + c}\right).
\]
References


[5] T. Oda, The standard $(g, K)$-modules of $Sp(2, \mathbb{R})$, Preprint Series, UTMS 2007-3, Graduate School of Mathematical Sciences, University of Tokyo.


