Generalized Whittaker functions of degenerate principal series

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Abstract

In the theory of modular forms, modular forms with weights are important objects. For automorphic forms on $SL(2, \mathbb{R})$, the notion of weights are translated to characters of $SO(2)$. Hence for general cases, $K$-types of admissible representations can be seen as a generalization of weights of corresponding automorphic forms. In this paper, we consider degenerate principal series representations and define a class of their $K$-types which are called strongly spherical (Definition 3.2). And we give a characterization of generalized Whittaker functions with strongly spherical $K$-types of degenerate principal series representation (Theorem 5.2). The contents in this paper will appear with concrete proofs in [2].

1 Notation and preliminaries

In this section we give a quick review of some definitions and well known facts in the representation theory of Lie groups.

Let $G$ be a connected real semisimple Lie group, $K$ a maximal compact subgroup and $\theta$ the associated Cartan involution. Throughtout this paper we assume that $G$ is split over $\mathbb{R}$ and has a complexification $G_{\mathbb{C}}$. The differentiation of $\theta$ is also written by same symbol. The associated Cartan decomposition of Lie algebra $\mathfrak{g}$ of $G$ is denoted by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Here $\mathfrak{k}$ and $\mathfrak{s}$ are eigenspaces of $\theta$ with eigenvalues 1 and $-1$ respectively.

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Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{s}$ and $\Sigma$ the root system of $(\mathfrak{g}, \mathfrak{a})$. Its Weyl group $W$ is isomorphic with $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. Fix a positive system $\Sigma^+$ of $\Sigma$ and denote the set of simple roots by $\Pi = \{\alpha_1, \ldots, \alpha_r\}$. Let $\mathfrak{n}$ be the sum of the root space $\mathfrak{g}_{\alpha} = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for any } H \in \mathfrak{a} \}$ for $\alpha \in \Sigma^+$, i.e., $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{\alpha}$. Then we have an Iwasawa decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $G = KAN$ where $A = \exp \mathfrak{a}$ and $N = \exp \mathfrak{n}$. Also we define $\tilde{\mathfrak{n}} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}$ and $\tilde{N} = \exp \tilde{\mathfrak{n}}$. Let us denote the Killing form on $\mathfrak{g}$ by $B$. For $\lambda \in \mathfrak{a}^*$, we take $H_\lambda \in \mathfrak{a}$ satisfying the equations $\lambda(H) = B(H_\lambda, H)$ for any $H \in \mathfrak{a}$. We introduce an inner product $\langle \ , \ \rangle$ on $\mathfrak{a}^*$ defined by $\langle \mu, \nu \rangle = B(H_\mu, H_\nu)$ for $\mu, \nu \in \mathfrak{a}^*$.

We denote the centralizer of $A$ in $K$ by $M$. Then a minimal parabolic subgroup $P$ is defined by $P = MAN$. Let $\Theta \subset \Pi$ be a finite subset and define the parabolic subgroup $P_\Theta$ associated to $\Theta$ as follows. Let $\mathfrak{a}_\Theta = \{H \in \mathfrak{a} \mid \alpha(H) = 0 \text{ for any } \alpha \in \Theta\}$ and $\mathfrak{a}_\Theta^\perp$ the orthogonal complement of $\mathfrak{a}_\Theta$ in $\mathfrak{a}$ with respect to the Killing form. Furthermore let $\mathfrak{m}_\Theta = \bigoplus_{\alpha \in \Sigma^+ \setminus \Theta} \mathfrak{g}_{\alpha}$ and $\mathfrak{m}_\Theta = \mathfrak{a}_\Theta^\perp \oplus \bigoplus_{\alpha \in \Sigma^+ \cap \operatorname{span}(\Theta)} \mathfrak{g}_{\alpha}$. Then we can define the parabolic subalgebra associated to $\Theta$ by $\mathfrak{p}_\Theta = \mathfrak{m}_\Theta \oplus \mathfrak{a}_\Theta \oplus \mathfrak{n}_\Theta$. Let $L_\Theta = Z_G(\mathfrak{a}_\Theta)$, $K_\Theta = L_\Theta \cap K$ and $M_\Theta = K_\Theta \exp(\mathfrak{m}_\Theta \cap \mathfrak{s})$. Then we can define the parabolic subgroup associated to $\Theta$ by $P_\Theta = M_\Theta A_\Theta N_\Theta$. If $\Theta = \emptyset$, the parabolic subgroup $P_\emptyset = M_\emptyset A_\emptyset N_\emptyset$ equals to the minimal parabolic subgroup $P = MAN$ defined above.

We write $\mathfrak{g}_C$, $\mathfrak{k}_C$ etc. as the complexifications of $\mathfrak{g}$, $\mathfrak{k}$ etc. Let $U(\mathfrak{g})$, $U(\mathfrak{k})$ etc. be the universal enveloping algebras of complexifications of $\mathfrak{g}$, $\mathfrak{k}$ etc. Also let $Z(\mathfrak{g})$, $Z(\mathfrak{k})$ be the centers of universal enveloping algebras $U(\mathfrak{g})$, $U(\mathfrak{k})$ respectively. As it is well-known, there is an inclusion

$$Z(\mathfrak{g}) \subset U(\mathfrak{a}) \oplus \tilde{\mathfrak{n}}_C U(\mathfrak{g}).$$

Let $\sigma: Z(\mathfrak{g}) \to U(\mathfrak{a})$ be the projection map along this decomposition. Put $\rho = \operatorname{tr}(\operatorname{Ad}|_{\mathfrak{n}}) \in \mathfrak{a}_C^*$, then we can define the $\rho$-shifted map $\sigma': Z(\mathfrak{g}) \to U(\mathfrak{a})$ by $\sigma'(X)(\lambda) = \sigma(X)(\lambda - \rho)$ for $X \in Z(\mathfrak{g})$ and $\lambda \in \mathfrak{a}_C^*$. It is well known that this map gives an algebra isomorphism

$$\sigma': Z(\mathfrak{g}) \longrightarrow U(\mathfrak{a})^W,$$

which is called Harish-Chandra isomorphism. For $\lambda \in \mathfrak{a}_C^*$, we can define a character of $Z(\mathfrak{g})$ by

$$\chi_\lambda : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$$

$$X \mapsto \sigma'(X)(\lambda).$$

For $C^\infty(G, E)$, the space of smooth functions from $G$ to a finite dimensional vector space $E$, we can consider natural actions of $G$ and $\mathfrak{g}$ by left (right)
translations and left (right) derivations, i.e.,

\[
L_g f(x) = f(g^{-1}x), \quad R_g f(x) = f(xg), \quad (1.1)
\]

\[
L_X f(x) = \frac{d}{dt}L_{(\exp tX)}f(x)|_{t=0}, \quad R_X = \frac{d}{dt}R_{(\exp tX)}f(x)|_{t=0}, \quad (1.2)
\]

where \( x, g \in G, X \in \mathfrak{g} \) and \( f \in C^\infty(G, E) \).

Let \( (\pi, E) \) be a continuous representation of \( G \) where \( E \) is a Hausdorff locally convex complete topological vector space. We write the space of \( K \)-finite vectors of \( E \) by \( E_K \).

2 Poisson transform on vector bundle.

The Poisson transform is a continuous \( G \)-homomorphism from a spherical principal series representation to the space of right \( K \)-invariant functions on \( G \). As a generalization of this, we will define a vector-valued Poisson transform and determine its image.

Let \((\tau, V_\tau)\) be an irreducible unitary representation of \( K \) and \( \lambda \) an element of \( \mathfrak{a}_C^* \). Then we consider the induced representation \( \pi_{\tau,\lambda} \) realized as follows. The representation space is

\[
\mathcal{H}_{\tau,\lambda}^\infty = \{ f \in C^\infty(G, V_\tau) \mid f(gman) = \tau(m)^{-1}a^{\lambda-\rho}f(g) \text{ for } (m,a,n,g) \in M \times A \times N \times G \}
\]

and \( G \) acts on this space by left translation, i.e., \( \pi_{\tau,\lambda}(g)f(x) = L_g f(x) = f(g^{-1}x) \) for \( f \in \mathcal{H}_{\tau,\lambda}^\infty \) and \( g \in G \). This is an admissible representation of \( G \) with infinitesimal character \( \chi_\lambda \). Also we denote the space of \( K \)-finite vectors of \( \mathcal{H}_{\tau,\lambda}^\infty \) by \( H_{\tau,\lambda} \) which becomes a \( (\mathfrak{g}_C, K) \)-module naturally.

Also we consider another induced representation. The representation space is

\[
C_\tau^\infty(G/K; \chi_\lambda) = \{ f \in C^\infty(G, V_\tau) \mid f(gk) = \tau(k)^{-1}f(g), (k,g) \in K \times G, \quad R_X f = \chi_\lambda(X)f \text{ for } X \in Z(\mathfrak{g}) \}
\]

and \( G \) acts on this space by left translation. We denote the space of its \( K \)-finite vectors by \( C_\tau^\infty(G/K; \chi_\lambda)_K \).

We define the generalized Harish-Chandra C-function as follows,

\[
C(\lambda, \tau) = \int_{\overline{N}} \tau(k(\overline{n}))e^{-(\lambda+\rho)H(\overline{n})}d\overline{n}.
\]
Here \( g = k(g) \exp H(g)n(g) \) for \( k(g) \in K, H(g) \in \mathfrak{a} \) and \( n(g) \in N \). It is known that this integral is absolutely convergent by the operator norm of \( \text{End}(V_\tau) \) in \( \{ \lambda \in a_\mathbb{C}^* \mid \text{Re} \langle \lambda, \alpha \rangle > 0 \text{ for any } \alpha \in \Sigma^+ \} \). It is meromorphically continued in all \( a_\mathbb{C}^* \) (cf. [4]).

Since \( M \) is the finite abelian group, \( V_\tau \) can be decomposed as the direct sum of 1-dimentional representations of \( M \). Therefore we can take a basis \( \{ v_1, \ldots, v_l \} \) of \( V_\tau \) so that there exist \( l \)-dimentional representation \( \sigma_i \) \((i = 1, \ldots, l)\) of \( M \) such that \( \tau(m)v_i = \sigma_i(m)v_i \) \((i = 1, \ldots, l)\) for \( m \in M \). Also we take the dual basis \( \{ v_1^*, \ldots, v_l^* \} \) of \( V_\tau^* = \text{Hom}_{\mathbb{C}}(V_\tau, \mathbb{C}) \), i.e., each \( v_i \) satisfies \( v_i^*(v_j) = \delta_{ij} \) for \( i, j = 1, \ldots, l \). We regard \( V_\tau^* \) as a representation space of \( M \) by the contragradient representation.

**Definition 2.1 (Poisson transform).** We define the \( G \)-homomorphism \( \mathcal{P}_{\tau, \lambda} \) from \( \mathcal{H}_{\tau, \lambda}^\infty \) to \( C_\tau^\infty(G/K; \chi_\lambda) \) by

\[
\mathcal{P}_{\tau, \lambda} : \mathcal{H}_{\tau, \lambda}^\infty \rightarrow C_\tau^\infty(G/K; \chi_\lambda) \\
f \mapsto \int_K \tau(k)f(gk)dk
\]

This is called the Poisson transform.

We see that \( \mathcal{P}_{\tau, \lambda} \) gives a bijection between the \( K \)-finite subspaces for generic \( \lambda \in a_\mathbb{C}^* \).

**Theorem 2.2.** We put following assumptions.

1. \( \lambda \in a_\mathbb{C}^* \) is regular and dominant, i.e.,

\[
2\frac{\langle \lambda, \beta \rangle}{\langle \beta, \beta \rangle} \notin \{0, -1, -2, \ldots\} \text{ for any } \beta \in \Sigma^+.
\]

2. The determinant of \( C(\tau, \lambda) \in \text{End}(V_\tau) \) is nonzero.

Then \( \mathcal{P}_{\tau, \lambda} \) gives a \((\mathfrak{g}_\mathbb{C}, K)\)-isomorphism,

\[
\mathcal{P}_{\tau, \lambda} : H_{\tau, \lambda} \approx C_\tau^\infty(G/K; \chi_\lambda)_K.
\]

**Remark 2.3.** This theorem is first proved by An Yang [5] in more general settings. However Yang put a stronger assumption

\[
2\frac{\langle \lambda, \beta \rangle}{\langle \beta, \beta \rangle} \notin \mathbb{Z} \text{ for any } \beta \in \Sigma.
\]

This is too strong for our purpose in this paper. Therefore we need a refined theorem under the weaker condition as above.
3 Strongly spherical $K$-types and vector valued Poisson transforms of degenerate principal series representations

Our purpose of this note is to give a characterization of the vector-valued generalized Whittaker functions of degenerate principal series. To do this, we need the Poisson transforms on degenerate principal series representations. Hence we need to restrict the vector-valued Poisson transform to degenerate principal series representations and determine their images.

Take a finite subset $\Theta \subset \Pi$ and let $P_\Theta$ be the corresponding parabolic subgroup of $G$. For $\lambda \in (a_\Theta^*_C)_C$, we define a character $\lambda_\Theta$ of $p_\Theta$ by

$$
\lambda_\Theta: \quad p_\Theta \rightarrow \mathbb{C},
X + H \rightarrow \lambda(H),
$$

where $X \in m_\Theta + n_\Theta$ and $H \in a_\Theta$. We take a character $\Lambda_\Theta$ of $P_\Theta$ whose differentiation is $\lambda_\Theta$. Then we define a degenerate principal series representation of $G$ as follows. The representation space is $C^\infty(G/P_\Theta; \Lambda_\Theta) = \{ f \in C^\infty(G) \mid f(gp) = \Lambda_\Theta(p)f(g) \text{ for } p \in P_\Theta, g \in G \}$. The action of $G$ on this space is defined by left translation. We denote the space of $K$-finite vectors of $C^\infty(G/P_\Theta; \Lambda_\Theta)$ by $H_{\Theta, \lambda}$.

**Definition 3.1** (annihilator ideal). We define a left ideal of $U(g)$ by

$$
J_\Theta(\lambda) = \sum_{X \in (p_\Theta)_C} U(g)(X - \lambda_\Theta(X))
$$

and also define a two-sided ideal

$$
I_\Theta(\lambda) = \bigcap_{g \in G} \text{Ad}(g)J_\Theta(\lambda).
$$

This two-sided ideal $I_\Theta(\lambda)$ is studied by H. Oda and T. Oshima in [3] and they give explicit generators of $I_\Theta(\lambda)$. This ideal is very important tool to investigate $C^\infty(G/P_\Theta; \Lambda_\Theta)$, because we can show that for any $X \in I_\Theta(\lambda)$ and $f \in C^\infty(G/P_\Theta; \Lambda_\Theta)$, we have $R_X f = 0$, i.e., $I_\Theta(\lambda)$ is the annihilator ideal of $C^\infty(G/P_\Theta; \Lambda_\Theta)$. Also it is known that $I_\Theta(\lambda)$ is the annihilator of the generalized Verma module $U(g)/J_\Theta(\lambda)$.

We define the notion of strongly spherical $K$-types.
Definition 3.2 (Strongly spherical $K$-type). Let $(\tau, V_\tau)$ be a irreducible unitary representation of $K$ such that $\dim \text{Hom}_K(V_\tau, H_{\Theta,\lambda}) \neq 0$. We call this representation $\tau$ a strongly spherical $K$-type of $H_{\Theta,\lambda}$ if the dimension of $V_\tau^{\mathfrak{m} \ominus \cap \mathfrak{k}} = \{v \in V_\tau | \tau(X)v = 0 \text{ for } X \in \mathfrak{m}_\Theta \cap \mathfrak{k}\}$ is equal to 1.

Remark 3.3. If $\Theta = \emptyset$, i.e., $P_\Theta$ is minimal parabolic subgroup, this condition says $V_\tau$ is 1-dimensional because $\mathfrak{m}_\Theta$ is trivial. On the other hand, if $(K, M_\Theta \cap K)$ is a symmetric pair, it is easy to see that every irreducible unitary representation of $K$ is strongly spherical.

For these strongly spherical $K$-types, we can consider vector valued Poisson transform of degenerate principal series. And we can determine its image. For an irreducible representation $(\tau, V_\tau)$ of $K$, we define a space

$$C_\tau^\infty(G/K; I_{\Theta}(\lambda)) = \{ f \in C^\infty(G, V_\tau) | f(gk) = \tau(k^{-1})f(g), R_Xf = 0 \text{ for } g \in G, k \in K, X \in I_{\Theta}(\lambda) \}.$$ 

This is a $G$-representation by the left translation.

Theorem 3.4. We use the notations as above. For $\lambda \in (a_\Theta^*)_\mathbb{C}$, we assume that

1. $\lambda + \rho$ is regular and dominant.

2. $\det C(\tau, \lambda + \rho) \neq 0$.

Let $(\tau, V_\tau)$ be a strongly spherical $K$-type of $H_{\Theta,\lambda}$. Then the restriction of $\mathcal{P}_{\tau,\lambda}$ to $H_{\Theta,\lambda}$ gives a following $(\mathfrak{g}_\mathbb{C}, K)$-isomorphism,

$$\mathcal{P}_{\Theta,\lambda} : H_{\Theta,\lambda} \rightarrow \text{C}_{\tau}^\infty(G/K; I_{\Theta}(\lambda))_K \quad \phi \mapsto \int_K \tau(k)\phi(gk) dk.$$ 

Here we note that we can see $a_\Theta^* \subset a^*$ by the Killing form $B$.

Proof. By the assumption, we have the $(\mathfrak{g}_\mathbb{C}, K)$-isomorphism

$$\mathcal{P}_{\tau,\lambda} : H_{\tau,\lambda} \rightarrow \text{C}_{\tau}^\infty(G/K; \chi_{\lambda})_K \quad \phi \mapsto \int_K \tau(k)\phi(gk) dk.$$ 

Since $H_{\Theta,\lambda}$ is a $(\mathfrak{g}_\mathbb{C}, K)$-submodule of $H_{\tau,\lambda}$, we have

$$\mathcal{P}_{\tau,\lambda}(H_{\Theta,\lambda}) \subset \text{C}_{\tau}^\infty(G/K; I_{\Theta}(\lambda))_K.$$
Here we notice that since it is easy to show that $\sum_{X \in Z(\mathfrak{g})} U(\mathfrak{g})(X - \chi_\lambda(X)) \subset I_\Theta(\lambda)$, we have $C^\infty(G/K; I_\Theta(\lambda)) \subset C^\infty(G/K; \chi_\lambda)$. It remains to show that $H_{\Theta, \lambda} \supset P^{-1}_{\tau, \lambda}(C^\infty(G/K; I_\Theta(\lambda))_K)$. To show this, we take an arbitrary element $u \in C^\infty(G/K; I_\Theta(\lambda))$. We can see $\lambda \in (\mathfrak{a}_n^*)_C$ as an element of $\mathfrak{a}_C$, hence we denote this by $\lambda_\Theta \in \mathfrak{a}_C$. We define a character of the Borel subalgebra of $\mathfrak{g}_C$, $b = a_C + n_C$ as follows,

$$
\lambda_\Theta : b \rightarrow C \\
H + X \mapsto \lambda(H)
$$

where $H \in a_C$ and $X \in n_C$. We define a left ideal of $U(\mathfrak{g})$ by $J(\lambda_\Theta) = \sum_{X \in b} U(\mathfrak{g})(X - \lambda_\Theta(X))$. Then for any $X \in J(\lambda_\Theta)$ and $f \in H_{\tau, \lambda}$ we have $R_X f = 0$. Hence $P^{-1}_{\tau, \lambda} u$ satisfies that $R_X P^{-1}_{\tau, \lambda} u = 0$ for any $J(\lambda_\Theta)$ because $J(\lambda_\Theta) = I_\Theta(\lambda) + J(\lambda_\Theta)$ by the result of Oda and Oshima (Theorem 3.12 in [3]). This implies that there exists a representation $\sigma$ of $M_\Theta$ which satisfies that $\text{Hom}_{M_\Theta}(\sigma, \tau) \neq \{0\}$ and differentiation of $\sigma$ is trivial. And $P^{-1}_{\tau, \lambda} u \in C^\infty\text{-Ind}_{P_0}^G(\sigma \otimes e^{-\lambda} \otimes 1_{N_\Theta})$. However since $\dim V_{\tau, \Theta} = 1$, $\sigma$ must be equal to $\Lambda_{\Theta}M_\Theta$.

\[\box\]

### 4 Maximal globalization

The vector-valued Poisson transform gives a $(\mathfrak{g}_C, K)$-isomorphism from the degenerate principal series $H_{\Theta, \lambda}$ to $C^\infty(G/K; I_\Theta(\lambda))_K$ if $\tau$ is a strongly spherical $K$-type of $H_{\Theta, \lambda}$. Furthermore, we see that this $(\mathfrak{g}_C, K)$-isomorphism extends to the continuous $G$-isomorphism.

Let $X$ be an admissible $(\mathfrak{g}_C, K)$-module with finite length. We consider the space of $(\mathfrak{g}_C, K)$-homomorphisms from the dual $(\mathfrak{g}_C, K)$-module $X^*$ to $C^\infty(G)$, $\text{Hom}_{(\mathfrak{g}_C, K)}(X^*, C^\infty(G))$ where $G$ acts on $C^\infty(G)$ by left translation. Since $C^\infty(G)$ has a uniformly convex topology and $X^*$ has a countably many basis, we can define the complete locally convex topology on $\text{Hom}_{(\mathfrak{g}_C, K)}(X^*, C^\infty(G))$. On the other hand, $G$ can also act on $C^\infty(G)$ by right translation. This action is continuous on the topology of $\text{Hom}_{(\mathfrak{g}_C, K)}(X^*, C^\infty(G))$. The space of $K$-finite elements of $\text{Hom}_{(\mathfrak{g}_C, K)}(X^*, C^\infty(G))$ can be identified with $(X^*)^* \cong X$ by the evaluation at the origin, i.e., for $I \in \text{Hom}_{(\mathfrak{g}_C, K)}(X^*, C^\infty(G))$, $X^* \ni v \mapsto I(v)(e) \in \mathbb{C}$ is a linear form of $X^*$. Hence $\text{Hom}_{(\mathfrak{g}_C, K)}(X^*, C^\infty(G))$ is a continuous $G$ representation and its $K$-finite subspace is $X$, i.e., $\text{Hom}_{(\mathfrak{g}_C, K)}(X^*, C^\infty(G))$ is a globalization of $X$. This is called the maximal globalization [1].

Let us return to our setting. In the previous section we see that there is a
$(\mathfrak{g}_\mathbb{C}, K)$-isomorphism

$$
P_{\Theta, \lambda}: \quad H_{\Theta, \lambda} \rightarrow C^\infty_\tau(G/K; I_\Theta(\lambda))_K$$

$$\phi \quad \mapsto \quad \int_K \tau(k)\phi(gk)\,dk.$$ 

This $(\mathfrak{g}_\mathbb{C}, K)$-isomorphism can be extended to $G$-isomorphism as follows. If $(\tau, V_\tau)$ is a strongly spherical $K$-type of $(\mathfrak{g}_\mathbb{C}, K)$-module $H_{\Theta, \lambda}$, it is multiplicity free by definition. We fix a $K$-projection $p_\tau: H_{\Theta, \lambda} \rightarrow V_\tau$. We define a $K$-embedding $\iota_\tau: V_\tau^* \hookrightarrow H_{\Theta, \lambda}^*$ as the dual map of $p_\tau$.

**Theorem 4.1.** We assume that

1. $\lambda_\Theta + \rho$ is regular and dominant.

2. $\det C(\lambda + \rho, \tau) \neq 0$.

Let $(\tau, V_\tau)$ be a strongly spherical $K$-type of $H_{\Theta, \lambda}$. Then we have the following topological $G$-isomorphism.

$$\Phi: \quad \text{Hom}_{(\mathfrak{g}_\mathbb{C}, K)}(H_{\Theta, \lambda}^*, C^\infty(G)) \rightarrow C^\infty_\tau(G/K; I_\Theta(\lambda))$$

$$I \quad \mapsto \quad \sum_{i=1}^l I(\iota_\tau(v_i^*))(g)v_i.$$ 

5 **Generalized Whittaker models**

Finally we give the main theorem of this note. We can give a characterization of vector-valued generalized Whittaker functions as solutions of system of differential equations which comes from $I_\Theta(\lambda)$.

Let $U$ be a closed subgroup of $N$ and $(\eta, V_\eta)$ an irreducible unitary representation of $U$. We consider a representation of $G$ induced from $\eta$. The representation space is

$$C^\infty_\eta(U\backslash G) = \{ f: G \rightarrow V_\eta^\infty \text{ smooth} \mid f(ug) = \eta(u)f(g) \text{ for all } u \in U, g \in G \}.$$ 

Here $V_\eta^\infty$ stands for the space of smooth vectors of $V_\eta$. We note that $V_\eta^\infty$ has a Hausdorff complete locally convex topology and we can define the derivation of $f: G \rightarrow V_\eta^\infty$ by the convergence on the topology of $V_\eta^\infty$.

**Definition 5.1** (Generalized Whittaker model). Let $X$ be an admissible $(\mathfrak{g}_\mathbb{C}, K)$-module with finite length. Let $U$ be a closed subgroup of $N$ and $(\eta, V_\eta)$ an
irreducible unitary representation of $U$. We consider the space of $(\mathfrak{g}_\mathbb{C}, K)$-homorphisms from $X$ to $C_\eta^\infty(U \backslash G)$,

$$\text{Hom}_{(\mathfrak{g}_\mathbb{C}, K)}(X, C_\eta^\infty(U \backslash G)).$$

If $\text{Hom}_{(\mathfrak{g}_\mathbb{C}, K)}(X, C_\eta^\infty(U \backslash G)) \neq \{0\}$, we say $X$ has generalized Whittaker models.

We consider genelarized Whittaker models of $H_{\Theta, \lambda}$. Let $(\tau, V_\tau)$ be a strongly spherical $K$-type of $H_{\Theta, \lambda}$. Take a irreducible unitary representation $(\eta, V_\eta)$ of $N$. For the algebraic tensor product $V_\eta^\infty \otimes V_\tau$, we can define a natural topology comes from $V_\eta^\infty$ because $V_\tau$ is finite dimensional. Hence we can consider the following space of smooth functions from $G$ to $V_\eta^\infty \otimes V_\tau$,

$$C_{\eta, \tau}^\infty(U \backslash G/K) = \{ f : G \to V_\eta^\infty \otimes V_\tau \text{ smooth } | f(ugk) = \eta(u) \otimes \tau(k^{-1})f(g) \text{ for } u \in U, g \in G, k \in K, \}.$$

Also we define

$$C_{\eta, \tau}(U \backslash G/K; I_{\Theta}(\lambda)) = \{ f \in C_{\eta, \tau}^\infty(U \backslash G/K) | R_X f = 0 \text{ for } X \in I_{\Theta}(\lambda) \}.$$

As a collorary of Theorem 4.1, we have the following characterization of genelarized Whittaker models.

**Theorem 5.2.** We use the same notations as Theorem 4.1. We assume that

1. $\lambda_\Theta + \rho$ is regular and dominant.

2. $\det C(\lambda + \rho, \tau) \neq 0$.

Let $(\tau, V_\tau)$ be a strongly spherical $K$-type of $H_{\Theta, \lambda}$. Then we have the following linear isomorphism.

$$\Phi : \text{Hom}_{(\mathfrak{g}_\mathbb{C}, K)}(H_{\Theta, \lambda}^*, C_\eta^\infty(U \backslash G)) \to C_{\eta, \tau}^\infty(U \backslash G/K; I_{\Theta}(\lambda))$$

$$I \mapsto \sum_{i=1}^l I(\iota_{\tau}(v_i^*))g)v_i.$$

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