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The mapping class group and the Meyer function for plane curves

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In this note, a new example of Meyer function and its application to the signature of 4-manifolds are presented.

1 The first MMM class and Meyer's signature cocycle

Let Σ_g be a closed oriented C^∞-surface of genus g and $\mathrm{Diff}^+\Sigma_g$ the group of all orientation preserving diffeomorphisms of Σ_g, endowed with C^∞-topology. The mapping class group of Σ_g, denoted by Γ_g, is defined to be the group of connected components of $\mathrm{Diff}^+\Sigma_g$.

By the result of Earle-Eells[2], each connected component of $\mathrm{Diff}^+\Sigma_g$ is contractible for $g \geq 2$. It follows that the classifying space $\mathrm{BDiff}^+\Sigma_g$ is the Eilenberg-MacLane space $K(\Gamma_g, 1)$. Let $p\colon E \to B$ be an oriented Σ_g bundle. For each $i \geq 1$, the i-th MMM class $e_i = e_i(p)$ is defined by

$$e_i(p) := p_!(e^{i+1}) \in H^{2i}(B).$$

Here, $e \in H^2(E)$ is the Euler class of the tangent bundle along the fiber of p and $p_!$ is the Gysin map. Since these classes are natural with respect to bundle maps between oriented Σ_g bundles, one can think of e_i as the cohomology class in

$$H^{2i}(\mathrm{BDiff}^+\Sigma_g) \cong H^{2i}(\Gamma_g).$$

In this note, we consider the first MMM class e_1, which is closely related to the signature of 4-manifolds as follows.

Let $p\colon E \to B$ be an oriented Σ_g bundle over a closed oriented surface B. Then the total space E is a closed 4-manifold endowed with the natural orientation. By the Hirzebruch signature formula, we have

$$\mathrm{Sign}(E) = \frac{1}{3}\langle e_1(p), [B]\rangle. \tag{1}$$

There is also a 2-cocycle of Γ_g using the signature of 4-manifolds. This is *Meyer's signature cocycle* τ_g [9]. Here we briefly recall its definition.

Let P denote the pair of pants, i.e., $P = S^2 \setminus \bigcup_{i=1}^3 \mathrm{Int}D_i$ where $D_i, i = 1, 2$, and 3 are the three disjoint closed disks in the 2-sphere S^2. Choose a base point $p_0 \in \mathrm{Int}P$ and fix a based loop ℓ_1 and ℓ_2 such that ℓ_i is free homotopic to the loop traveling once the boundary ∂D_i by counter clockwise manner ($i = 1, 2$). For $(f_1, f_2) \in \Gamma_g \times \Gamma_g$, we can construct an oriented Σ_g bundle $E(f_1, f_2)$ over P such that the topological monodromy $\pi_1(P) \to \Gamma_g$ sends $[\ell_i]$ to f_i for $i = 1, 2$. $E(f_1, f_2)$ is a compact C^∞-manifold of dimension 4 endowed with the natural orientation. Then the signature of $E(f_1, f_2)$ is defined and we set

$$\tau_g(f_1, f_2) := -\mathrm{Sign}(E(f_1, f_2)).$$

By the Novikov additivity of the signature τ_g turns out to be a 2-cocycle of Γ_g, and the equation (1) shows that

$$3[\tau_g] = e_1 \in H^2(\Gamma_g).$$

2 Triviality of e_1 over rationals

Let $p\colon E \to B$ be an oriented Σ_g bundle or continuous family of compact Riemann surface of genus g.

Now we are interested in the triviality of the rational cohomology class $e_1(p) \in H^2(B;\mathbb{Q})$. If this is the case, $[\tau_g]$ pulled back to $H^2(\pi_1(B);\mathbb{Q})$ vanishes and there exists a $\mathbb{Q}$-valued 1-cochain $\phi\colon \pi_1(B) \to \mathbb{Q}$ cobounding τ_g pulled back to $\pi_1(B)$ by the topological monodromy of p. Moreover, if $H^1(\pi_1(B);\mathbb{Z}) = 0$, such a 1-cochain is unique. Then we call ϕ the *Meyer function* of $\pi_1(B)$ with respect to $p\colon E \to B$.

There are several examples:

1. W. Meyer [9] showed that $[\tau_g] \in H^2(\Gamma_g;\mathbb{Z})$ is torsion for $g = 1, 2$. Thus, $e_1 = 0 \in H^2(\Gamma_g;\mathbb{Q})$ for $g = 1, 2$. In the case $g = 1$, Γ_1 is isomorphic to $SL(2;\mathbb{Z})$. Meyer also gave an explicit formula for the Meyer function $\phi_1\colon SL(2;\mathbb{Z}) \to \frac{1}{3}\mathbb{Z}$ using the Rademacher function.

2. The hyperelliptic mapping class group $\Gamma_g^H \subset \Gamma_g$ is defined as the centralizer of a hyperelliptic involution $\iota \in \Gamma_g$. As was shown by F. Cohen [4] and N. Kawazumi [6] independently, Γ_g^H is $\mathbb{Q}$-acyclic. In particular, $e_1 = 0 \in H^2(\Gamma_g^H;\mathbb{Q})$. Later, H. Endo [3] directly showed that the existence and the uniqueness of the Meyer function $\phi_g^H : \Gamma_g^H \to \frac{1}{2g+1}\mathbb{Z}$ using a finite presentation of Γ_g^H by J. Birmann-H. Hilden.

3. In contrast, W. Meyer [9] showed that $[\tau_g] \in H^2(\Gamma_g;\mathbb{Z})$ has infinite order therefore $e_1 \in H^2(\Gamma_g;\mathbb{Q})$ is non-trivial for $g \geq 3$. In fact, J. Harer [5] showed that $H^2(\Gamma_g;\mathbb{Z}) \cong \mathbb{Z}$ for $g \geq 3$, and combining this with Meyer's computation, it follows that $[\tau_g] \in H^2(\Gamma_g;\mathbb{Z}) \cong \mathbb{Z}$ is equal to 4 times a generator.

4. D. Mumford [10] observed that if $p\colon E \to B$ is a family of non-hyperelliptic curves of genus 3, $e_1(p) = 0 \in H^2(B;\mathbb{Q})$. His proof uses the Grothendieck-Riemann-Roch formula.

In this note, we give an alternative proof of Mumford's observation, and generalize it to the case of family of plane curves. Our approach is to show the existence and the uniqueness of the Meyer function of the group $\Pi(4)$ defined in the next section, which is universal for families of non-hyperelliptic curves of genus 3. Our approach is purely topological, although some algebraic geometry is used to compute examples.

3 The mapping class group for plane curves

In this section we construct the group $\Pi(d)$ and state the main result of this note.

Henceforth d is a fixed integer ≥ 2. Let V be the complex vector space of homogeneous polynomials of degree d in the determinates x, y, and z, and let $\mathbb{P} = \mathbb{P}(V)$ be the projectivization of V. Each point $a \in \mathbb{P}$ determines the complex plane curve $C_a \subset \mathbb{P}^2$ of degree

d. Let D be the set of all points $a \in \mathbb{P}$ such that the corresponding curve C_a is singular. D is called the discriminant locus and known to be irreducible and of codimension 1. Set

$$\mathcal{F} := \{(a,p) \in (\mathbb{P} \setminus D) \times \mathbb{P}^2;\ p \in C_a\}.$$

The group of automorphisms of $\mathbb{P}^2$, namely $PGL(3)$, acts naturally on $\mathbb{P}\setminus D$ as change of variables. Then the first projection $\mathcal{F} \to (\mathbb{P} \setminus D)$ is a complex analytic family of compact Riemann surfaces of genus 3 and the projection map is $PGL(3)$-equivariant. Here, the action of $PGL(3)$ on $\mathcal{F}$ is diagonal.

Taking Borel construction $(\mathbb{P} \setminus D)_{PGL(3)} = EPGL(3) \times_{PGL(3)} (\mathbb{P} \setminus D)$, we obtain a continuous family

$$p_u \colon \mathcal{F}_{PGL(3)} \to (\mathbb{P} \setminus D)_{PGL(3)}$$

of compact Riemann surfaces of genus 3.

We denote by $\Pi(d)$ the fundamental group of $(\mathbb{P} \setminus D)_{PGL(3)}$ and call this group *the mapping class group for plane curves of degree d*. Let

$$\rho\colon \Pi(d) \to \Gamma_g$$

be the topological monodromy of p_u. Here, $g = \frac{1}{2}(d-1)(d-2)$.

Recall that the usual mapping class group Γ_g is the fundamental group of the classifying space $\mathrm{BDiff}^+\Sigma_g$. The name "mapping class group" for $\Pi(d)$ comes from the following universal property of $(\mathbb{P} \setminus D)_{PGL(3)}$.

Theorem 3.1. *For any topological space B, there is a natural bijection*

$$[B, (\mathbb{P} \setminus D)_{PGL(3)}] \cong \frac{\{\text{family of non}-\text{singular plane curves of degree } d \text{ over } B\}}{\sim \text{isotopy}} \tag{2}$$

induced by pulling back p_u. Here, the left hand side is the set of homotopy classes of continuous maps from B to $(\mathbb{P} \setminus D)_{PGL(3)}$.

The following are the main results of this note:

Theorem 3.2. $\rho^*([\tau_g]) = 0 \in H^2(\Pi(d); \mathbb{Q})$.

Theorem 3.3.

$$H_1(\Pi(d); \mathbb{Z}) = \begin{cases} \mathbb{Z}/3(d-1)^2\mathbb{Z} & \text{if } d \equiv 0 \bmod 3, \\ \mathbb{Z}/(d-1)^2\mathbb{Z} & \text{if } d \equiv 1 \text{ or } 2 \bmod 3. \end{cases}$$

As a consequence of Theorems 3.2 and 3.3, there exists the unique 1-cochain $\phi^d\colon \Pi(d) \to \mathbb{Q}$ such that $\delta\phi^d = \rho^*\tau_g$. We call this cochain *the Meyer function for plane curves of degree d*. One can easily see that ϕ^d is a class function on $\Pi(d)$.

Let σ be an element of $\pi_1(\mathbb{P} \setminus D)$ traveling once around D. Such σ is called a *lasso* around D. We also denote by σ the image of σ by the natural surjection $\pi_1(\mathbb{P}\setminus D) \to \Pi(d)$. σ is well-defined up to conjugacy.

Proposition 3.4. *For $d \geq 3$,*

$$\phi^d(\sigma) = -\frac{d+1}{3(d-1)}.$$

By this proposition, we can see the order of the integral cohomology class $\rho^*[\tau_g] \in H^2(\Pi(d);\mathbb{Z})$.

Here we briefly explain how to prove Theorem 3.2. Let $\widetilde{D} \subset V$ be the union of all lines in $D \subset \mathbb{P}$. There is a natural map $V \setminus \widetilde{D} \to \mathbb{P} \setminus D \to (\mathbb{P} \setminus D)_{PGL(3)}$. We can see that this map induces an injective homomorphism

$$H^2(\Pi(d);\mathbb{Q}) \to H^2(\pi_1(V \setminus \widetilde{D});\mathbb{Q}).$$

Moreover we can construct a 1-cochain $c\colon \pi_1(V \setminus \widetilde{D}) \to \mathbb{Z}$ cobounding τ_g pulled back to $\pi_1(V \setminus \widetilde{D})$. The construction is based on the signature of 4-manifolds. Thus we also have $\rho^*([\tau_g]) = 0 \in H^2(\Pi(d);\mathbb{Q})$. For details, see Section 3 of [7].

4 The local signature

As an application, we define the local signature for each fiber germs of 4-dimensional fiber spaces whose general fibers are non-hyperelliptic curves of genus 3. We first introduce a class of 4-manifolds we consider. By a *4-dimensional non-hyperelliptic fibration of genus 3* is meant a following data:

1. E(*resp.* B) is an oriented 4(resp. 2)-manifold and $\pi\colon E \to B$ is a C^∞-map,
2. there exist finitely many points $b_1, \dots, b_n \in \mathrm{Int}(B)$ such that the restriction of π to $B \setminus \{b_i\}_i$ has a structure of C^∞-family of non-hyperelliptic curves of genus 3.

Let $\mathcal{F}_i$ be the (possibly singular) fiber germ around b_i and D_i a small closed 2-disk centered at b_i. Now for the case $d = 4$, the right hand side of (2) in Theorem 3.1 is naturally isomorphic to

$$\frac{\{\text{continuous family of non}-\text{hyperelliptic curves of genus 3 over } B\}}{\sim \text{isotopy}}.$$

Thus, by restricting π to $D_i \setminus \{b_i\}$ we obtain the classifying map $g^i\colon D_i \setminus \{b_i\} \to (\mathbb{P} \setminus D)_{PGL(3)}$ (determined up to homotopy). Define

$$\mathrm{loc.sig}^{\mathbb{Q}}(\mathcal{F}_i) := \phi^4(g^i_*(\partial D_i))_*(\gamma)) + \mathrm{Sign}(N(\pi^{-1}(b_i))) \in \frac{1}{9}\mathbb{Z}.$$

Here, $N(\pi^{-1}(b_i))$ denotes a fiber neighborhood of the singular fiber $\pi^{-1}(b_i)$. It is easy to see that this value only depends on the germ of π around b_i (note that ϕ^4 is a class function on $\Pi(4)$). We remark here that this definition originates from Y. Matsumoto [8] where Lefschetz fibrations of genus 2 are discussed.

By using this local signature, we can now formulate the signature formula in our setting:

Theorem 4.1 (The signature formula). *Let $p\colon E \to B$ be a 4-dimensional non-hyperelliptic fibration of genus 3. If E is closed,*

$$\mathrm{Sign}(E) = \sum_i \mathrm{loc.sig}^{\mathbb{Q}}(\mathcal{F}_i).$$

In this case of non-hyperelliptic family of genus 3, T. Ashikaga-K. Konno [1] and K. Yoshikawa [11] have already defined local signature independently. The definition of [1] is algebro geometric and that of [11] is complex analytic. Computing some examples of values of our local signature, we observe that they coincide with those computed in [1] and [11]. For details of the computations, see [7].

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