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On a theorem of de Franchis

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1 Introduction

Let X be a compact Riemann surface of genus $g\,(>1)$. De Franchis [1] stated the following:

Theorem 1 (de Franchis) *(a) For a fixed compact Riemann surface Y of genus >1, the number of nonconstant holomorphic maps $X\rightarrow Y$ is finite.*

(b) There are only finitely many compact Riemann surfaces Y_i of genus >1 which admit a nonconstant holomorphic map from X.

The second statement (b) is often attributed to Severi. After knowing the finiteness of maps, we may ask if there exists a upper bound depending only on some topological invariant, for example, the genus g. Related to the statement (a), the author [4] showed that the bound is smaller than $(cg)^{2g}$ for some constant c.

Now, we consider a bound for holomorphic maps when Y is not fixed, that is, we estimate the number of all nonconstant holomorphic maps from X to other Riemann surfaces. Let $f_i : X\rightarrow Y_i$ be nonconstant holomorphic maps for $i=1,2$. We say that f_1 and f_2 are isomorphic if and only if there is a conformal map $h:Y_1\rightarrow Y_2$ such that $h\circ f_1=f_2$. Let $\mathcal{I}_\gamma(X)$ denote the set of all isomorphic classes of nonconstant holomorphic maps into compact Riemann surfaces of genus $\gamma>1$, and denote $\mathcal{I}(X)=\bigcup_{g>\gamma>1}\mathcal{I}_\gamma(X)$. By the theorem of de Franchis, we see that $\sharp\mathcal{I}(X)$ is finite. In 1983 Howard and Sommese [2] first showed that there is a bound on $\sharp\mathcal{I}(X)$ depending only on g.

Let

$$M(g)=\max_X\{\sharp\mathcal{I}(X)\},$$

where the maximum is taken over all Riemann surfaces X of genus g. It is an interesting problem to determine the exact rate of growth of $M(g)$. The author [5] showed

$$M(g) \leq (cg)^{5g}$$

for some constant c and it was the best upper bound depending only on g.

In this note we will improve the bound and show

$$M(g) \leq (cg)^{2g}$$

for some constant c.

On the other hand, Kani [3] also constructed a sequence of Riemann surfaces of genara $g_1 < g_2 < \ldots < g_n < \ldots$, such that the number of isomorphic classes of nonconstant holomorphic maps of each Riemann surface is larger than $\exp(c(\log(g_n))^2)$ for some constant $c > 0$ (independent of n). It implies that $M(g)$ cannot be bounded by any polynomial in g.

2 The bound

In the following, we will refer to [5] for all of the notation and lemmata. In [5], the leading term of the upper bound was depend on Lemma 3 (p.3060) and the Proposition (p.3062). We improve them as follows.

Lemma 3' *Let $f_1 : X \longrightarrow Y_1$ be a holomorphic map of degree d, and $\mathfrak{f}_1 : \mathrm{J}(X) \longrightarrow \mathrm{J}(Y_1)$ be the homomorphism induced by f_1. Take an arbitrary $u \in {}^t\mathfrak{f}_1(\widehat{\mathrm{J}(\mathrm{Y}_1)})$. Then, the number of isomorphic classes of holomorphic maps $f_i : X \longrightarrow Y_i$ of degree d such that the dual map ${}^t\mathfrak{f}_i : \widehat{\mathrm{J}(Y_i)} \longrightarrow \widehat{\mathrm{J}(X)}$ of the induced homomorphism $\mathfrak{f}_i$ satisfies $u \in {}^t\mathfrak{f}_i(\widehat{\mathrm{J}(\mathrm{Y}_i)})$ is at most $\binom{2g-2}{d} \times \binom{4g-4}{d}$.*

In [5], the conclusion was $\binom{2g-2}{d} \times (2g-1)^d$ which is now replaced by $\binom{2g-2}{d} \times \binom{4g-4}{d}$.

Proof. The assumption means that there exist holomorphic differentials ϕ_1 on Y_1 and ϕ_i on Y_i such that their pull backs satisfy $f_{1*}\phi_1 = f_{i*}\phi_i$.

Then, for a zero p_{01} of ϕ_1, the number of possible $f_1^{-1}(p_{01})$ (counting multiplicities) that can occur is at most $\begin{pmatrix} 2g-2 \\ d \end{pmatrix}$. After determining $\phi = f_{1*}\phi_1$ and $f_1^{-1}(p_{01})$, we can show that there are at most $\begin{pmatrix} 4g-4 \\ d \end{pmatrix}$ possible isomorphic classes of holomorphic maps of degree d as follows.

Let $f_i : X \longrightarrow Y_i$ be holomorphic maps ($i = 1, 2$). Suppose that there are holomorphic differentials ϕ_1 and ϕ_2 on Y_1 and Y_2, respectively, with $f_{1*}\phi_1 = f_{2*}\phi_2$, and there is a zero p_{01} (resp. p_{02}) of ϕ_1 (resp. ϕ_2) satisfying $f_1^{-1}(p_{01}) = f_2^{-1}(p_{02})$. We put $\phi = f_{1*}\phi_1 = f_{2*}\phi_2$.

Let $\tilde{p}_0 \in f_1^{-1}(p_{01}) = f_2^{-1}(p_{02})$. Take a sufficiently small neighbourhood $U_{\tilde{p}_0}$ (resp. $U_{p_{0i}}$) of $\tilde{p}_0$ (resp. p_{0i}) so that there is no zero of ϕ (resp. ϕ_i) on $U_{\tilde{p}_0}$ (resp. $U_{p_{0i}}$) except $\tilde{p}_0$ (resp. p_{0i}), and that $f_i(U_{\tilde{p}_0}) \subset U_{p_{0i}}$ ($i = 1, 2$). We may take a local coordinate z (resp. z_i) on $U_{\tilde{p}_0}$ (resp. $U_{p_{0i}}$) such that $z(\tilde{p}_0) = 0$ (resp. $z_i(p_{0i}) = 0$) and the differential is written as

$$\phi = z^m dz \qquad (\text{resp.}\ \phi_i = z_i^{n_i} dz_i).$$

Recalling that $f_1^{-1}(p_{01}) = f_2^{-1}(p_{02})$, we see $n_1 = n_2$ and we will denote it by n for brevity. We take two real lines $\gamma_i : [0, a) \longrightarrow U_{p_{0i}}$ with $\gamma_i(t) = t \in \mathbb{R}$ in the local coordinates z_i ($i = 1, 2$). For an arbitrary $\tilde{p} \in U_{\tilde{p}_0} \setminus \{\tilde{p}_0\}$,

$$\int_0^{\tilde{p}} z^m dz = \int_0^{f_1(\tilde{p})} z_1^n dz_1 = \int_0^{f_2(\tilde{p})} z_2^n dz_2,$$

hence the number of possible positions for the set of lifts of γ_1 (thus also those of γ_2) in $U_{\tilde{p}_0}$ is at most $m + 1$. Accordingly, the total number of possible positions for the set of all the lifts of γ_1 is at most $\begin{pmatrix} 4g-4 \\ d \end{pmatrix}$.

Let $\{\tilde{p}_{0j}\}_{j=1}^N = {f_1}^{-1}(p_{01}) (= {f_2}^{-1}(p_{02}))$. Suppose that, for every $\tilde{p}_{0j} \in {f_1}^{-1}(p_{01})$, $U_{\tilde{p}_{0j}} \bigcap f_1^{-1}(\gamma_1) = U_{\tilde{p}_{0j}} \bigcap f_2^{-1}(\gamma_2)$, that is, the set of lifts of γ_1 coincide with that of γ_2. Then, it is easy to see that we can define a local conformal map $h : f_1(U_{\tilde{p}_{0j}}) \longrightarrow f_2(U_{\tilde{p}_{0j}})$ such that $h \circ f_1|_{\bigcup_j U_{\tilde{p}_{0j}}} = f_2|_{\bigcup_j U_{\tilde{p}_{0j}}}$. We want to extend it to a global conformal map from Y_1 to Y_2, and actually it is possible. Indeed, for an arbitrary point $p \in Y_1$, we will draw a curve c from p_{01} to p avoiding branch points of f_1 other than possibly at p_{01} and p. Let $\tilde{c}$ and $\tilde{c}'$ be two lifts of c by f_1. Then, we see that $f_2(\tilde{c}) = f_2(\tilde{c}')$ since $h \circ f_1$ is well-defined near $\tilde{p}_{0j}$ ($j = 1, \ldots, N$). It implies that h is well-defined on Y_1. It is easy to see that h is invertible. □

Proposition' *Let $f_i : X \longrightarrow Y_i$ be nonconstant holomorphic maps, and $\mathcal{F}_i$ be the rational representations of the endomorphisms associated with f_i ($i = 1, 2$). Suppose* *that, for some $k < 2g$,*

$$\begin{cases} {}^t\mathcal{F}_1 a_1 = \ldots = {}^t\mathcal{F}_1 a_{k-1} = 0, \\ {}^t\mathcal{F}_2 a_1 = \ldots = {}^t\mathcal{F}_2 a_{k-1} = 0, \end{cases}$$

and that there exists some integer $l > 2g-2$ such that ${}^t\mathcal{F}_1 a_k \equiv {}^t\mathcal{F}_2 a_k \,(mod.\ l)$ holds. Then ${}^t\mathcal{F}_1 a_k = {}^t\mathcal{F}_2 a_k$.

If, in addition, Y_1 and Y_2 are of the same genus γ, then the assumption $l > 2g-2$ can be replaced by $l > (2g-2)/(\gamma-1)$.

In [5], we assumed $l > (2g-2)^2$. But in Proposition', we only need $l > 2g-2$.

Proof. Let $D = \mathcal{F}_1 - \mathcal{F}_2$. Then, D is the rational representation of some endomorphism of $\mathrm{J}(X)$. By an easy calculation, we see ${}^tD' = {}^tD$. We note that ${}^tD'x, a_1, \ldots, a_{k-1}$ are linearly independent for any vector $x \in \mathbb{R}^{2g}$ if ${}^tD'x$ is not zero. Indeed, using Lemma 1 in [5], we see $({}^tD'x, a_j)_X = (x, {}^tDa_j)_X = 0$ for $j = 1, \ldots, k-1$ by the assumption. Thus, ${}^tD'x, a_1, \ldots, a_{k-1}$ are linearly independent. By the assumption, ${}^tD'a_k \equiv 0 \,(\mathrm{mod.}\ l)$ thus the vector ${}^tD'a_k$ can be written in the form ${}^tD'a_k = l \times n$, where $n \in \mathbb{Z}^{2g}$. Thus, if it is not 0, then

$$||{}^tD\,a_k|| \geq l\lambda_k.$$

We also have

$$||{}^tDa_k|| \leq ||{}^t\mathcal{F}_1 a_k|| + ||{}^t\mathcal{F}_2 a_k|| \leq d_1||a_k|| + d_2||a_k||,$$

where d_i is the degree of f_i $(i = 1, 2)$. The first inequality is just the triangle inequality, and the second one is obtained by Lemma 2.

Therefore, we have

$$||{}^tDa_k|| \leq ||a_k||(d_1 + d_2) = (d_1 + d_2)\lambda_k.$$

By Riemann-Hurwitz formula, $d_i \leq g-1$ and we see that ${}^tDa_k$ must be 0 since $l > 2(g-1)$.

A little modification of above argument lead us to the conclusion for the case Y_1 and Y_2 are of the same genus γ. $\square$

Now we will get the improved bound. Just the same consideration as in[5, p.3063], we have

$$\sharp\mathcal{I}_\gamma(X) < \sum_{d>1}(2g-2\gamma+1) \times \{(\frac{2g-2}{\gamma-1}) + 1\}^{2g} \times \begin{pmatrix} 2g-2 \\ d \end{pmatrix} \times \begin{pmatrix} 4g-4 \\ d \end{pmatrix}.$$

Observing $\begin{pmatrix} m \\ d \end{pmatrix} \leq 2^m$, we see that the right hand side is smaller than

$$\{(\frac{2g-2}{\gamma-1})+1\}^{2g} \times 2^{2g-2} \times 2^{4g-4} \times (2g-2\gamma+1)(g-\gamma)/(\gamma-1).$$

Summing up for all possible γ, we get

$$M(g) \leq (cg)^{2g}$$

for some constant c.

References

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