Singular domains in higher dimensional complex dynamics

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This article aims to extend the fundamental Cremer theorem from the iteration theory of one complex variable to the setting of higher-dimensional dynamics over more general valued fields, not necessarily C. This article is an announcement of the preprint [Oku2].

Projective spaces over valued fields. Let $K$ be a commutative algebraically closed field which is complete and nondiscrete with respect to a non-trivial absolute value (or valuation) $|\cdot|$. This $|\cdot|$ is said to be non-Archimedean if $\forall z, \forall w \in K, |z - w| \leq \max\{|z|, |w|\}$. Otherwise, $|\cdot|$ is said to be Archimedean and $K$ is then topologically isomorphic to $\mathbb{C}$ (with Hermitian norm). We extend $|\cdot|$ to $K^\ell (\ell \in \mathbb{N})$ as the maximum norm $|Z| = |Z|_\ell = \max_{j=1,\ldots,\ell} |z_j|$ for $Z = (z_1, \ldots, z_\ell)$. Let $\pi : K^{n+1} \setminus \{O\} \rightarrow \mathbb{P}^n(K)$ be the canonical projection and set $\ell(n) \in \mathbb{N}$ so that $\wedge^2 K^{n+1} \cong K^{\ell(n)}$. The chordal distance $[\cdot, \cdot]$ on $\mathbb{P}^n(K)$ is defined as

$$[z, w] := \frac{|Z \wedge W|_{\ell(n)}}{|Z|_{n+1}|W|_{n+1}},$$

where $Z \in \pi^{-1}(z), W \in \pi^{-1}(w)$ (cf. [KS]). For $z_0 \in \mathbb{P}^n(K)$ and $r > 0$, we consider the ball

$$\overline{B}(z_0, r) := \{z \in \mathbb{P}^n(K); [z, z_0] \leq r\}.$$

Nonlinearity of morphisms. Let $f : \mathbb{P}^n(K) \rightarrow \mathbb{P}^n(K)$ be a (finite) morphism, i.e., there is a homogeneous polynomial map $F : K^{n+1} \rightarrow K^{n+1}$ over $K$, which is called a lift of $f$, such that $F^{-1}(O) = \{O\}$ and satisfies

$$\pi \circ F = f \circ \pi.$$
The degree \( d = \deg f \) is that of \( F \) as homogeneous polynomial map. As in the case of \( K = \mathbb{C} \), the Fatou set \( F(f) \) is the largest open set at each point of which the family \( \{ f^k; k \in \mathbb{N} \} \) is equicontinuous.

The Julia set \( J(f) \) is defined by \( \mathbb{P}^n(K) \setminus F(f) \). In non-Archimedean case, \( J(f) \) may be empty even if \( d \geq 2 \). One of the main results is

**Theorem 1** (nonlinearity of morphisms). Let \( f : \mathbb{P}^n(K) \to \mathbb{P}^n(K) \) be a morphism of degree \( d \geq 1 \). If there are a ball \( \overline{B}(z_0, r) \subset \mathbb{P}^n(K) \) and a morphism \( g : \mathbb{P}^n(K) \to \mathbb{P}^n(K) \) such that

\[
\lim_{k \to \infty} \frac{1}{d^k} \log \sup_{\overline{B}(z_0, r)} [f^k, g] = -\infty,
\]

then either \( f \) is linear or \( J(f) = \emptyset \).

We give a few applications of Theorem 1.

**Analytic linearization over a field \( K \).** Consider the \( K \)-algebra

\[
\mathcal{O}_k \cong K \{ X_1, \ldots, X_k \} = \{ f = \sum c_I X^I; \lim_{|I| \to \infty} |c_I|^{1/|I|} =: r_f^{-1} < \infty \}
\]

of all germs of analytic functions at the origin \( O \in K^k \). Here \( I = (i_1, \ldots, i_k) \in \mathbb{Z}_{\geq 0}^k \) is a multi-index, \( X_1^{i_1} \cdots X_k^{i_k} \) is denoted by \( X^I \) and we put \( |I| := i_1 + \cdots + i_k \). For germ of analytic map \( \phi = (f_1, \ldots, f_n) \in (\mathcal{O}_n)^n \), we identify the linear part of \( \phi - \phi(O) \) at \( O \) with

\[
A_\phi := \left( \frac{\partial f_i}{\partial X_j}(O) \right)_{i,j=1,\ldots,n} \in M(n, K) \cong \text{End}(K^n).
\]

We also denote the operator norm on \( M(n, K) \) by \( | \cdot | \).

A germ \( \phi = (f_1, \ldots, f_n) \in (\mathcal{O}_n)^n \) fixing \( O \) is (analytically) linearizable if there is \( H \in (\mathcal{O}_n)^n \) fixing \( O \) such that \( A_H = I_n \) (unit matrix) and \( H \) satisfies the Schröder (or Poincaré) equation

\[
\phi \circ H = H \circ A_\phi.
\]

From Siegel and Sternberg ([Sie], [Ste]) and its non-Archimedean version by Herman-Yoccoz [HY], \( \phi \) is linearizable if \( A_\phi \) is diagonalizable and its eigenvalues \( \lambda_1, \ldots, \lambda_n \) satisfy the Diophantine condition: there exist \( C > 0 \) and \( \beta \geq 0 \) such that for every \( I \in \mathbb{Z}_{\geq 0}^n \) (multi-index) with \( |I| \geq 1 \),

\[
|(\lambda_1, \ldots, \lambda_n)^I - 1| \geq \frac{C}{|I|^\beta}.
\]
On the other hand, consider an inverse of a coordinate chart
\[ \sigma : K^n \ni (z_1, \ldots, z_n) \mapsto (1 : z_1 : \cdots : z_n) \in \mathbb{P}^n(K). \]
When a morphism \( f : \mathbb{P}^n(K) \to \mathbb{P}^n(K) \) fixes a point \( z_0 \in \mathbb{P}^n(K) \), assuming that \( z_0 = \sigma(O) \) without loss of generality, we say \( f \) to be \textit{linearizable} at \( z_0 \) if the germ \( \phi_f \in (\mathcal{O}_n)^n \) of the analytic map \( \sigma^{-1} \circ f \circ \sigma : \mathbb{P}^n(O, r) \to K^n \) is linearizable. The following is regarded as a higher dimensional version of the Cremer condition [Cre, p. 157].

**Theorem 2** (nonresonance). Let \( f : \mathbb{P}^n(K) \to \mathbb{P}^n(K) \) be a morphism of degree \( d \geq 2 \) which fixes \( z_0 \in \mathbb{P}^n(K) \), and suppose that \( J(f) \neq \emptyset \). If \( f \) is linearizable at \( z_0 \) and \( |A_{\phi_f}| \leq 1 \), then
\[ \liminf_{k \to \infty} \frac{1}{d^k} \log |(A_{\phi_f})^k - I_n| > -\infty. \]
If in addition \( A_{\phi_f} \) is diagonalizable, then its eigenvalues \( \lambda_1, \ldots, \lambda_n \) satisfy
\[ \liminf_{k \to \infty} \frac{1}{d^k} \log \max_{j=1,\ldots,n} |\lambda_j^k - 1| > -\infty. \]

**Singular domain over the field** \( \mathbb{C} \). Let \( f : \mathbb{P}^n = \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^n \) be a morphism, which is now holomorphic, of degree \( d \geq 2 \).

Each component \( D \) of \( F(f) \), which is called a \textit{Fatou component} of \( f \), is Stein and Kobayashi hyperbolic [Ued1]. In particular, \( D \) is holomorphically separable and the biholomorphic automorphisms \( \text{Aut}(D) \) is a Lie group. When there is a sequence \( (f^{kj}) \subset \{f^k\} \) which converges to \( \text{Id}_D \) locally uniformly on \( D \), we have \( f^p|D = D \) for some \( p \in \mathbb{N} \) and moreover \( f^q|D \in \text{Aut}(D) \). Following Fatou [Fat, §28], we call such \( D \) a \textit{singular domain} (un domaine singulier) of \( f \). A singular domain is also called a \textit{Siegel domain} or rotation domain. When \( n = 1 \), a singular domain \( D \) is either a Siegel disk or an Herman ring. When \( n \geq 2 \), a partial analogue is known: let \( G \) be the closed subgroup generated by \( f^p|D \) in \( \text{Aut}(D) \), and \( G_0 \) the component of \( G \) containing \( \text{Id}_D \). Then there is a Lie group isomorphism \( G_0 \to \mathbb{T}^s \) for some \( s \in [1, n] \), which maps \( f^q|D \) for some \( q \in \mathbb{N} \) to \( (e^{2\pi i \alpha_1}, \ldots, e^{2\pi i \alpha_s}) \) for some \( \alpha_1, \ldots, \alpha_s \in \mathbb{R} \setminus \mathbb{Q} \) (see [FS1], [Ued2], [Mih]). In the maximal case of \( s = n \), we say the singular domain \( D \) to be of \textit{maximal type}.

A singular domain \( D \) of maximal type is exactly a generalization of one-dimensional Siegel disks and Herman rings: setting \( \lambda_j := e^{2\pi i \alpha_j} \) (\( j = 1, \ldots, n \)), we have by [BBD, Theorem 1] a biholomorphic homeomorphism \( \Phi \) from a Reinhardt domain \( U \subset \mathbb{C}^n \) to \( D \) such that the Schröder equation
\[ f^q(\Phi(w_1, \ldots, w_n)) = \Phi(\lambda_1 w_1, \ldots, \lambda_n w_n) \quad \text{on} \quad U \]
holds.
Theorem 3 (a priori bound). Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a holomorphic map of degree $d \geq 2$. If a singular domain $D$ of $f$ is of maximal type, then under the same notation as in the above, $D$ satisfies

$$\lim_{k \to \infty} \frac{1}{d^k} \log \max_{j=1, \ldots, n} |\lambda_j^k - 1| = 0.$$ 

In the case of $n = 1$, every singular domain of $f$ is of maximal type. In this case, Theorem 3 is essentially proved in [FS2, p. 169] by pluripotential theory, and in [Oku1, Main Theorem 3] by a Nevanlinna theoretical argument. Both proofs contain some one-dimensional arguments which are not easily extended to higher dimensions. Our proof of Theorem 3 is based on a proof of Theorem 1, which dispenses with pluripotential theory.

Finally, we give a vanishing result on the Valiron deficiency

$$\delta_V(\text{Id}_{\mathbb{P}^n}, (f^k)) := \limsup_{k \to \infty} \frac{1}{d^k} \int_{\mathbb{P}^n} \log \frac{1}{[f^k, \text{Id}]} d\omega_{FS}^n$$

(cf. [DO]). Here $\omega_{FS}$ denotes the Fubini-Study Kähler form on $\mathbb{P}^n$.

Theorem 4 (a vanishing theorem). Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a holomorphic map of degree $\geq 2$. If every singular domain of $f$ is of maximal type, then

$$\delta_V(\text{Id}_{\mathbb{P}^n}, (f^k)) = 0.$$ 

We expect that the assertion of Theorem 4 still remains true with no maximality assumption on singular domains.

References


