数理解析研究所講究録
第 1661 巻 2009 年 104-114

GLOBAL GENERATION OF THE DIRECT IMAGES OF RELATIVE PLURICANONICAL SYSTEMS

Hajime TSUJI*

April 24, 2009

Abstract

In this paper I summerize the results and the outline of the proofs in [T8, T9]. The new feature here is the use of Monge-Ampère foliation associated with the curvature current arising from the canonical measures.

1 Introduction

Let $f : X \longrightarrow Y$ be a surjective projective morphism of smooth projective varieties with connected fibers. In this paper we shall call such a fiber space an **algebraic fiber space** for simplicity. We set $K_{X/Y} := K_X \otimes f^*K_Y^{-1}$ and call it the relative canonical line bundle of $f : X \longrightarrow Y$.

Let $f : X \to Y$ be an algebraic fiber space. It is well known that the direct image $f_*\mathcal{O}_X(mK_{X/Y})$ is semipositive for every $m \geqq 1$ in certain algebraic senses (cf. [Ka1, Ka3, V1, V2]). In this paper, we shall discuss the result in [T9] which proves that $f_*\mathcal{O}_X(m!K_{X/Y})$ is globally generated on the complement of the discriminant locus of f for every sufficiently large m ([T9]). The proof uses the results in [T7, T8].

The main difficulty to prove the global generation is the fact that the direct image $f_*\mathcal{O}_X(mK_{X/Y})$ is only semipositive and not strictly positive (= ample) in general. In the former approach due to Y. Kawamata and E. Viehweg their semipositity is the weak semipositivity which corresponds to the nefness in the case of line bundles. Since the weak semipositivity is rather weak, I have strengthen it to the curvature semipositivity in the sense of current in [T8] including the case of KLT pairs. This new semipositivity is the crucial tool to prove the global generation.

The idea to prove the global generation of $f_*\mathcal{O}_X(m!K_{X/Y})$ is to distinguish the null direction of the positivity of $f_*\mathcal{O}_X(m!K_{X/Y})$ as a Monge-Ampère foliation and realize the direct image $f_*\mathcal{O}_X(m!K_{X/Y})$ (or its certain symmetric power) as the pull back of an ample vector bundle on a certain moduli space via the moduli map.

Here we note that the curvature semipositivity plays an essential role to define the Monge-Ampère foliations.

*Partially supported by Grant-in-Aid for Scientific Reserch (S) 17104001

This paper is a reserch announcement of my articles ([T8, T9]). For the detail see [T8, T9].

2 Analytic Zariski decompositions

To state the main result, we introduce the notion of analytic Zariski decompositions.

Definition 2.1 *Let M be a compact complex manifold and let L be a holomorphic line bundle on M. A singular hermitian metric h on L is said to be an* **analytic Zariski decomposition(AZD** *in short), if the followings hold.*

1. *Θ_h is a closed positive current.*
2. *for every $m \geq 0$, the natural inclusion:*

$$H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)) \to H^0(M, \mathcal{O}_M(mL)) \tag{2.1}$$

is an isomorphim. □

Remark 2.2 *If an AZD exists on a line bundle L on a smooth projective variety M, L is pseudoeffective by the condition 1 above.* □

It is known that for every pseudoeffective line bundle on a compact complex manifold, there exists an AZD on F (cf. [T1, T2, D-P-S]). The advantage of the AZD is that we can handle pseudoeffective line bundle L on a compact complex manifold X as a singular hermitian line bundle with semipositive curvature current as long as we consider the ring $R(X, L) := \oplus_{m \geqq 0} H^0(X, \mathcal{O}_X(mL))$.

3 Statement of the main results

Now we state the main results

Theorem 3.1 *Let $f : X \longrightarrow Y$ be an algebraic fiber space and let Y° be the complement of the discriminat locus of f in Y. Then we have the followings :*

(1) **Global generation**: *There exist positive integers b and m_0 such that for every integer m satisfying $b \,|\, m$ and $m \geqq m_0$, $f_*\mathcal{O}_X(mK_{X/Y})$ is globally generated over Y°.*

(2) **Weak semistability 1**: *Let r denote* rank $f_*\mathcal{O}_X(mK_{X/Y})$ *and let $X^r := X \times_Y X \times_Y \cdots \times_Y X$ be the r-times fiber product over Y. Let $f^r : X^r \longrightarrow Y$ be the natural morphism.*

Let $\Gamma \in |mK_{X^r/Y} - f^{r*} \det f_*\mathcal{O}_X(mK_{X/Y})|$ be the effective divisor corresponding to the canonical inclusion :

$$f^{r*}(\det f_*\mathcal{O}_X(mK_{X/Y})) \hookrightarrow f^{r*} f^r_* \mathcal{O}_{X^r}(mK_{X^r/Y}) \hookrightarrow \mathcal{O}_{X^r}(mK_{X^r/Y}). \tag{3.1}$$

Then Γ does not contain any fiber $X^r_y (y \in Y^\circ)$ such that if we we define the number δ_0 by

$$\delta_0 := \sup\{\delta \,|\, (X^r_y, \delta \cdot \Gamma_y) \text{ is KLT for all } y \in Y^\circ\}, \tag{3.2}$$

then for every $\varepsilon < \delta_0$

$$(3.3) \qquad f_*\mathcal{O}_X(mK_{X/Y}) \succeq \frac{m\varepsilon}{(1+m\varepsilon)r} \det f_*\mathcal{O}_X(mK_{X/Y})$$

holds over Y°, where $\succeq$ denotes that the fractional sheaf

$$f_*\mathcal{O}_X(mK_{X/Y}) \otimes \det f_*\mathcal{O}_X(mK_{X/Y})^{-\frac{m\varepsilon}{(1+m\varepsilon)r}}$$

is weakly positive ([V1]).

(3) **Weak semistability 2**: *There exists a singular hermitian metric $H_{m,\varepsilon}$ on $(1+m\varepsilon)K_{X^r/Y} - \varepsilon \cdot f^{r*} \det f_*\mathcal{O}_X(mK_{X/Y})^{**}$ such that*

   (a) $\sqrt{-1}\,\Theta_{H_{m,\varepsilon}} \geqq 0$ holds on X^r in the sense of current.

   (b) For every $y \in Y^\circ$, $H_{m,\varepsilon}|X_y^r$ is well defined and is an AZD (cf. Definition 2) of

$$(3.4) \qquad (1+m\varepsilon)K_{X^r/Y} - \varepsilon \cdot f^{r*} \det f_*\mathcal{O}_X(mK_{X/Y})^{**}|X_y.$$

□

Remark 3.2 *The 3rd assertion implies the 2nd assertion.* □

The major advantage of Theorem 3.1 is that in Theorem 3.1 $f_*\mathcal{O}_X(mK_{X/Y})$ is globally generated over the complement of the discriminant locus of f, while the former results [Ka1, Ka3, V1, V2] imply the weak semipositivity of $f_*\mathcal{O}_X(mK_{X/Y})$.

We also have the following log version of Theorem 3.1.

Theorem 3.3 *Let $f : X \longrightarrow Y$ be an algebraic fiber space and let D be an effective $\mathbb{Q}$ divisor on X such that (X, D) is KLT. Let Y° denote the complement of the discriminant locus of f. We set*

$$(3.5) \qquad Y_0 := \{y \in Y | y \in Y^\circ, (X_y, D_y) \text{ is a KLT pair}\}$$

(1) **Global generation**: *There exist positive integers b and m_0 such that for every for every integer m satisfying $b\,|\,m$ and $m \geqq m_0$, $m(K_{X/Y}+D)$ is Cartier and $f_*\mathcal{O}_X(m(K_{X/Y}+D))$ is globally generated over Y_0.*

(2) **Weak semistability 1**: *Let r denote* rank $f_*\mathcal{O}_X(\lfloor m(K_{X/Y}+D)\rfloor)$. *Let $X^r := X \times_Y X \times_Y \cdots \times_Y X$ be the r-times fiber product over Y and let $f^r : X^r \longrightarrow Y$ be the natural morphism. And let D^r denote the divior on X^r defined by $D^r = \sum_{i=1}^r \pi_i^* D$, where $\pi_i : X^r \longrightarrow X$ denotes the projection: $X^r \ni (x_1, \cdots, x_n) \mapsto x_i \in X$.*

   There exists a canonically defined effective divisor Γ (depending on m) on X^r which does not contain any fiber $X_y^r (y \in Y^\circ)$ such that if we we define the number δ_0 by

$$(3.6) \qquad \delta_0 := \sup\{\delta \,|\, (X_y^r, D_y^r + \delta\Gamma_y) \text{ is KLT for all } y \in Y^\circ\},$$

   then for every $\varepsilon < \delta_0$

$$(3.7) \quad f_*\mathcal{O}_X(\lfloor m(K_{X/Y}+D)\rfloor) \succeq \frac{m\varepsilon}{(1+m\varepsilon)r} \det f_*\mathcal{O}_X(\lfloor m(K_{X/Y}+D)\rfloor)$$

   holds over Y_0.

(3) **Weak semistability 2**: *There exists a singular hermitian metric* $H_{m,\varepsilon}$ *on*

$$(3.8) \qquad (1+m\varepsilon)(K_{X^r/Y}+D^r)-\varepsilon\cdot f^*\det f_*\mathcal{O}_X(\lfloor m(K_{X/Y}+D)\rfloor)^{**}$$

such that

(a) $\sqrt{-1}\,\Theta_{H_{m,\varepsilon}} \geqq 0$ *holds on* X *in the sense of current.*

(b) *For every* $y \in Y_0$, $H_{m,\varepsilon}|X_y^r$ *is well defined and is an AZD of*

$$(3.9) \qquad (1+m\varepsilon)(K_{X^r/Y}+D^r)-\varepsilon\cdot f^{r*}\det f_*\mathcal{O}_X(\lfloor m(K_{X/Y}+D)\rfloor)^{**}|X_y$$

□

The main ingredient of the proof of Theorems 3.1 and 3.3 is the plurisubharmonic variation property of canonical measures ([T7]). The new feature of the proof is the use of the Monge-Ampère foliations arising from the canonical measures and the weak semistability of the direct images of relative pluricanonical systems. One may consider these new tools as substitutes of the local Torelli theorem for minimal models with semiample canonical divisors in [Ka2].

The scheme of the proof is as follows. For an algebraic fiber space $f : X \longrightarrow Y$ with $\mathrm{Kod}(X/Y) \geqq 0$, we take the relative canonical measure $d\mu_{can,X/Y}$. Then the null distribution of the curvature $\Theta_{d\mu^{-1}_{can,X/Y}}$ of the singular hermitian metric $d\mu^{-1}_{X/Y}$ on $K_{X/Y}$ defines a singular Monge-Ampère foliation on X. The important fact here is that the leaf of the foliation is complex analytic ([B-K]) (although it is not clear that the foliation itself is complex analytic apriori). By using the weak semistability of $f_*\mathcal{O}_X(m!K_{X/Y})$, we may prove that this singular foliation actually descends to a singular foliation $\mathcal{G}$ on Y. Let us define the (singular) hermitian metric h_m on $f_*\mathcal{O}_X(m!K_{X/Y})$ defined by

$$(3.10) \qquad h_m(\sigma,\sigma') := \int_{X/Y}\sigma\cdot\overline{\sigma'}\cdot d\mu_{X/Y}^{-(m!-1)}.$$

Then we see that $(f_*\mathcal{O}_X(m!K_{X/Y}), h_m)$ is flat along the leaves of $\mathcal{G}$ on Y. Taking m sufficiently large, we see that the relative canonical model of $f : X \longrightarrow Y$ is locally trivial along the leaves. Then we see that the leaves of $\mathcal{G}$ consists of the fiber of the moduli map to the moduli space of relative canonical models marked with the metrized Hodge line bundles. Then the global generation property of $f_*\mathcal{O}_X(mK_{X/Y})$ follows from the Nakai-Moishezon type argument.

4 Canonical measures on KLT pairs of nonnegative Kodaira dimension

In [Ka1], Kawamata proved the semipositivity of the direct image $f_*\mathcal{O}_X(mK_{X/Y})$ for an algebraic fiber space $f : X \longrightarrow Y$ over a smooth projective curve Y in the sense that every quotient of $f_*\mathcal{O}_X(mK_{X/Y})$ has semipositive degree.

Let $f : X \longrightarrow Y$ an algebraic fiber space such that there exists a a nonempty Zariski open subset Y_0 such that f is smooth over Y_0 and $K_{X/Y}$ is f-semiample over Y_0. In [V1], E. Viehweg proved that $f_*\mathcal{O}_X(mK_{X/Y})$ is weakly positive for every $m \geqq 1$ over Y_0, i.e., for every ample line bundle A and positive integer a, there exists a positive integer b such that $S^{ab}(f_*\mathcal{O}_X(mK_{X/Y})) \otimes A^b$ is globally generated over Y_0. And he also proved that $f_*\mathcal{O}_X(mK_X)$ is weakly semistable, i.e., there exists a positive rational number ϵ such that $f_*\mathcal{O}_X(mK_{X/Y}) \otimes (\det f_*\mathcal{O}_X(mK_{X/Y}))^{-\epsilon}$ is weakly positive on Y_0. Later Y. Kawamata generalized his result to the case of family of KLT pairs ([Ka3, p.175, Theorem 1.2]).

In [T8], I have refined these semipositivity as a logarithmic plurisubharmonicity of relative canonical measures. The advantage of this refinement is that we may distinguish the null direction of the semipositivity as the Monge-Ampère foliation as well as the canonicity of the metric.

Let (X, D) be a KLT pair of nonnegative Kodaira dimension, i.e., $|m!(K_X + D)| \neq \emptyset$ for every sufficiently large m.

Let $f : X - \cdots \to Y$ be the Iitaka fibration associated with the log canonical divisor $K_X + D$. By replacing X and Y by suitable modifications, we may assume the followings:

(1) X,Y are smooth and f is a morphism with connected fibers.

(2) $\operatorname{Supp} D$ is a divisor with normal crossings.

(3) There exists an effective divisor Σ on Y such that f is smooth over $Y - \Sigma$, $\operatorname{Supp} D^h$ is relatively normal crossings over $Y - \Sigma$ and $f(D^v) \subset \Sigma$, where D^h, D^v denote the horizontal and the vertical component of D respectively.

(4) There exists a positive integer m_0 such that for every $m \geqq m_0$, $m!(K_X+D)$ is Cartier and $f_*\mathcal{O}_X(m!(K_X + D))^{**}$ is a line bundle on Y, where $**$ denotes the double dual.

We note that adding effective exceptional $\mathbb{Q}$-divisors does not change the log canonical ring. Such a modification exists by [F-M, p.169,Proposition 2.2]. We define the $\mathbb{Q}$-line bundle $L_{X/Y,D}$ on Y by

$$L_{X/Y,D} = \frac{1}{m_0!} f_*\mathcal{O}_X(m_0!(K_X + D))^{**}. \tag{4.1}$$

$L_{X/Y}$ is independent of the choice of m_0. Similarly as before we may define the singular hermitian metric $h_{L_{X/Y,D}}$ on $L_{X/Y,D}$ by

$$h^{m!}_{L_{X/Y,D}}(\sigma, \sigma)(y) := \left(\int_{X_y} |\sigma|^{\frac{2}{m!}}\right)^{m!}, \tag{4.2}$$

where $y \in Y - \Sigma$ and $X_y := f^{-1}(y)$. We call the singular hermitian $\mathbb{Q}$-line bundle $(L_{X/Y,D}, h_{L_{X/Y,D}})$ **the metrized Hodge $\mathbb{Q}$-line bundle** of the Iitaka fibration $f : X \longrightarrow Y$ associated with the KLT pair (X, D). We note that since (X, D) is KLT, $h_{L_{X/Y,D}}$ is well defined. By the same strategy as in the proof of Theorem 3.1 and [T7, Theorem 1.6], we have the following theorem :

Theorem 4.1 *([T8, Theorem 1.7]) In the above notations, there exists a unique singular hermitian metric on h_K on $K_Y + L_{X/Y,D}$ and a nonempty Zariski open subset U of Y such that*

(1) h_K *is an AZD of* $K_Y + L_{X/Y,D}$.

(2) f^*h_K *is an AZD of* $K_X + D$.

(3) h_K *is* C^∞ *on* U.

(4) $\omega_Y = \sqrt{-1}\,\Theta_{h_K}$ *is a Kähler form on* U.

(5) $-\mathrm{Ric}_{\omega_Y} + \sqrt{-1}\,\Theta_{L_{X/Y,D}} = \omega_Y$ *holds on* U. □

The following theorem is the fundamental tool to prove Theorems 3.1 and Theorem 3.3.

Theorem 4.2 *[T8, Theorem 1.8] Let* $f : X \longrightarrow Y$ *be an algebraic fiber space and let* D *be an effective* $\mathbb{Q}$*-divisor on* X. *Suppose that there exists a nonempty Zariski open subset* Y_0 *of* Y *such that*

(1) f *is smooth over* Y_0,

(2) For every $y \in Y_0$, $(X_y, D_y)(X_s := f^{-1}(y), D_y := D \cap X_y)$ *is a KLT pair of nonnegative Kodaira dimension.*

Let $d\mu_{can,X/Y}$ *be the relative canonical measure defined by*

$$d\mu_{can,X/Y}|X_y := d\mu_{can,y} \quad (y \in Y_0) \tag{4.3}$$

where $d\mu_{can,y}$ *denotes the canonical measure on* $(X_y, D_y)(y \in Y_0)$ *constructed as in Theorem 4.1. Then the singular hermitian metric*

$$h_K^\circ|X_y := d\mu_{can,y}^{-1} \cdot h_{\sigma_D}|X_y \quad (y \in Y_0) \tag{4.4}$$

on $K_{X/Y} + D|f^{-1}(Y_0)$ *extends to a singular hermitian metric* h_K *on* $K_{X/Y} + D$ *and has semipositive curvature in the sense of current everywhere on* X. □

5 Special case of Theorem 3.1

Here to indicate the strategy of the proof of Theorem 3.1, we shall prove the following special case of Theorem 3.1.

Theorem 5.1 *Let* $f : X \longrightarrow Y$ *be an algebraic fiber space. Let* Y° *be the complement of the discriminant locus of* f. *Suppose that* $K_{X/Y}$ *is* f*-ample over* Y°. *Then there exists a positive integer* m_0 *such that for every* $m \geqq m_0$, $f_*\mathcal{O}_X(mK_{X/Y})$ *is globally generated over* Y°. □

Sketch of the proof of Theorem 3.1. First we note that by applying Theorem 4.2 to $K_{X^r/Y} + \varepsilon\Gamma$, we see that $f_*\mathcal{O}_X(m!K_{X/Y})$ is weakly semistable in the sense of Theorem 3.1.

Let $\omega_{X/Y}$ be the canonical relative Kähler-Einstein current on $f : X \longrightarrow Y$. Then by the implicit function theorem, we see that $\omega_{X/Y}$ is C^∞ over $X^\circ := f^{-1}(Y^\circ)$. Let n denote the relative dimension $\dim X - \dim Y$ of $f : X \longrightarrow Y$. Then the relative canonical measure

$$d\mu_{can,X/Y} := \frac{1}{n!}\omega_{X/Y}^n \tag{5.1}$$

is considered to be a relative volume form on $f : X \longrightarrow Y$. And by [T7], we see that

$$\omega_{X/Y} = -\mathrm{Ric}\, d\mu_{can,X/Y} \tag{5.2}$$

is a closed positive current on X and is C^∞ on X° by the implicit function theorem.

Now we consider the Monge-Ampère foliation

$$\mathcal{F} = \{v \in TX | \omega_{X/Y}(v, \bar{v}) = 0\}. \tag{5.3}$$

Then by the weak semistability above, we see that the foliation $\mathcal{F}$ decend to a Monge-Ampère foliation $df(\mathcal{F})$. More precisely, for $m \gg 1$, the L^2-metric

$$h_m(\sigma, \sigma') := \int_{X/Y} \sigma \cdot \bar{\sigma}' \cdot d\mu_{can,X/Y}^{-(m!-1)} \tag{5.4}$$

on $f_*\mathcal{O}_X(m!K_{X/Y})$ induces a metric $\det h_m$ on $\det f_*\mathcal{O}_X(mK_{X/Y})$ and has semipositive curvature on Y in the sense of current. And $\det h_m$ defines a Monge-Ampère foliation on Y. We see that the this foliation is nothing but $df(\mathcal{F})$. Now we shall consider the leaf L of $df(\mathcal{F})$. By [B-K], we know that L is a complex submanifold at generic point on Y. Then we see that along the leaf L, the restricted family $f|f^{-1}(L) : f^{-1}(L) \longrightarrow L$ is locally trivial as follows. First we note that

$$\mathrm{trace}\sqrt{-1}\,\Theta_{h_m} = \sqrt{-1}\Theta_{\det h_m} \tag{5.5}$$

holds and the lefthand side is semipositive. Hence $(f_*\mathcal{O}_X(m!K_{X/Y}), h_m)|L$ is flat over L. This implies that moving m we see that the relative canonical ring is locally trivialized on L, hence $f|f^{-1}(L) : f^{-1}(L) \longrightarrow L$ is locally holomorphically trivial.

Let $\mathcal{M}_{can}$ denote the moduli space of canonically polarized varieties with only canonical singularities. Then we see that the leaf L is nothing but the fiber of the moduli map:

$$\mu : Y_0 \longrightarrow \mathcal{M}_{can}. \tag{5.6}$$

Hence in particular L is closed. And the curvature current $\Theta_{\det h_m}$ decends to a closed semipositive current on the image $\mu(Y_0)$. Now we shall take a compactification $\overline{\mathcal{M}_{can}}$ of $\mathcal{M}_{can}$. This is certainly possible, since $\mathcal{M}_{can}$ is quasiprojective. We see that for some positive integer r, the r-times symmetric powers $S^r(\det f_*\mathcal{O}_X(m!K_{X/Y}))$ and $S^r(f_*\mathcal{O}_X(m!K_{X/Y}))$ to coherent sheaves $\det \mathcal{F}_m$ and $\mathcal{F}_m$ on the closure $\overline{\mu(Y_0)}$ in $\overline{\mathcal{M}_{can}}$ respectively. We note that on every irreducible (possibly incomplete) curve C in $\mu(Y_0)$ the restriction: $\mu_*(\sqrt{-1}\Theta_{\det h_m})|C$ is generically strictly positive by the argument as above. Hence by the Nakai-Moishezon type argument as in [Sch-T], we see that $\left(\det f_*\mathcal{O}_X(mK_{X/Y})\right)^{\otimes r}$ decends to an ample line bundle on $\mu(Y_0)$ and extends to a coherent sheaf $\det \mathcal{F}_m$ on the closure $\overline{\mu(Y_0)}$. Then by the weak semistability of $f_*\mathcal{O}_X(m!K_{X/Y})$ we see that $\mathcal{F}_m$ is an ample vector bundle on $\mu(Y_0)$ in the sense that it is globally generated by a global section of $\mathcal{F}_m$ on the closure $\overline{\mu(Y_0)}$. Hence some symmetric power $f_*\mathcal{O}_X(m!K_{X/Y})$ is globally generated over Y_0 for every sufficiently large m. Then by the finite generation of relative canonical bundles, we see that $f_*\mathcal{O}_X(m!K_{X/Y})$ is globally generated over Y_0 for every sufficiently large m. This completes the proof of Theorem 5.1.

□

6 Scheme of the proof of Theroems 3.1 and 3.3

Here we shall indicate the scheme of the proof for general case. Let $f : X \longrightarrow Y$ be an algebraic fiber space with nonnegative relative Kodaira dimension. Let $d\mu_{can,X/Y}$ be the relative canonical measure and we define the L^2-metric h_m on $f_*\mathcal{O}_X(m!K_{X/Y})$ similar to (5.4). Let $h : Z \longrightarrow Y$ be the relative canonical models ([B-C-H-M]). Then we have the commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f"'] & & Z \arrow[dl, "h"] \\ & Y & \end{tikzcd}$$

Taking a suitable modification we may and do assume the followings :

1. g is a morphism,
2. Z is smooth.
3. $g_*\mathcal{O}_X(m!K_{X/Z})^{**}$ is a line bundle on Z for every sufficiently large m.

Let $(L_{X/Y}, h_{L_{X/Y}}) \longrightarrow Z$ be the Hodge $\mathbb{Q}$-line bundle and let Y° be the complement of the discriminant locus of $h : Z \longrightarrow Y$. We consider the moduli space:

$$\mathcal{M} := \{[(Z_y, (L_{X/Y}, h_{L_{X/Y}})|Z_y)|y \in Y^\circ\},$$

where $[(Z_y, (L_{X/Y}, h_{L_{X/Y}})|Z_y)]$ denotes the equivalence class with respect to the equivalence relation:

$$(Z_y, (L_{X/Y}, h_{L_{X/Y}})|Z_y) \sim (Z_{y'}, (L_{X/Y}, h_{L_{X/Y}})|Z_{y'}),$$

if and only if there exists a biholomorphism $\varphi : Z_y \longrightarrow Z_{y'}$ and a bundle isomorphism $\tilde{\varphi} : aL_{X/Y}|Z_y \longrightarrow aL_{X/Y}|Z_{y'}$ such that the following commutative diagram :

$$\begin{tikzcd} aL_{X/Y}|_{Z_y} \arrow[r, "\tilde{\varphi}"] \arrow[d] & aL_{X/Y}|Z_{y'} \arrow[d] \\ Z_y \arrow[r, "\varphi"'] & Z_{y'} \end{tikzcd}$$

and

$$\tilde{\varphi}^*(h_L|Z_{y'}) = h_L|Z_y \tag{6.1}$$

holds, where a denotes the minimal positive integer such that $aL_{X/Y}$ is Cartier. We call $\mathcal{M}$ the moduli space of metrized canonical models. By the theory of variation of Hodge structures ([G]), we see that $\mathcal{M}$ has a natural algebraic space structure. We shall use $\mathcal{M}$ as the substitute of $\mathcal{M}_{can}$ in the previous section.

The relative canonical measure $d\mu_{can,X/Y}$ is C^∞ on a nonempty Zariski open subset of X by the dynamical construction of canonical measures ([T7]) and the parameter dependence of the Bergman projections.

Then we may define the (singular) Monge-Ampère foliation $\mathcal{F}$ on X associated with the closed positive current:

$$\sqrt{-1}\partial\bar{\partial}\log d\mu_{can,X/Y}.$$

Again by the weak semistability of $K_{X/Y}$, we have that $df(\mathcal{F})$ defines a (singular) foliation on Y associated with the closed positive current $\sqrt{-1}\Theta_{\det h_m}$ for every sufficiently large m. Here we have used the weak stability, since the regularity of $\det h_m$ seems to be unclear.

Then as in the previous section, we see that for any leaf L, $f|f^{-1}(L)$: $f^{-1}(L) \longrightarrow L$ has locally trivial metrized canonical model, i.e., the moduli map

$$\mu : Y_0 \longrightarrow \mathcal{M} \tag{6.2}$$

is constant on L. It is easy to see that the leaf of the foliation $df(\mathcal{F})$ is nothing but the fiber of the moduli map $\mu : Y_0 \longrightarrow \mathcal{M}$.

Now we proceed as in the last section. We see that by using the weak semistability of $f_*\mathcal{O}_X(m!K_{X/Y})$, some symmetric power of $f_*\mathcal{O}_X(m!K_{X/Y})$ decends to an ample vector bundle on $\mathcal{M}$ and is globally generated over on $\mathcal{M}$ by global sections on some compactificaion $\overline{\mathcal{M}}$. Hence again by finite generation of canonical rings ([B-C-H-M]), we conclude that $f_*\mathcal{O}_X(m!K_{X/Y})$ is globally generated over Y_0 for every sufficiently large m. This completes the proof of Theorem 3.1.

The proof of Theorem 3.3 is quite similar. □

Remark 6.1 *If the general fiber of $f : X \longrightarrow Y$ is of general type, then $\mathcal{M}$ is nothing but the moduli space of the canonical models of the fibers. Hence in particular we obtain that the moduli space of the canonical models of general type is quasiprojective. This gives an alternative proof of this result in [V1, V2].*

□

References

[Au] Aubin, T.: Equation du type Monge-Ampère sur les varieté kählerienne compactes, C.R. Acad. Paris **283** (1976), 459-464.

[B-K] Bedford, E. and Kalka, M.: Foliations and complex Monge-Amp‘ere equations, Comm. Pure and Appl. Math. **30**(1977), 543-571.

[B1] Berndtsson, B.: Subharmonicity properties of the Bergman kernel and some other functions associated to pseudoconvex domains, math.CV/0505469 (2005).

[B2] Berndtsson, B.: Curvature of vector bundles and subharmonicity of vector bundles, math.CV/050570 (2005).

[B3] Berndtsson, B.: Curvature of vector bundles associated to holomorphic fibrations, math.CV/0511225 (2005).

[B-P] Berndtsson, B. and Paun, M. : Bergman kernels and the pseudoeffectivity of relative canonical bundles, math.AG/0703344 (2007).

[B-C-H-M] Birkar, C.-Cascini, P.-Hacon,C.-McKernan, J.: Existence of minimal models for varieties of log general type, arXiv:math/0610203.

[D-P-S] Demailly, J.P.- Peternell, T.-Schneider, M. : Pseudo-effective line bundles on compact Kähler manifolds, International Jour. of Math. **12** (2001), 689-742.

[F-M] Fujino, O. and Mori, S.: Canonical bundle formula, J. Diff. Geom. **56** (2000), 167-188.

[G] Griffiths, Ph.: Periods of integrals on algebraic manifolds III: Some global differential-geometric properties of the period mapping, Publ. Math., Inst. Hautes Etud. Sci. **38** 125–180 (1970).

[Ka1] Kawamata, Y.: Kodaira dimension of Algebraic fiber spaces over curves, Invent. Math. **66** (1982), pp. 57-71.

[Ka2] Kawamata, Y., Minimal models and the Kodaira dimension, Jour. für Reine und Angewande Mathematik **363** (1985), 1-46.

[Ka3] Kawamata, Y.: On effective nonvanishing and base point freeness, Kodaira's issue, Asian J. Math. **4**, (2000), 173-181.

[N] Nadel, A.M.: Multiplier ideal sheaves and existence of Kähler-Einstein metrics of positive scalar curvature, Ann. of Math. **132**(1990),549-596.

[Sch] Schmid, W.: Variation of Hodge structure: the singularities of the period mapping. Invent. math. **22**, 211–319 (1973).

[S-T] Song, J. and Tian, G. : Canonical measures and Kähler-Ricci flow, math. ArXiv0802.2570 (2008).

[Sch-T] Schumacher, G.-Tsuji, H.: Quasiprojectivity of the moduli space of polarized projective manifolds, Ann. of Math **156** (2004).

[T0] Tsuji H.: Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type. Math. Ann. **281** (1988), no. 1, 123–133.

[T1] Tsuji H.: Analytic Zariski decomposition, Proc. of Japan Acad. **61**(1992), 161-163.

[T2] Tsuji, H.: Existence and Applications of Analytic Zariski Decompositions, Trends in Math., Analysis and Geometry in Several Complex Variables(Katata 1997), Birkhäuser Boston, Boston MA.(1999), 253-272.

[T3] Tsuji, H.: Deformation invariance of plurigenera, Nagoya Math. J. **166** (2002), 117-134.

[T4] Tsuji, H.: Dynamical construction of Kähler-Einstein metrics, math.AG/0606023 (2006).

[T5] Tsuji, H.: Canonical singular hermitian metrics on relative canonical bundles, math.ArXiv0704.0566 (2007).

[T6] Tsuji, H.: Extension of log pluricanonical forms from subvarieties, math.ArXiv0709.2710 (2007).

[T7] Tsuji, H.: Canonical measures and dynamical systems of Bergman kernels, arXjv.0805.1829 (2008).

[T8] Tsuji, H.: Ricci iterations and canonical Kähler-Einstein currents on log canonical pairs, math.ArXiv0903.5445 (2009).

[T9] Tsuji, H.: Global generation of the direct images of pluri log canonical bundles, manuscript (2009).

[V1] Viehweg, E.: Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces. In: Algebraic Varieties and Analytic Varieties, Advanced Studies in Pure Math. **1**(1983), 329-353. II. The local Torelli map. In: Classification of Algebraic and Analytic Manifolds, Progress in Math. **39**(1983), 567-589.

[V2] Viehweg, E.: Quasi-projective Moduli for Polarized Manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge. Band 30 (1995).

[Y1] Yau, S.-T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, Comm. Pure Appl. Math. **31** (1978),339-441.

[Y2] Yau, S.-T.: A general Schwarz lemma for Kähler manifolds, Amer. J. of Math. **100** (1978), 197-303.

Author's address
Hajime Tsuji
Department of Mathematics
Sophia University
7-1 Kioicho, Chiyoda-ku 102-8554
Japan
e-mail address: tsuji@mm.sophia.ac.jp or h-tsuji@h03.itscom.net