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Hypergeometric groups for complete intersections associated to Calabi–Yau varieties in weighted projective spaces

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Abstract

Let Y be a smooth Calabi–Yau complete intersection in a weighted projective space. We show that the space of quadratic invariants of the hypergeometric group associated with the mirror manifold X_t of Y in the sense of Batyrev and Borisov is one-dimensional and spanned by the Gram matrix of a classical generator of the derived category of coherent sheaves on Y with respect to the Euler form. This is a part of collaboration with Kazushi Ueda.

1 Introduction

Let $(q_0, \dots, q_N)$ and $(d_1, \dots, d_r)$ be sequences of positive integers such that

$$Q := q_0 + \cdots + q_N = d_1 + \cdots + d_r,$$

and consider a smooth complete intersection Y of degree $(d_1, \dots, d_r)$ in the weighted projective space $\mathbb{P} = \mathbb{P}(q_0, \dots, q_N)$. It is a Calabi–Yau manifold of dimension $n = N - r \geq 1$. The derived category $D^b \operatorname{coh} \mathbb{P}$ of coherent sheaves is known [2, 1] to have a full strong exceptional collection

$$(\widetilde{\mathcal{E}}_i)_{i=Q}^{1} = (\mathcal{O}_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(1), \dots, \mathcal{O}_{\mathbb{P}}(Q)).$$

Let $(\widetilde{\mathcal{F}}_i)_{i=1}^{Q}$ be the full exceptional collection dual to $(\widetilde{\mathcal{E}}_i)_{i=Q}^{1}$ so that

$$\chi(\widetilde{\mathcal{E}}_i, \widetilde{\mathcal{F}}_j) = \delta_{ij},$$

where

$$\chi(\mathcal{E}, \mathcal{F}) = \sum_k (-1)^k \dim \operatorname{Ext}^k(\mathcal{E}, \mathcal{F}),$$

is the Euler form.

A set $\{\mathcal{F}_i\}_{i=1}^{Q}$ of objects in a triangulated category $\mathcal{D}$ is said to be a *classical generator* if $\mathcal{D}$ is the smallest subcategory containing $\{\mathcal{F}_i\}_{i=1}^{Q}$ which is closed under shifts, cones and direct summands [5]. When $\mathcal{D}$ is the derived category $D^b \operatorname{coh} Y$ of coherent sheaves on

Y, the set $\{\overline{\mathcal{F}}_i\}_{i=1}^{Q}$ of restrictions $\overline{\mathcal{F}}_i$ of $\widetilde{\mathcal{F}}_i$ to Y is a classical generator by Kontsevich, as explained in Seidel [18, Lemma 5.4].

The mirror of Y is identified by Batyrev and Borisov [3] as a toric complete intersection whose affine part is given by

$$X_t = \{(x_0, \cdots, x_N) \in \mathbb{T}^{N+1} \mid f_0(x) + t = 0, f_1(x) + 1 = 0, \ldots, f_r(x) + 1 = 0\}, \quad (1)$$

where

$$f_0(x) = x_0^{q_0} x_1^{q_1} \ldots x_N^{q_N}$$

and

$$f_k(x) = \sum_{i \in S_k} x_i$$

for $1 \le k \le r$. Here

$$\{0, 1, \ldots, N\} = S_1 \amalg \cdots \amalg S_r$$

is a partition of $\{0, 1, \ldots, N\}$ into r disjoint subsets such that

$$d_k = \sum_{i \in S_k} q_i.$$

The period integral

$$I(t) = \int_\gamma \frac{x_0^{q_0} \ldots x_N^{q_N}}{df_0 \wedge \cdots \wedge df_r} \frac{dx_0}{x_0} \wedge \cdots \wedge \frac{dx_n}{x_n} \quad (2)$$

of the holomorphic volume form on X_t for a middle-dimensional (vanishing) cycle $\gamma \in H_n(X_t)$, $\boldsymbol{q} = (q_0, q_2, \cdots, q_N)$ and $\mathbf{1} = (1, 1, \cdots, 1)$ satisfies the hypergeometric differential equation

$$\left[\prod_{\nu=0}^{N} \prod_{a_0}^{q_\nu - 1} (q_\nu \theta_t - a) - t \prod_{k=1}^{r} \prod_{b=1}^{d_k} (d_k \theta_t + b)\right] I = 0. \quad (3)$$

where $\theta_t = t\frac{\partial}{\partial t}$. We remark that the submodule of $H_n(X_t)$ consisting of its vanishing cycles has rank Q. Define the *hypergeometric group* $H(q_0, \ldots, q_N; d_1, \ldots, d_r)$ as the subgroup of $GL(Q, \mathbb{Z})$ generated by

$$h_0 = \begin{pmatrix} 0 & 0 & \ldots & 0 & -A_Q \\ 1 & 0 & \ldots & 0 & -A_{Q-1} \\ 0 & 1 & \ldots & 0 & -A_{Q-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -A_1 \end{pmatrix} \quad (4)$$

and

$$h_\infty^{-1} = \begin{pmatrix} 0 & 0 & \ldots & 0 & -B_Q \\ 1 & 0 & \ldots & 0 & -B_{Q-1} \\ 0 & 1 & \ldots & 0 & -B_{Q-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -B_1 \end{pmatrix}, \quad (5)$$

where

$$\prod_{k=1}^{r}(\lambda^{d_k}-1)=\lambda^Q+A_1\lambda^{Q-1}+A_2\lambda^{Q-2}+\cdots+A_Q \tag{6}$$

and

$$\prod_{\nu=0}^{N}(\lambda^{q_\nu}-1)=\lambda^Q+B_1\lambda^{Q-1}+B_2\lambda^{Q-2}+\cdots+B_Q \tag{7}$$

are characteristic polynomial of the monodromy at zero and infinity. When the monodromy representation of a Pochhammer hypergeometric equation is irreducible, Levelt [15] shows that the monodromy group is conjugate to the hypergeometric group $(H(q_0,\dots,q_N;d_1,\dots,d_r)$. Especially when the roots of the characteristic polynomial at $t=0$ and $t=\infty$ are mutually distinct, the irreducibility of the monodromy is ensured [4, Theorem 3.5]. Although the monodromy representation of (3) is reducible, we show in section 2 that the monodromy group of (3) coincides with $H(q_0,\dots,q_N;d_1,\dots,d_r)$:

Theorem 1. *For any sequences $(d_1,\dots,d_r)$ and $(q_0,\dots,q_N)$ of positive integers such that*

$$Q:=q_0+\cdots+q_N=d_1+\cdots+d_r$$

and $N-r\geq 1$, the monodromy group of (3) is given by the hypergeometric group $H(q_0,\dots,q_N;d_1,\dots,d_r)$.

An element $h\in H(q_0,\dots,q_N;d_1,\dots,d_r)$ acts naturally on the space of $Q\times Q$ matrices by

$$H(q_0,\dots,q_N;d_1,\dots,d_r)\ni h:X\mapsto h\cdot X\cdot h^T,$$

where h^T is the transpose of h. We prove the following in sections 3 and 4:

Theorem 2. *The space of matrices invariant under the action of $H(q_0,\dots,q_N;d_1,\dots,d_r)$ is one-dimensional and spanned by the Gram matrix*

$$\left(\chi(\overline{\mathcal{F}}_i,\overline{\mathcal{F}}_j)\right)_{i,j=1}^{Q}$$

of the classical generator $\{\overline{\mathcal{F}}_i\}_{i=1}^{Q}$ with respect to the Euler form.

This theorem is a variation of theorems of Horja [11, Theorem 4.9], which he attributes to Kontsevich, and of Golyshev [9, §3.5]. The main difference between their result and ours is in the rank of the hypergeometric differential equation, which is Q in our case and $n+1<Q$ in their case, that corresponds to the rank of the submodule of vanishing cycles of X_t that survive after its compactification.

2 Monodromy of hypergeometric equation

Let h_0, h_1 and h_∞ be the global monodromy matrix of the hypergeometric differential equation (3) around the origin, one and infinity with respect to some basis of solutions. Recall that a vector $v\in\mathbb{C}^Q$ is said to be *cyclic* with respect to $h\in GL(Q,\mathbb{C})$ if the set $\{h^i\cdot v\}_{i=0}^{Q-1}$ spans $\mathbb{C}^Q$. The following lemma is used by Levelt [15] to compute the monodromy of hypergeometric functions (see also Beukers and Heckman [4, Theorem 3.5]).

Lemma 3. *Assume that there exists a vector satisfying*

$$h_0^i v = h_\infty^{-i} v, \qquad i = 0, 1, \ldots, Q-2, \tag{8}$$

which is cyclic with respect to h_0. *Then the monodromy group of* (3) *is isomorphic to* $H(q_0, \ldots, q_N; d_1, \ldots, d_r)$.

Proof. The condition (8) shows that the action of h_0 and h_∞^{-1} with respect to the basis $\{h_\infty^{-i} v\}_{i=0}^{Q-1}$ of $\mathbb{C}^Q$ is given by

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & * \\ 1 & 0 & \ldots & 0 & * \\ 0 & 1 & \ldots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & * \end{pmatrix}.$$

The last line is determined by the characteristic equations

$$\det(\lambda - h_0) = \lambda^Q + A_1 \lambda^{Q-1} + A_2 \lambda^{Q-2} + \cdots + A_Q$$

and

$$\det(\lambda - h_\infty^{-1}) = \lambda^Q + B_1 \lambda^{Q-1} + B_2 \lambda^{Q-2} + \cdots + B_Q.$$

□

Hence the proof of Theorem 1 is reduced to the following:

Proposition 4. *There exists a vector* v *in the space of solutions of* (3) *which is cyclic with respect to* h_0 *and satisfies* (8).

The rest of this section is devoted to the proof of Proposition 4. The hypergeometric differential equation (3) has regular singularities at $t = 0, \infty$ and $\lambda = \prod_{i=0}^N q_i^{q_i} \Big/ \prod_{k=1}^r d_k^{d_k}$. To simplify notations, we introduce another variable z by $t = \lambda z$. Then the local exponents are given by

$$\begin{array}{llll} \dfrac{b}{d_k}, & k = 1, \ldots, r, \quad b = 1, \ldots, d_k & & \text{at } z = \infty, \\ \dfrac{a}{q_\nu}, & \nu = 1, \ldots, N, \quad a = 0, \ldots, q_\nu - 1 & & \text{at } z = 0, \text{ and} \\ 0,\ 1,\ 2, \ldots, Q-2, \ \dfrac{n-1}{2} & & & \text{at } z = 1. \end{array} \tag{9}$$

Let

$$1 > \rho_1 > \rho_2 > \cdots > \rho_p = 0$$

be the characteristic exponents of (3) at $z = 0$ so that

$$\{\rho_1, \cdots, \rho_p\} = \bigcup_{0 \le \nu \le N} \left\{0, \frac{1}{q_\nu}, \ldots, \frac{q_\nu - 1}{q_\nu}\right\}.$$

Let further

$$\mu_\alpha = \# \left\{ (q_\nu, a) \,\middle|\, \rho_\alpha = \frac{a}{q_\nu}, \quad 0 \le a \le q_\nu - 1, \quad 0 \le \nu \le N \right\}$$

be the multiplicity of the exponent ρ_α and put

$$e_\alpha = \exp(2\pi\sqrt{-1}\rho_\alpha), \qquad 1 \le \alpha \le p.$$

Introduce the matrices

$$M_0 = \begin{pmatrix} \rho_1 \,\mathrm{id}_{\mu_1} + J_{\mu_1,-} & 0 & \cdots & 0 \\ 0 & \rho_2 \,\mathrm{id}_{\mu_2} + J_{\mu_2,-} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho_p \,\mathrm{id}_{\mu_p} + J_{\mu_p,-} \end{pmatrix}$$

$$E_0 = \begin{pmatrix} e_1 \,\mathrm{id}_{\mu_1} + J_{\mu_1,-} & 0 & \cdots & 0 \\ 0 & e_2 \,\mathrm{id}_{\mu_2} + J_{\mu_2,-} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_p \,\mathrm{id}_{\mu_p} + J_{\mu_p,-} \end{pmatrix}$$

where $J_{i,-}$ is a $i \times i$ matrix defined by

$$J_{i,-} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

As the two matrices $e^{2\pi\sqrt{-1}M_0}$ and E_0 have the same Jordan normal form, we see easily the following statement:

Corollary 5. *There is a basis*

$$\boldsymbol{X}(z) = (X_1(z), \ldots X_Q(z))$$

of solutions to (3) *such that the monodromy around* $z = 0$ *is given by*

$$\boldsymbol{X}(z) \to \boldsymbol{X}(z) \cdot E_0.$$

Define σ_α by

$$\sigma_i = \sum_{\alpha=1}^{i} \mu_\alpha$$

for $i = 1, \ldots, p$. Here we remark that we have chosen the above basis in such a way that $z^{-\rho_i} X_{\sigma_i}(z)$ were holomorphic at $z = 0$.

Lemma 6. $X_{\sigma_i}(z)$ *is singular at* $z = 1$ *for any* $1 \le i \le p$.

Proof. Assume that $X_{\sigma_i}(z)$ is holomorphic at $z = 1$. Since $X_{\sigma_i}(z)$ is a solution to (3), its only possible singular points on $\mathbb{C}$ are $z = 0$ and 1, so that $z^{-\rho_i} X_{\sigma_i}(z)$ in fact turns out to be an entire function. Since (3) has a regular singularity at infinity, $X_{\sigma_i}(z)$ has at most polynomial growth at infinity. This implies that $z^{-\rho_i} X_{\sigma_i}(z)$ is a polynomial, which cannot be the case since the series defining $X_{\sigma_i}(z)$ around the origin is infinite. This is a direct consequence of the fact that none of the expressions b/d_k in (9) coincides with a negative integer. □

Lemma 7. *There is a fundamental solution* $\boldsymbol{Y}(z) = (Y_1(z), \ldots, Y_Q(z))$ *with* $X_Q(z) = Y_Q(z)$ *of* (3) *around* $z = 1$ *such that* $Y_i(z)$ *is holomorphic for* $i = 1, \ldots, Q-1$.

Proof. $Y_Q(z)$ has the series expansion

$$Y_Q(z) = (z-1)^{\frac{n-1}{2}} \sum_{m \geq 0} G'_m (z-1)^m + \sum_{m \geq 0} G''_m (z-1)^m.$$

when n is even, and

$$Y_Q(z) = (z-1)^{\frac{n-1}{2}} \log(z-1) \left(\sum_{m \geq 0} G'_m (z-1)^m \right) + \sum_{m \geq 0} G''_m (z-1)^m.$$

when n is odd. These expressions together with local exponents (9) show the statement. □

Lemma 6 and Lemma 7 implies the following:

Lemma 8. *One can choose a fundamental solution* $\boldsymbol{Y}(z) = (Y_1(z), \ldots, Y_Q(z))$, *around* $z = 1$ *so that the connection matrix*

$$\boldsymbol{X}(z) = \boldsymbol{Y}(z) \cdot L_1 \tag{10}$$

is given by

$$L_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ c_1 & c_2 & \cdots & c_{Q-1} & 1 \end{pmatrix} \tag{11}$$

where $c_{\sigma_i} \neq 0$ *for any* $i = 1, \ldots, p$.

When n is odd, the monodroymy of Y_Q around $s = 1$ is given by

$$Y_Q(z) \to Y_Q(z) + 2\pi\sqrt{-1}(z-1)^{(n-1)/2} \sum_{m=0}^{\infty} G'_m (z-1)^m.$$

The second term is holomorphic at $z = 1$ and can be expressed as a linear combination of the components of $Y_1(z)$. Hence the monodromy $z = 1$ is given by

$$\boldsymbol{Y}(z) \to \boldsymbol{Y}(z) \cdot E_1$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & c'_1 \\ 0 & 1 & \cdots & 0 & c'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c'_{Q-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

When n is even,

$$Y_Q(z) \to -Y_Q(z) + 2\sum_{m=0}^{\infty} G'_m(z-1)^m,$$

so that the monodroymy around $z = 1$ is given by

$$\boldsymbol{Y}(z) \to \boldsymbol{Y}(z) \cdot E_1.$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & c'_1 \\ 0 & 1 & \cdots & 0 & c'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c'_{Q-1} \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

Note that the monodromy of $\boldsymbol{Y}(z)$ around $z = 0$ is given by

$$\begin{aligned} \boldsymbol{Y}(z) &= \boldsymbol{X}(z) \cdot L_1^{-1} \\ &\to \boldsymbol{X}(z) \cdot E_0 \cdot L_1^{-1} = \boldsymbol{Y}(z) \cdot L_1 \cdot E_0 \cdot L_1^{-1}. \end{aligned}$$

By a straightforward calculation, we have the following:

Proposition 9. *The monodromy matrices h_0, h_1 and h_∞ around $z = 0$, 1 and ∞ with respect to the solution basis $\boldsymbol{Y}(z)$ of* (3) *are given by*

$$h_0 = E_0 + \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \gamma_1 & \gamma_2 & \cdots & 1 & 0 \end{pmatrix},$$

$$(\gamma_1, \cdots, \gamma_{Q-2}, 1, 0) = (c_1, \cdots, c_{Q-1}, 1)(E_0 - id_Q),$$

$$h_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & g_1 \\ 0 & 1 & \cdots & 0 & g_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & g_{Q-1} \\ 0 & 0 & \cdots & 0 & (-1)^{n-1} \end{pmatrix},$$

$$h_\infty^{-1} = h_0 + \begin{pmatrix} 0 & 0 & \cdots & 0 & \delta_1 \\ 0 & 0 & \cdots & 0 & \delta_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \delta_Q \end{pmatrix},$$

$$(\delta_1, \cdots, \delta_Q) = (g_1, \cdots, g_{Q-1}, (-1)^{n-1})h_0^T + (0, \cdots, 0, -1).$$

Lemma 10. *Let $v = (v_1, \ldots, v_Q)^T$ be a column vector and define a $Q \times Q$ matrix by*

$$T = (v, h_0 \cdot v, \ldots, h_0^{Q-1} \cdot v).$$

Then one has

$$\det T = \prod_{1 \le \beta < \alpha \le p} (e_\alpha - e_\beta)^{\mu_\alpha \cdot \mu_\beta} \cdot \prod_{\alpha=1}^{p} (v_{\sigma_{\alpha-1}+1})^{\mu_\alpha}.$$

Proof. First we introduce a $i \times i$ matrix defined by

$$J_{i,+} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let $T(\alpha, j) \in \mathrm{SL}_Q(\mathbb{C})$ be the block diagonal matrix defined by

$$T(\alpha, j) = \begin{pmatrix} \mathrm{id}_{Q-j-1} & 0 \\ 0 & \mathrm{id}_{j+1} - e_\alpha \cdot J_{j+1,+} \end{pmatrix}.$$

Then

$$\begin{aligned} T \cdot T(1, Q-1) \cdot T(1, Q-2) \cdot \cdots \cdot T(1, Q-\mu_1) \\ \cdot T(2, Q-\mu_1-1) \cdot \cdots \cdot T(2, Q-\mu_1-\mu_2) \\ \cdot T(p, Q-\sigma_{p-1}-1) \cdot \cdots \cdot T(p, 1) \end{aligned}$$

is a lower-triangular matrix whose i-th diagonal component for $\sigma_{\alpha-1} < i \leq \sigma_\alpha$ is given by

$$\prod_{\beta<\alpha} (e_\alpha - e_\beta)^{\mu_\beta} \cdot v_{\sigma_{\alpha-1}+1}.$$

□

Corollary 11. *$v = (v_1, \ldots, v_Q)^T$ is a cyclic vector with respect to h_0 if and only if the condition*

$$\prod_{\alpha=1}^{p} v_{\sigma_{\alpha-1}+1} \neq 0 \tag{12}$$

is satisfied.

Lemma 12. *If $v \in \mathbb{C}^Q$ satisfies*

$$h_\infty^{-i} \cdot v = h_0^i v, \tag{13}$$

then (12) *holds.*

Proof. Since the cokernel of $h_\infty^{-1} - h_0$ is spanned by the last coordinate vector $(0, \ldots, 0, 1) \in \mathbb{C}^Q$, the equations (13) for $v = (\boldsymbol{v}, 0)$ where $\boldsymbol{v} = (v_1, \cdots, v_{Q-1})$ can be rewritten as

$$\Sigma \cdot \boldsymbol{v} = 0$$

where Σ is a $(Q-1) \times (Q-1)$ matrix whose j-th row vector is given by the first $Q-1$ component vector of the following

$$(\gamma_1, \cdots, \gamma_{Q-2}, 1, 0)(h_0)^{j-1}.$$

Define a block diagonal $(Q-1)\times(Q-1)$ matrix by

$$S(\alpha,j)=\begin{pmatrix}\mathrm{id}_{Q-j-2} & 0\\ 0 & S'\end{pmatrix}$$

where $S'\in \mathrm{SL}_{j+1}(\mathbb{C})$ is given by

$$S'=\begin{pmatrix}1 & 0 & \cdots & 0 & 0\\ -e_\alpha & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & \cdots & -e_\alpha & 1\end{pmatrix}.$$

Then the components of the matrix

$$\begin{aligned}\widetilde{\Sigma}=S(1,1)\cdots S(1,\mu_1-1)\cdot S(2,\mu_1)\cdots S(2,\sigma_2-1)\cdot S(3,\sigma_2)\cdots S(3,\sigma_3-1)\\ \cdots S(p,\sigma_{p-1})\cdots S(p,\sigma_p-2)\cdot\Sigma\end{aligned}$$

are zero below the anti-diagonal (i.e., $\widetilde{\Sigma}_{ij}=0$ if $i+j>Q$) and the i-th anti-diagonal component $\widetilde{\Sigma}_{i,Q-i-1}$ for $\sigma_{\alpha-1}<i\le\sigma_\alpha$ is given by

$$\prod_{\beta>\alpha}(e_\alpha-e_\beta)^{\mu_\beta}c_{\sigma_\alpha}.$$

The $(Q-1)$-st equation

$$(\mathrm{const})\cdot v_1+\prod_{\beta>1}(e_1-e_\beta)^{\mu_\beta}c_{\mu_1}v_2$$

together with Lemma 8 implies that $v_2=0$ if $v_1=0$. By repeating this type of argument, one shows that $v_1=0$ implies $\boldsymbol{v}=0$. Moreover, one can run the same argument by interchanging the role of (x_1,e_1,c_{μ_1}) with $(v_{\sigma_{\alpha-1}+1},e_\alpha,c_{\sigma_\alpha})$ to show that $v_{\sigma_{\alpha-1}+1}=0$ implies $\boldsymbol{v}=0$. Hence a non-trivial solution to (13) must satisfy (12). □

3 Invariants of the hypergeometric group

We prove the following in this section:

Proposition 13. *Let $(q_0,\dots,q_N)$ and $(d_1,\dots,d_r)$ be sequences of positive integers such that $Q:=\sum_{i=0}^N q_i=\sum_{k=1}^r d_r$. Then the space of $Q\times Q$ matrices invariant under the action*

$$H(q_0,\dots,q_N;d_1,\dots,d_r)\ni h: X\mapsto h\cdot X\cdot h^T$$

is at most one-dimensional.

Proof. Let X be a $Q\times Q$ matrix invariant under the hypergeometric group $H=H(q_0,\dots,q_N;d_1,\dots,d_r)$ so that

$$h\cdot X\cdot h^T=X$$

for any $h \in H$. Let $e_1 = (1, 0, \ldots, 0)$ be the first coordinate vector. Since $\{(h_0^T)^i e_1\}_{i=0}^Q$ spans $\mathbb{C}^Q$, X_{ij} is determined by the H-invariance once we know X_{i1} for $i = 1, \ldots, Q$. Consider the relation

$$X = h_1 \cdot X \cdot h_1^T. \tag{14}$$

Since

$$\begin{aligned}
(h_1 \cdot X \cdot h_1^T)_{i1} &= \sum_{k,l=1}^{Q} (h_1)_{ik} X_{kl} (h_1)_{1l} \\
&= \sum_{k,l=1}^{Q} (h_1)_{ik} X_{kl} (-1)^{N+1-r} \delta_{1l} \\
&= \sum_{k=1}^{Q} (-1)^{N+r+1} (h_1)_{ik} X_{k1} \\
&= (-1)^{N+r+1} ((h_1)_{i1} X_{11} + X_{i1}),
\end{aligned}$$

the first column of the above equation reduces to

$$(-1)^{N+r+1}((h_1)_{i1} X_{11} + X_{i1}) = X_{i1}$$

for $2 \le i \le Q$. This equation implies

$$X_{i1} = -\frac{1}{2}(h_1)_{i1} X_{11}$$

if $N + r$ is even, and

$$X_{11} = 0$$

if $N+r$ is odd. In the latter case, fix $j \neq 1$ such that $(h_1)_{j1} = (-1)^r (B_{Q-j+1} - A_{Q-j+1}) \neq 0$ and consider the j-th row of (14). Since

$$\begin{aligned}
(h_1 \cdot X \cdot h_1^T)_{ij} &= \sum_{k,l=1}^{Q} (h_1)_{ik} X_{kl} (h_1)_{jl} \\
&= \sum_{k=1}^{Q} (h_1)_{ik} (X_{k1} (h_1)_{j1} + X_{kj} (h_1)_{jj}) \\
&= \sum_{k=1}^{Q} (h_1)_{ik} (X_{k1} (h_1)_{j1} + X_{kj}) \\
&= (h_1)_{i1} (X_{11} (h_1)_{j1} + X_{1j}) + (X_{i1} (h_1)_{j1} + X_{ij}) \\
&= (h_1)_{i1} X_{1j} + X_{i1} (h_1)_{j1} + X_{ij},
\end{aligned}$$

the second column of (14) reduces to

$$(h_1)_{i1} X_{1j} + (h_1)_{j1} X_{i1} = 0$$

for $2 \le i \le Q$. Since $(h_1)_{j1} \neq 0$, the solution to the above equation is given by

$$X_{i1} = -\frac{(h_1)_{1i}}{(h_1)_{j1}} X_{1j}$$

for $2 \le i \le N$. This shows that the space of H-invariant matrices is at most one-dimensional. □

4 Coherent sheaves on Calabi-Yau complete intersections in weighted projective spaces (by Kazushi Ueda)

The results of this section belong to Kazushi Ueda. Here we describe an invariant bilinear form of the hypergeometric group in terms of Euler characteristic of coherent sheaves on Calabi-Yau complete intersections in weighted projective spaces.

Let Y be a smooth complete intersection of degree $(d_1, \dots, d_r)$ in $\mathbb{P}(q_0, \dots, q_N)$. We have the Koszul resolution

$$0 \to \mathcal{O}(-d_1 - \cdots - d_r) \to \bigoplus_{i=1}^{r} \mathcal{O}(-d_1 - \cdots - \widehat{d_i} - \cdots - d_r)$$

$$\to \cdots \to \bigoplus_{1 \le i < j \le r} \mathcal{O}(-d_i - d_j) \to \bigoplus_{i=1}^{r} \mathcal{O}(-d_i) \to \mathcal{O} \to \mathcal{O}_Y \to 0$$

of the structure sheaf $\mathcal{O}_Y$ of Y. By tensoring this sequence with $\mathcal{O}(i)$, we obtain a locally-free resolution of $\mathcal{O}_Y(i)$ for any $i \in \mathbb{Z}$. By Kontsevich (cf. Seidel [18, Lemma 5.4]), $\{\mathcal{O}_Y, \mathcal{O}_Y(1), \dots, \mathcal{O}_Y(N)\}$ is a classical generator of the bounded derived category $D^b \operatorname{coh} Y$ of coherent sheaves on Y.

Let $(\widetilde{\mathcal{E}}_i)_{i=Q}^{1}$ be the full strong exceptional collection on $D^b \operatorname{coh} \mathbb{P}(q_0, \dots, q_N)$ given as

$$(\widetilde{\mathcal{E}}_Q, \dots, \widetilde{\mathcal{E}}_1) = (\mathcal{O}, \dots, \mathcal{O}(Q-1)),$$

and $(\widetilde{\mathcal{F}}_1, \dots, \widetilde{\mathcal{F}}_Q)$ be its right dual exceptional collection so that

$$\operatorname{Ext}^k(\widetilde{\mathcal{E}}_i, \widetilde{\mathcal{F}}_j) = \begin{cases} \mathbb{C} & i = j, \text{ and } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Equip the Grothendieck group $K(\mathbb{P}(q_1, \dots, q_N))$ with the *Euler form*

$$\chi(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}) = \sum_i (-1)^i \dim \operatorname{Ext}^i(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}).$$

Note that the Euler form on $K(\mathbb{P}(q_1, \dots, q_N))$ is neither symmetric nor anti-symmetric, whereas that on $K(Y)$ is either symmetric or anti-symmetric depending on the dimension of Y. The bases $\{[\widetilde{\mathcal{E}}_i]\}_{i=1}^{Q}$ and $\{[\widetilde{\mathcal{F}}_i]\}_{i=1}^{Q}$ of $K(\mathbb{P}(q_1, \dots, q_N))$ are dual to each other in the sense that

$$\chi(\widetilde{\mathcal{E}}_i, \widetilde{\mathcal{F}}_j) = \delta_{ij}.$$

We will write the restrictions of $\widetilde{\mathcal{E}}_i$ and $\widetilde{\mathcal{F}}_i$ to Y as $\overline{\mathcal{E}}_i$ and $\overline{\mathcal{F}}_i$ respectively. Unlike $\{[\widetilde{\mathcal{E}}_i]\}_{i=1}^{Q}$ and $\{[\widetilde{\mathcal{F}}_i]\}_{i=1}^{Q}$, $\{[\overline{\mathcal{E}}_i]\}_{i=1}^{Q}$ and $\{[\overline{\mathcal{F}}_i]\}_{i=1}^{Q}$ are not bases of $K(Y)$. Put

$$\overline{X}_{ij} = \chi([\overline{\mathcal{F}}_i], [\overline{\mathcal{F}}_j])$$

and let $(a_{ij})_{i,j=1}^{Q}$ be the transformation matrix between two bases $\{[\widetilde{\mathcal{E}}_i]\}_{i=1}^{Q}$ and $\{[\widetilde{\mathcal{F}}_i]\}_{i=1}^{Q}$ so that

$$[\widetilde{\mathcal{F}}_i] = \sum_{j=1}^{Q} [\widetilde{\mathcal{E}}_j] a_{ji}.$$

The following is the main result in this section:

Theorem 14. *$\overline{X}$ is an invariant of the hypergeometric group $H(q_0,\dots,q_N;d_1,\dots,d_r)$.*

We divide the proof into three steps.

Lemma 15. *Let Φ be an autoequivalence of $D^b\operatorname{coh}Y$ such that its action on $\{[\overline{\mathcal{F}}_i]\}_{i=1}^Q$ is given by*

$$[\overline{\mathcal{F}}_i] \mapsto \sum_{j=1}^{Q} h_{ij}[\overline{\mathcal{F}}_j].$$

Then $\overline{X}$ is invariant under the action of $h = (h_{ij})_{i,j=1}^{Q}$;

$$\overline{X} = h \cdot \overline{X} \cdot h^T.$$

Proof. Since Φ induces an isometry of $K(Y)$, one has

$$\begin{aligned}\overline{X}_{ij} &= \chi([\overline{\mathcal{F}}_i],[\overline{\mathcal{F}}_j]) \\ &= \chi([\Phi(\overline{\mathcal{F}}_i)],[\Phi(\overline{\mathcal{F}}_j)]) \\ &= \sum_{k,l=1}^{Q} h_{ik}\chi([\overline{\mathcal{F}}_k],[\overline{\mathcal{F}}_l])h_{jl} \\ &= \sum_{k,l=1}^{Q} h_{ik}\overline{X}_{kl}h_{jl}\end{aligned}$$

for any $1 \le i,j \le Q$. □

Remark 16. Since $\{[\overline{\mathcal{F}}_i]\}_{i=1}^Q$ are not linearly independent, the choice of h in Lemma 15 is not unique.

Lemma 17. *The action of the autoequivalence of $D^b\operatorname{coh}Y$ defined by the tensor product with $\mathcal{O}_Y(-1)$ on $\{\overline{\mathcal{F}}_i\}_{i=1}^Q$ is given by h_∞;*

$$[\overline{\mathcal{F}}_i \otimes \mathcal{O}_Y(-1)] = \sum_{i=1}^{Q}(h_\infty)_{ij}[\overline{\mathcal{F}}_j].$$

Proof. Since tensor product with $\mathcal{O}(-1)$ commutes with restriction, it suffices to show

$$[\widetilde{\mathcal{F}}_i \otimes \mathcal{O}(-1)] = \sum_{i=1}^{Q}(h_\infty)_{ij}[\widetilde{\mathcal{F}}_j].$$

Since $\{[\widetilde{\mathcal{E}}_i]\}_{i=1}^Q$ and $\{[\widetilde{\mathcal{F}}_i]\}_{i=1}^Q$ are dual bases, this is equivalent to

$$[\widetilde{\mathcal{E}}_i \otimes \mathcal{O}(-1)] = \sum_{i=1}^{Q}[\widetilde{\mathcal{E}}_j](h_\infty^{-1})_{ji},$$

which follows from the exact sequence on $\mathbb{P}(q_0,\dots,q_N)$ obtained by sheafifying the Koszul resolution

$$\begin{aligned}0 \to \Lambda^N V \otimes \operatorname{Sym}^* V^* \to \cdots \to \Lambda^2 V \otimes \operatorname{Sym}^* V^* \\ \to V \otimes \operatorname{Sym}^* V^* \to \operatorname{Sym}^* V^* \to \mathbb{C} \to 0,\end{aligned}$$

where V is a graded vector space such that $\mathbb{P}(q_0,\dots,q_N) = \operatorname{Proj}(\operatorname{Sym}^* V^*)$. Here, one has to be careful with our choice $\widetilde{\mathcal{E}}_i = \mathcal{O}(Q-i)$ of numbering on $\mathcal{E}_i$. □

Lemma 18. *The action of the autoequivalence of $D^b \operatorname{coh} Y$ given by the spherical twist $T^{\vee}_{\overline{\mathcal{F}}_1}$ along $\overline{\mathcal{F}}_1$ is given on $\{\overline{\mathcal{F}}_i\}_{i=1}^{Q}$ by h_1;*

$$[T^{\vee}_{\overline{\mathcal{F}}_1}(\overline{\mathcal{F}}_i)] = \sum_{i=1}^{Q} (h_1)_{ij} [\overline{\mathcal{F}}_j].$$

Proof. Recall that for a spherical object $\mathcal{E}$ and an object $\mathcal{F}$, the twist $T^{\vee}_{\mathcal{E}}\mathcal{F}$ of $\mathcal{F}$ along $\mathcal{E}$ is defined as the mapping cone

$$T^{\vee}_{\mathcal{E}}\mathcal{F} = \{F \to \hom(\mathcal{F}, \mathcal{E})^{\vee} \otimes \mathcal{F}\}$$

of the dual evaluation map. Since the induced action of the twist functor $T^{\vee}_{\mathcal{E}}$ on the Grothendieck group is given by the reflection

$$[T^{\vee}_{\mathcal{E}}(\mathcal{F})] = [\mathcal{F}] - \chi(\mathcal{F}, \mathcal{E})[\mathcal{E}],$$

it suffices to show that

$$(h_1 - \mathrm{id})_{ij} = \begin{cases} -\overline{X}_{i1} & \text{if } j = 1, \\ 0 & \text{otherweise.} \end{cases}$$

Note that

$$\begin{aligned}
(-1)^r \overline{X}_{i1} &= (-1)^r \chi(\overline{\mathcal{F}}_i, \overline{\mathcal{F}}_1) \\
&= (-1)^N \chi(\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_i) \\
&= (-1)^N \chi(\mathcal{O}_Y(-1)[N], \overline{\mathcal{F}}_i) \\
&= \chi(\mathcal{O}_Y(-1), \overline{\mathcal{F}}_i) \\
&= \chi(\overline{\mathcal{F}}_i(1)) \\
&= \chi(\widetilde{\mathcal{F}}_i(1)) - \sum_{k=1}^{r} \chi(\widetilde{\mathcal{F}}_i(1 - d_k)) + \sum_{1 \le k < l \le r} \chi(\widetilde{\mathcal{F}}_i(1 - d_k - d_l)) \\
&\quad - \cdots + (-1)^r \chi(\widetilde{\mathcal{F}}_i(1 - d_1 - \cdots - d_r))
\end{aligned}$$

and

$$\chi(\widetilde{\mathcal{F}}_i(1)) = \sum_{j=1}^{Q} \chi((h_{\infty}^{-1})_{ij} \widetilde{\mathcal{F}}_j) = \sum_{j=1}^{Q} (h_{\infty}^{-1})_{ij} \chi(\widetilde{\mathcal{E}}_Q, \widetilde{\mathcal{F}}_j) = (h_{\infty}^{-1})_{iQ} = -B_{Q-i+1}.$$

Since

$$\chi(\widetilde{\mathcal{F}}_i(j)) = \chi(\mathcal{O}(-j), \widetilde{\mathcal{F}}_i) = \chi(\widetilde{\mathcal{E}}_{Q+j}, \widetilde{\mathcal{F}}_i) = \delta_{Q+j,i}$$

for $-Q + 1 \le j \le 0$ and

$$\begin{aligned}
\prod_{k=1}^{r} (t^{d_k} - 1) &= t^Q - \sum_{k=1}^{r} t^{Q - d_k} + (-1)^2 \sum_{1 \le k < l \le r} t^{Q - d_k - d_l} \\
&\quad + \cdots + (-1)^{r-1} \sum_{k=1}^{r} t^{d_k} + (-1)^r \\
&= t^Q + A_1 t^{Q-1} + \cdots + A_{Q-1} t + A_Q,
\end{aligned}$$

it follows that

$$\chi(\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_i) - \chi(\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_i) = A_{Q-i+1},$$

and hence $\overline{X}_{i1} = -(h_1 - \mathrm{id})_{i1}$. □

Theorem 14 immediately follows from Lemmas 15, 17 and 18.

5 The Mellin transform of the period integrals

In this section we show that the period integral (2) satisfies the hypergeometric equation (3). First of all we recall the notion of the Leray coboundary cycle $\Gamma \in H_{N+1}(\mathbb{T}^{N+1} \setminus X_t)$ constructed as a $(r+1)$times successive S^1−bundle over a cycle $\gamma \in H_n(X_t)$. It is a cycle that avoids all the hypersurfaces $f_0(x) + t = 0$ and $f_1(x) + 1 = 0, \cdots, f_r(x) + 1 = 0$ [7, Theorem 2]. Without loss of generality, we can assume that $\Re e(f_0(x) + t)|_\Gamma < 0$, $\Re e(f_k(x) + 1)|_\Gamma < 0$, $1 \leq k \leq r$ out of a compact set

Theorem 19. *For a Leray coboundary cycle $\Gamma \in H_{N+1}(\mathbb{T}^{N+1} \setminus X_t)$ we consider the following residue integral:*

$$I^{(v)}_{x^i,\Gamma}(t) = \int_\Gamma x^{i+1}(f_0(x) + t)^{-v_0} \prod_{k=1}^{r} (f_k(x) + 1)^{-v_k} \frac{dx}{x^1}, \tag{15}$$

with the monomial $x^i := x_0^{i_0} \cdots x_N^{i_N}$, $x^1 := x_0 \cdots x_N$, $v = (v_0, v_1, \cdots, v_r)$. Then the integral $I^{(v)}_{x^i,\Gamma}(t)$ satisfies the following hypergeometric differential equation

$$\left[P^{(i)}(-\theta_t) - tQ^{(i)}(-\theta_t)\right] I^{(v)}_{x^i,\Gamma}(t) = 0, \tag{16}$$

for

$$P^{(i)}(-\theta_t) = \prod_{k=1}^{r} \prod_{p=0}^{d'_k-1} \prod_{a=0}^{q_{\lambda_{k-1}+p}-1} (-q_{\lambda_{k-1}+p}\theta_t + i_{\lambda_{k-1}+p} - a), \tag{17}$$

$$Q^{(i)}(-\theta_t) = \prod_{k=1}^{r} \prod_{b=1}^{d_k} (-d_k\theta_t - d_k + \sum_{p=0}^{d'_k-1} (i_{\lambda_{k-1}+p} + 1) - b) \tag{18}$$

with $\theta_t = t\frac{\partial}{\partial t}$. Here we used the notation $\lambda_k = \sum_{i=1}^{k} \sharp(S_i)$ and $d'_k = \sharp(S_k)$.

Proof. Let us consider the Mellin transform of the fibre integral (15)

$$M^{(v)}_{i,\Gamma}(z) := \int_\Pi t^z I^{(v)}_{x^i,\Gamma}(t) \frac{dt}{t}, \tag{19}$$

for a cycle avoiding the discriminant. For the Mellin transform (19), we have the following

$$\begin{aligned} M^{(v)}_{i,\Gamma}(z) =& g(z)\Gamma(z)\Gamma(v_0 - z) \prod_{k=1}^{r} \prod_{j=0}^{d'_k-1} \Gamma(q_{\lambda_{k-1}+j}(z - v_0) + i_{\lambda_{k-1}+j} + 1) \\ &\cdot\Gamma(-\sum_{j=0}^{d'_k-1} (i_{\lambda_{k-1}+j} + 1) - d_k(z - v_0) + v_k), \end{aligned} \tag{20}$$

with $g(z)$ a rational function in $e^{2\pi i z}$. As the period integral $I^{(v)}_{x^i,\Gamma}(t)$ can be expressed by the inverse Mellin transform,

$$I^{(v)}_{x^i,\Gamma}(t)=\int_{\check{\Pi}} t^{-z} M^{(v)}_{i,\Gamma}(z)dz,$$

for some cycle $\check{\Pi}$ encircling the poles of Γ function factors, the equation (16) immediately follows from (20).

To show (20) we make use of the so called Cayley trick. Namely we transform the integral (19) into the following form.

$$M^{(v)}_{i,\Gamma}(z)=\int_{\Pi\times\mathbf{R}^{r+1}_{+}\times\Gamma} x^{i+1} e^{y_0(f_0(x)+t)+\sum_{k=1}^{r} y_k(f_k(x)+1)} \prod_{k=0}^{r} y_k^{v_k} t^z \frac{dx}{x^1}\frac{dy}{y^1}\frac{dt}{t}, \tag{21}$$

with $\mathbb{R}_+$ the positive real axis in $\mathbb{C}_{y_p}$ for $p=0,\cdots,r$. Here we introduce new variables $T_0,\cdots T_{N+r+2}$,

$$T_0=y_0f_0(x),\ \ T_1=y_0t, T_2=y_0x_0^{q_0}, T_3=y_0x_1^{q_1},\cdots,T_{N+r+2}=y_r,$$

in such a way that the phase function of the right hand side of (21) becomes

$$y_0(f_0(x)+s)+\sum_{i=1}^{r} y_k(f_k(x)+1)=T_0+T_1+\cdots+T_{N+r+2}.$$

If we introduce the following notation,

$$\begin{aligned} &Log\ T:={}^t\,(log\ T_0,\cdots,log\ T_{N+r+2})\\ &\Xi:={}^t\,(x_0,\cdots,x_N,t,y_0,\cdots,y_r)\\ &Log\ \Xi:={}^t\,(\log\ x_0,\cdots,\log\ x_N,\log\ s,\log\ y_0,\cdots,log\ y_r),\end{aligned} \tag{22}$$

we have

$$Log\ T=\mathsf{L}\cdot Log\ \Xi,$$

for the following non-singular matrix L,

$$\mathsf{L}=\begin{bmatrix}
q_0 & q_1 & \cdots & q_{d'_1-1} & q_{d'_1} & q_{d'_1+1} & q_{d'_1+2} & \cdots & q_N & 0 & 1 & 0 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 & 0\\
1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0\\
0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & 1 & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 1\\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}. \tag{23}$$

The above relation is equivalent to

$$\mathsf{L}^{-1} \cdot Log\, T = Log\, \Xi,$$

for a non-singular matrix L^{-1} which has the following form;

$$\mathsf{L}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & 1 & q_0 & q_1 & \cdots & q_{d_1'-1} & -d_1 & q_{\ell_1} & q_{d_1'+1} & \cdots & q_N & -d_r \\ 1 & 0 & -q_0 & -q_1 & \cdots & -q_{d_1'-1} & d_1 & -q_{d_1'} & -q_{d_1'+1} & \cdots & -q_N & d_r \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{24}$$

If we set

$$(i_0+1, \cdots, i_N+1, z, v_0, \cdots, v_r) \cdot \mathsf{L}^{-1} = \big(\mathcal{L}_0(\boldsymbol{i}, z, \boldsymbol{v}), \cdots, \mathcal{L}_{N+r+2}(\boldsymbol{i}, z, \boldsymbol{v})\big), \tag{25}$$

then we can see that

$$\begin{aligned} M_{\boldsymbol{i},\Gamma}^{(\boldsymbol{v})}(z) &= \int_{\Pi\times\mathbb{R}_+^{r+1}\times\Gamma} x^{\boldsymbol{i}+1} e^{T_0+\cdots+T_{N+r+2}} y_0^{v_0} \cdots y_r^{v_r} t^z \frac{dx}{x^1}\frac{dy}{y^1}\frac{dt}{t} \\ &= (\det \mathsf{L})^{-1} \int_{\mathsf{L}_*(\Pi\times\mathbb{R}_+^{r+1}\times\Gamma)} e^{T_0+\cdots+T_{N+r+2}} \prod_{0\le a\le N+r+2} T_a^{\mathcal{L}_a(\boldsymbol{i},z,\boldsymbol{v})} \bigwedge_{0\le a\le N+r+2} \frac{dT_a}{T_a}. \end{aligned} \tag{26}$$

Here $\mathsf{L}_*(\Pi \times \mathbb{R}_+^{r+1} \times \Gamma)$ denotes a $(N+r+3)-$chain in $T_0 \cdots T_{N+r+2} \neq 0$ that obtained as a image of $\Pi \times \mathbb{R}_+^{r+1} \times \Gamma$ under the transformation induced by L. In view of the choice of the cycle Γ, we can apply the formula to calculate Gamma function to our situation:

$$\int_C e^{-T} T^{\sigma} \frac{dT}{T} = (1 - e^{2\pi i \sigma})\Gamma(\sigma),$$

for the unique nontrivial cycle C turning around $T = 0$ that begins and returns to $\Re e T \to +\infty$. Here one can consider the natural action $\lambda : C_a \to \lambda(C_a)$ defined by the relation,

$$\int_{\lambda(C_a)} e^{-T_a} T_a^{\sigma_a} \frac{dT_a}{T_a} = \int_{(C_a)} e^{-T_a} (e^{2\pi\sqrt{-1}} T_a)^{\sigma_a} \frac{dT_a}{T_a}.$$

In terms of this action, $\mathsf{L}_*(\Pi \times \mathbb{R}_+^{r+1} \times \Gamma)$ is shown to be homologous to a chain

$$\sum_{(j_0^{(\rho)}, \cdots, j_{N+r+2}^{(\rho)}) \in [0,Q]^{N+r+3}} m_{j_0^{(\rho)}, \cdots, j_{N+r+2}^{(\rho)}} \prod_{a=0}^{r-1} \lambda^{j_a^{(\rho)}}(\mathbb{R}_+) \prod_{a'=r}^{N+r+2} \lambda^{j_{a'}^{(\rho)}}(C_{a'}),$$

with $m_{j_0^{(\rho)},\cdots,j_{N+r+2}^{(\rho)}} \in \mathbb{Z}$. This explains the appearance of the factor $g(z)$ in front of the Γ function factors in (20).

The direct calculation of (25) shows that

$$\mathcal{L}_0(\boldsymbol{i},z,\boldsymbol{v}) = -z+v_0, \mathcal{L}_1(\boldsymbol{i},z,\boldsymbol{v}) = z,$$

$$\mathcal{L}_{\lambda_{k-1}+p+k+1}(\boldsymbol{i},z,\boldsymbol{v}) = q_{\lambda_{k-1}+p}(z-v_0)+i_{\lambda_{k-1}+p}+1, 0 \leq j \leq \ell_k - 1.$$

$$\mathcal{L}_{\lambda_k+k+1}(\boldsymbol{i},z,\boldsymbol{v}) = -d_k(z-v_0) - \sum_{p=0}^{d'_k-1}(i_{\lambda_{k-1}+p}+1)+v_k, 1 \leq k \leq r.$$

We remark here that $\sum_{0\leq a\leq N+r+2}\mathcal{L}_a(\boldsymbol{i},z.\boldsymbol{v}) = \sum_{k=0}^r v_k$ and the variable change $T_a \to -T_a$ in the integration of (26) would cause only multiplication by the factor $(-1)^{\sum_{k=0}^r v_k}$.

This shows the formula (20) and consequently (16) by virtue of the fact that the periodic function $g(z)$ plays no rôle in establishment of the differential equation satisfied by its Mellin inverse transform. □

As a result we get the Mellin transofrm $M_{\boldsymbol{i},\gamma}^{(\boldsymbol{v})}(z)$. Especially

$$M_{\boldsymbol{q}-1,\gamma}^{(1)}(z) = \frac{\prod_{\nu=1}^r \Gamma(q_\nu z)}{\prod_{k=1}^r \Gamma(d_k z)}. \tag{27}$$

always up to a periodic function factor. This formula has already been claimed in [8] (resp. [11]) in the case when $\mathbf{q} = \mathbf{1}$ (resp. $\mathbf{q}$ general).

References

[1] D. Auroux, L. Katzarkov, D. Orlov, *Mirror symmetry for weighted projective planes and their noncommutative deformations*, Ann. of Math. (2) 167 (2008), no. 3, 867–943.

[2] A. Beilinson, *Coherent sheaves on P^n and problems in linear algebra*, Funktsional. Anal. i Prilozhen. 12 (1978), no. 3, 68–69

[3] V. Batyrev, L. Borisov, *On Calabi-Yau complete intersections in toric varieties*, In *Higher-dimensional complex varieties (Trento, 1994)*, pp. 39–65. de Gruyter, Berlin, 1996.

[4] F. Beukers, G.Heckman, *Monodromy for the hypergeometric function ${}_nF_{n-1}$* , Inventiones Math.**95** (1989), pp.325–354.

[5] A. Bondal, M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. 3 (2006), no.1, 1–36.

[6] Lev A. Borisov, R.Paul Horja, *Mellin Barnes integrals as Fourier-Mukai transofrms*, Advances in Math. **207** (2006), pp.876-927.

[7] D.Fotiadi,M.Froissart,J.Lascoux,F.Pham, *Applications of an isotopy theorem*, Topology **4** (1965), pp.159-191.

[8] A.GIVENTAL'.*A mirror theorem for toric complete intersections,* Topological field theory, primitive forms and related topics (Kyoto, 1996), pp.141–175, Progr. Math., 160, Birkäuser Boston, Boston, MA, 1998

[9] V.V.GOLYSHEV, *Riemann-Roch Variations,* Izvestia Math. **65** (2001), no. 5, pp.853-887.

[10] D.GUZZETTI, *Stokes matrices and monodromy of the quantum cohomology of projective spaces,* Comm. in Math. Physics **207** (1999), no.2, pp.341-383.

[11] R.P.HORJA, *Hypergeometric Functions and Mirror Symmetry in Toric Varieties,* math.AG/9912109.

[12] M.KOHNO. *Global analysis in linear differential equations.* Mathematics and its Applications, 471. Kluwer Academic Publishers, Dordrecht, 1999. xvi+526 pp.

[13] M.KONTSEVICH, *Homological algebra of mirror symmetry,* Proceedings of ICM Zürich 1994, (1994) alg-geom/9411018.

[14] M.KONTSEVICH, *Lecture at École Normale Supérieure,* September 16 1998. Notes prises par J. Bellaiche, J.-F. Dat, I.Marin,G.Racinet et H.Randriambololona.

[15] A.H.M.LEVELT, *Hypergeometric functions,* Indagationes Math.**23**(1961), pp.361-403.

[16] E.LOOIJENGA, *Isolated singular points on complete intersections,* London Math. Soc. Lect. Notes Ser., 1984, No. 77, 200pp.

[17] CH.OKONEK,M.SCHNEIDER and H.SPINDLER *Vector bundles on complex projective spaces,* Progress in Mathematics, 3. Birkhäuser, Boston, Mass., 1980. vii+389 pp.

[18] P. SEIDEL, *Homological mirror symmetry for the quartic surface,* arXiv:math/0310414.

[19] S.TANABÉ *Invariant of the hypergeometric group associated to the quantum cohomology of the projective space,* Bulletin des Sciences mathématiques(tome 128,2004, pp.811-827).

[20] S.TANABÉ *Transformée de Mellin des intégrales- fibres associée aux singularités isolées d'intersection complète non-dégénérée,* arXiv:math/0405399.

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