

# An equivalence problem for second order PDE and double fibration as a flat model space

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## 1. INTRODUCTION

The purpose of the note is to announce my recent result. The detail will be given in the forthcoming paper [N].

We will study an equivalence problem of second order PDE for one unknown function of two variables. The equivalence problem for differential equations is simply explained as follows. We fix classes of differential equations and a group of coordinate transformations. Then, we consider a problem how differential equations change under coordinate transformations. We can also express this problem in terms of group actions. Let  $\mathcal{G}$  be a coordinate transformation group and  $X$  be a set of certain differential equations. Then the equivalence problem for differential equations in  $X$  is interpreted as the problem of determining the orbit decomposition with respect to the action of  $\mathcal{G}$  on  $X$ .

The equivalence problem is studied deeply by Sophus Lie and Élie Cartan, and many other authors. (See [M] for a detailed history of the equivalence problem.)

We consider an equivalence problem of second order PDE for one unknown function of two variables  $y = y(x_1, x_2)$ :

$$(1) \quad \frac{\partial^2 y}{\partial x_i \partial x_j} = f_{ij}(x_1, x_2, y, z_1, z_2),$$

where,  $f_{ij}$  ( $1 \leq i, j \leq 2$ ) satisfying  $f_{ij} = f_{ji}$  are  $C^\infty$  functions on  $J^1(\mathbb{R}^2, \mathbb{R}) := \{(x_1, x_2, y, z_1, z_2)\}$ , and  $z_1 = y_{x_1}$ ,  $z_2 = y_{x_2}$ . If  $f_{ij}$  all vanish, (1) is called the flat equation. For this PDE, we take the group  $\text{ScaleDiff}(\mathbb{R}^3)^{\text{cont}}$  of lifts of scale transformations on  $\mathbb{R}^3$  as a transformation group  $\mathcal{G}$ , where scale transformations are defined by:

$$\phi(x_1, x_2, y) = (X_1(x_1), X_2(x_2), Y(x_1, x_2, y)).$$

Since  $\phi$  is the transformation on  $J^0(\mathbb{R}^2, \mathbb{R})$ , we can characterize this one geometrically as follows. Scale transformations preserve not only fibers on  $J^0(\mathbb{R}^2, \mathbb{R})$ , but also the web-structure on the base space  $\mathbb{R}^2$  constructed by parallel translation of  $x_1$ -axis and  $x_2$ -axis. For the equivalence problem, we can not apply the Tanaka theory, because symmetry group is not semi-simple group. Thus, it is necessary to use the Cartan's classical method. We will calculate explicitly the curvatures for this equivalence problem by using Cartan's equivalence method. We obtain the necessary and sufficient condition when the second order PDE satisfying integrability condition is equivalent to the flat equation via a vanishing condition of these curvatures (**Theorem 2.3**).

We also discuss a duality associated with differential equations via double fibrations. In particular, we consider a duality between the coordinate space and the solution space of the flat equation. Double fibrations play an important role for a study of this duality. Moreover, these fibrations are usually described via some transformation groups appeared in equivalence problems. For the group  $\text{ScaleDiff}(\mathbb{R}^3)^{\text{cont}}$ , we can not obtain a fibration of compact type, because the group  $\text{ScaleDiff}(\mathbb{R}^3)^{\text{cont}}$  is too small. Hence,

it is natural to consider an existence problem of groups from which double fibration of compact-type is obtained as a flat model space. For this problem, we find a non-trivial group which gives a fibration of compact-type (**Theorem 3.4**).

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## 2. EQUIVALENCE PROBLEM AND LOCAL INVARIANT

In this section, we introduce an equivalence problem and explain the  $G$ -structure associated with this problem. For this purpose, we prepare some terminology and notation.

For functions of two variables  $y = y(x_1, x_2)$ , we consider the second order PDE (1) and diffeomorphisms  $\phi$  on  $\mathbb{R}^3$  of the form

$$\phi(x_1, x_2, y) = (X_1(x_1), X_2(x_2), Y(x_1, x_2, y)).$$

The map  $\phi$  of this form is called a scale transformation. A scale transformation  $\phi$  lifts naturally to a contact diffeomorphism  $\hat{\phi}$  of  $J^1(\mathbb{R}^2, \mathbb{R})$  defined by:

$$\hat{\phi}(x_1, x_2, y, z_1, z_2) = (X_1(x_1), X_2(x_2), Y(x_1, x_2, y), Z_1, Z_2),$$

where,  $Z_1 = \frac{Y_{x_1} + Y_y z_1}{(X_1)_{x_1}}$ ,  $Z_2 = \frac{Y_{x_2} + Y_y z_2}{(X_2)_{x_2}}$ . We introduce the following terminology:

$$\begin{aligned} \text{ScaleDiff}(\mathbb{R}^3) &:= \{ \text{Scale transformation on } \mathbb{R}^3 \}, \\ \text{Diff}(\mathbb{R}^3)^{\text{cont}} &:= \{ \text{The lift of } \text{Diff}(\mathbb{R}^3) \text{ to } J^1(\mathbb{R}^2, \mathbb{R}) \}, \\ \text{ScaleDiff}(\mathbb{R}^3)^{\text{cont}} &:= \{ \text{The lift of } \text{ScaleDiff}(\mathbb{R}^3) \text{ to } J^1(\mathbb{R}^2, \mathbb{R}) \}, \\ X &:= \{ \text{second order PDE (1)} \}. \end{aligned}$$

The main problem in the present note is the following.

**Problem 2.1.** *Examine the orbit decomposition under the action of  $\text{ScaleDiff}(\mathbb{R}^3)^{\text{cont}}$  on  $X$ .*

In order to resolve the above problem, we use a  $G$ -structure associated with the equation (1). First, we replace from data of second order PDE (1) to data of differential system ( $[G]$ ,  $[O1]$ ). We choose the following coframe of  $J^1(\mathbb{R}^2, \mathbb{R})$  corresponding to the equation (1),

$$\begin{aligned} \underline{\theta}_0 &:= dy - z_1 dx_1 - z_2 dx_2, \\ \underline{\theta}_1 &:= dz_1 - f_{11} dx_1 - f_{12} dx_2, \\ \underline{\theta}_2 &:= dz_2 - f_{21} dx_1 - f_{22} dx_2, \\ \underline{\omega}_1 &:= dx_1, \\ \underline{\omega}_2 &:= dx_2. \end{aligned} \tag{2}$$

We consider the Frobenius system

$$\mathcal{I} := \{ \underline{\theta}_0, \underline{\theta}_1, \underline{\theta}_2 \}_{\text{diff}} \text{ with } \underline{\omega}_1 \wedge \underline{\omega}_2 \neq 0 \tag{3}$$

constructed by this coframe. The correspondence between second order PDE (1) and the Frobenius system  $\mathcal{I}$  is described as follows. Consider vector fields on  $J^1(\mathbb{R}^2, \mathbb{R})$  which are annihilated by  $\underline{\theta}_i$ , while are not annihilated by  $\underline{\omega}_i$ . At any point on  $J^1(\mathbb{R}^2, \mathbb{R})$ , such

vector fields are generated by two vector fields  $v_1, v_2$ . The integral surfaces which are tangent to the 2-plane  $\text{span}\{v_1, v_2\}$  at any point are the graphs of solutions of the second order PDE (1). Then, the parameters  $(x_1, x_2)$  are regarded as a local coordinate system of this integral surface.

The Frobenius condition (integrability condition) of the Frobenius system  $\mathcal{I}$  is:

$$d\theta_i \equiv 0 \pmod{\theta_0, \theta_1, \theta_2} \quad (i = 0, 1, 2).$$

Then, the above integrability condition is equivalent to  $A = B = 0$ , where  $A$  and  $B$  are given by

$$A = (f_{11})_{x_2} - (f_{12})_{x_1} + (f_{11})_y z_2 + (f_{11})_{z_1} f_{12} + (f_{11})_{z_2} f_{22} \\ - (f_{12})_y z_1 - (f_{12})_{z_1} f_{11} - (f_{12})_{z_2} f_{12},$$

$$B = (f_{12})_{x_2} - (f_{22})_{x_1} + (f_{12})_y z_2 + (f_{12})_{z_1} f_{12} + (f_{12})_{z_2} f_{22} \\ - (f_{22})_y z_1 - (f_{22})_{z_1} f_{11} - (f_{22})_{z_2} f_{12}.$$

*Remark 2.2.* Hereafter, we discuss only the second order PDE (1) with respect to  $f_{ij}$  satisfying  $A = B = 0$ .

A family of integral surfaces of  $\mathcal{I}$  gives a 2-dimensional foliation on  $J^1(\mathbb{R}^2, \mathbb{R})$ . We describe an infinitesimal automorphism group of the foliation, and consider a principal bundle over  $J^1(\mathbb{R}^2, \mathbb{R})$  with this group as a structure group.

The contact lift  $\hat{\phi}$  of the scale transformation  $\phi$  preserving  $\mathcal{I}$  satisfies the following equations:

$$(4) \quad \begin{aligned} \hat{\phi}^* \theta_0 &= a \theta_0 & (a \neq 0), \\ \hat{\phi}^* \theta_1 &= b \theta_0 + c \theta_1 & (c \neq 0), \\ \hat{\phi}^* \theta_2 &= e \theta_0 + g \theta_2 & (g \neq 0), \\ \hat{\phi}^* \omega_1 &= h \omega_1 & (h \neq 0), \\ \hat{\phi}^* \omega_2 &= k \omega_2 & (k \neq 0). \end{aligned}$$

The equation (4) can be written in the following form:

$$(5) \quad \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ b & c & 0 & 0 & 0 \\ e & 0 & g & 0 & 0 \\ 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & k \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{bmatrix}$$

where,  $a, b, c, e, g, h, k$  are functions. Thus we have linear transformations of coframes determined by  $\hat{\phi}$ . Moreover, the lift  $\hat{\phi}$  of the scale transformation satisfies:

$$(6) \quad \begin{aligned} d\theta_0 &\equiv -\theta_1 \wedge \omega_1 - \theta_2 \wedge \omega_2 & (\text{mod } \theta_0), \\ d\theta_1 &\equiv 0 & (\text{mod } \theta_0, \theta_1, \theta_2), \\ d\theta_2 &\equiv 0 & (\text{mod } \theta_0, \theta_1, \theta_2). \end{aligned}$$

These relations give conditions  $a = ch = gk$ . From these conditions, we get the linear transformations of coframes of the following form:

$$(7) \quad \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} ch & 0 & 0 & 0 & 0 \\ b & c & 0 & 0 & 0 \\ e & 0 & g & 0 & 0 \\ 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & k \end{bmatrix} \begin{bmatrix} \underline{\theta}_0 \\ \underline{\theta}_1 \\ \underline{\theta}_2 \\ \underline{\omega}_1 \\ \underline{\omega}_2 \end{bmatrix}.$$

Therefore, we obtain the following 5-dimensional Lie group as infinitesimal automorphism group:

$$(8) \quad G := \left\{ \begin{bmatrix} ch & 0 & 0 & 0 & 0 \\ b & c & 0 & 0 & 0 \\ e & 0 & g & 0 & 0 \\ 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & k \end{bmatrix} \in GL(5, \mathbb{R}) \mid ch = gk \right\}.$$

Then, we choose the reduced  $G$ -bundle  $\mathcal{F}_G$  of the coframe bundle  $\mathcal{F}_{GL}(\mathbb{R}^5)$  over  $J^1(\mathbb{R}^2, \mathbb{R})$ . This bundle  $\mathcal{F}_G$  is called  $G$ -structure associated with the second order PDE (1). On the principal bundle  $\mathcal{F}_G$ , we compute curvatures (local invariant functions) for the equivalence problem using the Cartan's equivalence method. ([G], [O2], [St]).

Consequently, we obtain the following curvatures.

$$\begin{aligned} K_1 &= -\frac{c}{gh}(f_{11})_{\theta_2}, & K_2 &= -\frac{1}{gh}(f_{12})_{\theta_2\theta_2}, \\ K_3 &= \frac{1}{hk} \{ (f_{12})_{\theta_0} + (f_{12})_{\theta_2}(f_{12})_{\theta_1} - (f_{12})_{\theta_2\omega_2} \}, \\ K_4 &= -\frac{g}{ck}(f_{22})_{\theta_1}, & K_5 &= -\frac{1}{ck}(f_{12})_{\theta_1\theta_1}, \\ K_6 &= \frac{1}{hk} \{ (f_{12})_{\theta_0} + (f_{12})_{\theta_1}(f_{12})_{\theta_2} - (f_{12})_{\theta_1\omega_1} \}, \\ K_7 &= \frac{1}{ch^2} \{ (f_{11})_{\theta_2\theta_1\omega_2} + (f_{11})_{\theta_2\theta_2}(f_{22})_{\theta_1} + (f_{11})_{\theta_2\theta_1}(f_{22})_{\theta_2} \\ &\quad - (f_{12})_{\theta_2\theta_1}(f_{11})_{\theta_2} - (f_{12})_{\theta_2\theta_2}(f_{12})_{\theta_2} - (f_{12})_{\theta_2\theta_1}(f_{11})_{\theta_1} \\ &\quad + 2(f_{12})_{\theta_2\theta_1}(f_{12})_{\theta_2} \}, \\ K_8 &= \frac{1}{chk} \{ (f_{12})_{\theta_2\theta_1\omega_2} - (f_{12})_{\theta_1\theta_0} - (f_{12})_{\theta_1\theta_1}(f_{12})_{\theta_2} \}, \\ K_9 &= \frac{2(f_{12})_{\theta_2\theta_1} - (f_{11})_{\theta_1\theta_1}}{ch}, \end{aligned}$$

$$\begin{aligned}
K_{10} &= \frac{1}{hk} \left\{ -(f_{12})_{\varrho_0} - (f_{12})_{\varrho_1} (f_{12})_{\varrho_2} + (f_{11})_{\varrho_2} (f_{22})_{\varrho_1} + (f_{12})_{\varrho_2 \varrho_2} \right\}, \\
K_{11} &= \frac{1}{hk} \left\{ (f_{11})_{\varrho_1 \varrho_2} - 2(f_{12})_{\varrho_2 \varrho_2} \right\}, \\
K_{12} &= \frac{1}{ch^2} \left\{ (f_{11})_{\varrho_1 \varrho_0} - 2(f_{12})_{\varrho_2 \varrho_0} + (f_{11})_{\varrho_1 \varrho_1} (f_{12})_{\varrho_2} \right. \\
&\quad \left. + (f_{11})_{\varrho_1 \varrho_2} (f_{12})_{\varrho_1} - 2(f_{12})_{\varrho_1 \varrho_2} (f_{12})_{\varrho_2} - 2(f_{12})_{\varrho_2 \varrho_2} (f_{12})_{\varrho_1} \right\}, \\
K_{13} &= \frac{1}{hk} \left\{ 2(f_{12})_{\varrho_1 \varrho_1} - (f_{22})_{\varrho_2 \varrho_1} \right\}, \\
K_{14} &= \frac{1}{chk} \left\{ -2(f_{12})_{\varrho_1 \varrho_0} - 2(f_{12})_{\varrho_1 \varrho_1} (f_{12})_{\varrho_2} - 2(f_{12})_{\varrho_1 \varrho_2} (f_{12})_{\varrho_1} \right. \\
&\quad \left. + (f_{22})_{\varrho_2 \varrho_0} + (f_{22})_{\varrho_1 \varrho_2} (f_{12})_{\varrho_2} + (f_{22})_{\varrho_2 \varrho_2} (f_{12})_{\varrho_1} \right\}, \\
K_{15} &= \frac{2(f_{12})_{\varrho_1 \varrho_2} - (f_{22})_{\varrho_2 \varrho_2}}{gk}.
\end{aligned}$$

where, curvature functions are expressed by the dual frame:

$$\begin{aligned}
\partial_{\varrho_0} &:= \frac{\partial}{\partial y}, & \partial_{\varrho_1} &:= \frac{\partial}{\partial z_1}, & \partial_{\varrho_2} &:= \frac{\partial}{\partial z_2}, \\
\partial_{\varrho_1} &:= \frac{\partial}{\partial x_1} + z_1 \frac{\partial}{\partial y} + f_{11} \frac{\partial}{\partial z_1} + f_{12} \frac{\partial}{\partial z_2}, \\
\partial_{\varrho_2} &:= \frac{\partial}{\partial x_2} + z_2 \frac{\partial}{\partial y} + f_{21} \frac{\partial}{\partial z_1} + f_{22} \frac{\partial}{\partial z_2}.
\end{aligned}$$

From the general theory of  $G$ -structure, a vanishing condition of curvatures gives the following theorem:

**Theorem 2.3** ([N]). *Suppose that the second order PDE satisfies the integrability condition  $A = B = 0$ . Then, the equation (1) is (locally) equivalent to the flat equation under lifts of scale transformations if and only if the curvatures  $K_i$  vanish.*

We have some results from the theorem:

First, the functions  $f_{ij}$  satisfying  $A = B = K_i = 0$  are written as quadratic polynomials in  $z_1, z_2$ . Hence, if there is a polynomial of degree greater than 2 as a polynomial of  $z_1, z_2$  among  $f_{ij}$ , then corresponding equation (1) is not equivalent to the flat equation under lifts of scale transformations.

Next, we give some examples of equation which is equivalent to the flat equation.

**Corollary 2.4.** *Suppose that the functions  $f_{ij}$  in (1) are given in the following form:  $f_{11} = P(x_1, x_2, y)$ ,  $f_{12} = Q(x_1, x_2, y)$ ,  $f_{22} = R(x_1, x_2, y)$ . Then the equation (1) is (locally) equivalent to the flat equation under the lifts of scale transformations if and only if  $P_y = Q_y = R_y = 0$ ,  $P_{x_2} = Q_{x_1}$ ,  $Q_{x_2} = R_{x_1}$ .*

**Remark 2.5.** The conditions  $P_y = Q_y = R_y = 0$ ,  $P_{x_2} = Q_{x_1}$ ,  $Q_{x_2} = R_{x_1}$  in Corollary 2.4 are obtained by the integrability condition  $A = B = 0$ . Therefore, the second order PDE (1) for the functions  $f_{ij}$  given by the above form is equivalent to the flat equation if and only if it is integrable.

### 3. DUALITY ASSOCIATED WITH DIFFERENTIAL EQUATIONS

In this section, we discuss a duality between the coordinate space and the solution space associated with the following flat equation;

$$(9) \quad \frac{\partial^2 y}{\partial x_i \partial x_j} = 0 \quad (1 \leq i, j \leq 2).$$

For the purpose, we consider the following double fibration.

$$(10) \quad \begin{array}{ccc} & J^1(\mathbb{R}^2, \mathbb{R}) & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{R}^3 := \{(x_1, x_2, y)\} & & \mathbb{R}^3 := \{(a, b, c)\} \end{array}$$

where, projections  $\pi_1, \pi_2$  are defined by

$$\begin{aligned} \pi_1(x_1, x_2, y, z_1, z_2) &= (x_1, x_2, y), \\ \pi_2(x_1, x_2, y, z_1, z_2) &= (z_1, z_2, y - z_1 x_1 - z_2 x_2). \end{aligned}$$

We call the double fibration (10) the model space of the flat equation or flat model space. In this fibration, we regard the left base space as a coordinate space  $\mathbb{R}^3 := \{(x_1, x_2, y)\}$ , and a right base space as a solution space  $\mathbb{R}^3 := \{(a, b, c)\}$ . Solutions of (9) are written as  $y = ax_1 + bx_2 + c$  for real parameters  $a, b, c$ . Graphs of solutions are planes on  $\mathbb{R}^3$  or  $J^1(\mathbb{R}^2, \mathbb{R})$ , and the 3-parameter family of solutions yields a 2-dimensional foliation on  $J^1(\mathbb{R}^2, \mathbb{R})$ . Then the leaf space of this foliation is interpreted as a solution space of (9). We discuss the compactification of the flat model space. The fibration (10) can be embedded naturally into the following (global) double fibration:

$$(11) \quad \begin{array}{ccc} & \mathbb{F}_V(1, 3) & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{P}(V) & & \mathbb{P}(V^*) \cong Gr(3, 4) \end{array}$$

where,  $V = \mathbb{R}^4$ , and  $Gr(3, 4)$  is a Grassmannian manifold and  $\mathbb{F}_V(1, 3)$  is a flag variety:

$$\begin{aligned} Gr(3, 4) &= \{E \mid E \text{ is a hyperplane of } V := \mathbb{R}^4\}, \\ \mathbb{F}_V(1, 3) &= \{(l, E) \mid l \in \mathbb{R}P^3, E \in Gr(3, 4) \cong \mathbb{R}P^3, l \subset E\}. \end{aligned}$$

The, projections  $\pi_1, \pi_2$  are defined by

$$\pi_1([u], H) := [u], \quad \pi_2([u], H) := [f_H],$$

where,  $f_H$  is a linear functional satisfying  $\ker(f_H) = H$  of  $V^* \setminus \{0\}$ . (Since  $f_H$  is uniquely defined up to scalar multiplication,  $\pi_2$  is well-defined.) The double fibration (10) do not depend on coordinate transformation group  $\mathcal{G}$ . So, we introduce the flat model space depending on  $\mathcal{G}$ .

We fix a coordinate transformation group  $\mathcal{G} \subset \text{Diff}(\mathbb{R}^3)$ . First, we define the following symmetry group  $([O1], [O2])$ .

**Definition 3.1.** Let  $G$  be an isotropy subgroup of the flat equation (9) in  $\mathcal{G}$ . This group  $G$  is called symmetry group of the flat equation for  $\mathcal{G}$ .

In the case of  $\mathcal{G} = \text{Diff}(\mathbb{R}^3)$ , the symmetry group is  $SL(4, \mathbb{R})$  and the action on the coordinate space  $\mathbb{R}^3$  is given by:

$$(12) \quad \text{For } \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \in SL(4, \mathbb{R}),$$

$$(x_1, x_2, y) \mapsto \left( \frac{a_1x_1 + a_2x_2 + a_3y + a_4}{d_1x_1 + d_2x_2 + d_3y + d_4}, \frac{b_1x_1 + b_2x_2 + b_3y + b_4}{d_1x_1 + d_2x_2 + d_3y + d_4}, \frac{c_1x_1 + c_2x_2 + c_3y + c_4}{d_1x_1 + d_2x_2 + d_3y + d_4} \right).$$

Next, we introduce subgroups of  $SL(4, \mathbb{R})$  as follows:

$$H_i := \{g \in SL(4, \mathbb{R}) \mid g[e_i] = [e_i]\},$$

$$\bar{H}_i := \{g \in SL(4, \mathbb{R}) \mid {}^t g^{-1}[e_i] = [e_i]\},$$

where,  $e_i$  ( $i=1, \dots, 4$ ) are standard basis of  $\mathbb{R}^4$ , and  $[e_i]$  are corresponding elements in  $RP^3$ . The subgroups  $H_i$  are isotropy subgroups which preserve lines  $[e_i]$ , and the subgroups  $\bar{H}_i$  are isotropy subgroups which preserve hyperplanes spanned by  $e_j$  ( $j \neq i$ ) respectively. We used Cartan involution  $\tilde{\theta}(g) = {}^t g^{-1}$  in the definition of  $\bar{H}_i$ . We consider the following double fibration.

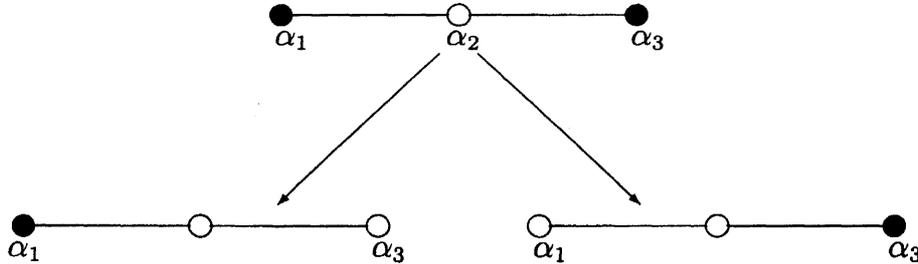
$$(13) \quad \begin{array}{ccc} & G/(G \cap H) & \\ & \swarrow \quad \searrow & \\ (G \cap H_4)/(G \cap H) & & (G \cap \bar{H}_1)/(G \cap H) \\ \swarrow \quad \searrow & & \swarrow \quad \searrow \\ G/(G \cap H_4) & & G/(G \cap \bar{H}_1) \end{array}$$

where,  $H = H_4 \cap \bar{H}_1$ . We call this fibration as a model space of the flat equation (9) with respect to  $\mathcal{G}$ .

In the case of  $\mathcal{G} = \text{Diff}(\mathbb{R}^3)$ , we obtain the following well-known fibration using corresponding symmetry group  $G = SL(4, \mathbb{R})$ .

$$(14) \quad \begin{array}{ccc} & SL(4, \mathbb{R})/H \cong \mathbb{F}_V(1, 3) & \\ & \swarrow \quad \searrow & \\ RP^2 \cong H_4/H & & \bar{H}_1/H \cong RP^2 \\ \swarrow \quad \searrow & & \swarrow \quad \searrow \\ RP^3 \cong SL(4, \mathbb{R})/H_4 & & SL(4, \mathbb{R})/\bar{H}_1 \cong RP^3 \end{array}$$

This fibration equals the fibration (11). Also, this fibration corresponds to the following picture by Dynkin diagrams:



From the picture, we have the fibration (14) by the following process ([Ya]). Let  $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$  be the simple root system and  $\Phi^+$  be the set of positive roots. If we choose a non-empty subset  $\Delta_1$  of  $\Delta$ , then we have the decomposition  $\Phi^+ = \cup_{k \geq 0} \Phi_k^+$ , where  $\Phi_k^+$  for  $k \geq 0$  are subset defined by

$$\Phi_k^+ := \left\{ \alpha = \sum_{i=1}^3 n_i(\alpha) \alpha_i \in \Phi^+ \mid \sum_{\alpha_i \in \Delta_1} n_i(\alpha) = k \right\}$$

Corresponding to the decomposition, we can choose a parabolic subalgebra as follows:

$$\mathfrak{p} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Phi_0^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \oplus \sum_{\alpha \in \Phi_k^+, k > 0} \mathfrak{g}_\alpha.$$

Let  $P \subset G = SL(4, \mathbb{R})$  be the parabolic Lie subgroup whose Lie algebra is equal to  $\mathfrak{p}$ , and put  $M := G/P$ . This space is called the model space of parabolic geometry associated with subset  $\Delta_1$  of simple root system  $\Delta$ . Now, we determine the topology of the above fibration by Dynkin diagrams using this construction. We choose subsets  $\Delta_1 = \{\alpha_1\}$ ,  $\Delta_2 = \{\alpha_3\}$ ,  $\Delta_{12} = \{\alpha_1, \alpha_3\}$  of simple root system. Corresponding to the choice, we have the following parabolic subalgebras:

$$\mathfrak{p}_1 = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \in \mathfrak{sl}(4, \mathbb{R}) \right\}, \quad \mathfrak{p}_2 = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in \mathfrak{sl}(4, \mathbb{R}) \right\},$$

$$\mathfrak{p}_{12} = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in \mathfrak{sl}(4, \mathbb{R}) \right\}.$$

We have corresponding parabolic subgroups  $P_1, P_2, P_{12} \subset G$  and model spaces  $G/P_1 \cong \mathbb{R}P^3$ ,  $G/P_2 \cong \mathbb{R}P^3$ ,  $G/P_{12} \cong \mathbb{F}_V(1, 3)$ , where  $V = \mathbb{R}^4$ . Thus, we can identify the above picture constructed by Dynkin diagrams with the fibration (14). We can use Tanaka theory for the equivalence problem associated with this case ([Ya]).

In the case of  $\mathcal{G} = \text{ScaleDiff}(\mathbb{R}^3)$ , we calculate the corresponding flat model space. From the action (12) of  $SL(4, \mathbb{R})$  on  $\mathbb{R}^3$ , we have the following symmetry group from restriction of variables associated with the scale transformation.

$$G = \left\{ \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in SL(4, \mathbb{R}) \right\}.$$

Then, we have the following.

**Proposition 3.2.** *We obtain the following double fibration as the flat model space associated with  $\mathcal{G} = \text{ScaleDiff}(\mathbb{R}^3)$ :*

$$(15) \quad \begin{array}{ccc} & G/(G \cap H) \cong \mathbb{R}^3 & \\ & \swarrow \quad \searrow & \\ (G \cap H_4)/(G \cap H) \cong \{0\} & & (G \cap \bar{H}_1)/(G \cap H) \cong \mathbb{R}^2 \\ & \swarrow \quad \searrow & \\ G/(G \cap H_4) \cong \mathbb{R}^3 & & G/(G \cap \bar{H}_1) \cong \mathbb{R} \end{array}$$

This fibration is degenerate. Since  $\mathcal{G} = \text{ScaleDiff}(\mathbb{R}^3)$  is very strongly restricted from  $\text{Diff}(\mathbb{R}^3)$ , this degeneration arises. Hence, we consider the following problem.

**Problem 3.3.** *Find a symmetry group for proper subgroup  $\mathcal{G}$  of  $\text{Diff}(\mathbb{R}^3)$ , from which has double fibration of compact-type as a flat model space.*

For the problem, we consider the following subgroup  $G \subset SL(4, \mathbb{R})$ :

$$(16) \quad G = \{g \in SL(4, \mathbb{R}) \mid g[e_3] = [e_3], {}^t g^{-1}[e_3] = [e_3]\},$$

$$(17) \quad = \left\{ \begin{pmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & * & 0 \\ * & * & 0 & * \end{pmatrix} \in SL(4, \mathbb{R}) \right\}.$$

Note that  $G$  is a subgroup invariant under Cartan involution  $\tilde{\theta}$ . We have the following fibration of compact-type with respect to the group:

**Theorem 3.4 ([N]).** *A double fibration constructed by the group (16) is the following fibration of compact-type.*

$$(18) \quad \begin{array}{ccc} & G/(G \cap H) \cong \mathbb{F}_{\mathbb{V}^3}(1, 2) & \\ & \swarrow \quad \searrow & \\ (G \cap H_4)/(G \cap H) \cong S^1 & & (G \cap \bar{H}_1)/(G \cap H) \cong S^1 \\ & \swarrow \quad \searrow & \\ G/(G \cap H_4) \cong \mathbb{R}P^2 & & G/(G \cap \bar{H}_1) \cong \mathbb{R}P^2 \end{array}$$

We note that the coordinate transformation group  $\mathcal{G}$  corresponding to this symmetry group (16) is constructed by the transformations of the form:

$$X_1 = X_1(x_1, x_2), \quad X_2 = X_2(x_1, x_2), \quad Y = \frac{y}{A(x_1, x_2)}.$$

## REFERENCES

- [BG] R. Bryant, P. Griffiths: Toward a geometry of differential equation, *Geometry, Topology, and physics*, International Press (1995), 1–76.
- [B] R. Bryant: Élie Cartan and geometric duality, *Journées Élie Cartan 1998 et 1999*, Intitut Élie Cartan 16 (2000), 5–20.
- [G] R. B. Gardner: The method of equivalence and its applications, *CBMS-NSF Regional Conf. Ser. in Appl. math*, 58, SIAM. Philadelphia (1989).
- [GTW] C. Grissom, G. Thompson, G. Wilkens: Linearization of second order ordinary differential equations via Cartan’s equivalence method, *J. Differential Equations*, **77**, (1989), 1–15.
- [HK] Hurtubise, J. C. ,Kamran, N: Projective connections, double fibrations, and formal neighborhood of lines, *Math. Ann.* **292**, (1992) no.3 383–409.
- [IL] Thomas, A. Ivey, J. M. Landsberg: *Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems*, American Mathematical Society 2003.
- [M] T. Morimoto: Two great books by Élie Cartan, *Pleasure of mathematics*, no.29, 98–104, Nippon-Hyouronsha, (2002) (Japanese).
- [N] T. Noda: Equivalence problem for second order PDE and double fibration as a flat model space, preprint.
- [O1] P. Olver: *Applications of Lie Groups to Differential Equations (Second Edition)*, Graduate Texts in Mathematics, Springer 2000.
- [O2] P. Olver: *Equivalence, Invariants, and Symmetry*, Cambridge University Press, 1995.
- [OSS] T. Ozawa, H. Sato, H. Suzuki: *Differential Equations and Schwarzian Derivatives, noncommutative geometry and physics 2005*, Proceedings of the International Sendai-Beijing Joint Workshop, (2007) 129–149.
- [Sa] H. Sato, *Orbit decomposition of space of differential equations, mathematics of 21 century ~ untrodden peak of geometry~Nippon-Hyouronsha*, (2004) 267–280 (Japanese).
- [St] Sternberg, *Lectures on differential geometry*, Chelsea (1983).
- [Ta] N. Tanaka: On the equivalence problems associated with simple graded Lie algebras, *Hokkaido Math. J.* **8** (1979), no. 1, 23–84.
- [Ya] K. Yamaguchi: Differential systems associated with simple graded Lie algebras, *Advanced Studies in Pure Math.*, **22** (1993), 413–494.
- [YY] Yoshikawa, Atsuko Yamada: Equivalence problem of third-order ordinary differential equations, *Internat. J. Math.* **17** (2006), no. 9, 1103–1125.

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