数理解析研究所講究録
第1664巻 2009年 105-116

4次元多様体の手術とEngel構造

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1 Introduction

Each geometric structure has each own geometry. In this article, geometry with Engel structure is discussed. Construction of manifolds with certain geometric structures, and classification of such structures on a manifold are interesting problems. In order to tackle such problems, some tools should be needed. However, there are not enough such tools on Engel structure and 4-dimensional topology, so far. Therefore, as a tool on Engel geometry, a new surgery of manifolds is introduced in this article.

Engel structures are interesting object for differential topology. An *Engel structure* is a distribution of rank 2 on a 4-dimensional manifold which is maximally non-integrable (see Section 2.1 for precise definition). A distribution is a subbundle of the underlying manifold. Engel structures have an important property like contact structures. All Engel structures are locally equivalent. Therefore, global study is important for Engel structures ([M], [Ad1],..., etc). Recently, sufficient condition for the existence of an Engel structure is obtained by Vogel [V]: There exists an Engel structure on a 4-dimensional manifold if and only if the manifold is parallelizable. Then Engel manifolds must be going to be studied as a object for global differential topology.

A contact structure on a 3-dimensional manifold is a distribution of rank 2 which is completely non-integrable. Engel structures and contact structures on 3-dimensional manifolds are so closely related that mutual contributions between Engel topology and 3-dimensional contact topology are expected. Although locally an Engel structure is considered as a prolongation of contact structure (see Sect 2.2), globally some Engel manifolds can not be constructed as prolongations of contact manifolds. Therefore, not only applications of contact topology but its own geometry is important. Contact topology have been developing remarkably these 15years. One of the reason is the relation between contact structures and open book structures (see Sect. 3). An motivation for this article is to look for such kind of structure for Engel structures.

In this article a new notion of handle, torus round handle, is introduced (see Sect. 5). It is affected by the notion "round handle" due to Asimov [As] (see Sect. 4). Roughly speaking, T^2-round handle is an ordinary handle times 2-dimensional torus. An application of a result, Theorem B, of this article to 4-dimensional manifold is the following ([Ad2]):

Theorem A. *Any closed orientable parallelizable* 4*-manifold is obtained from* $S^3 \times S^1$ *by* T^2*-round surgeries.*

In Sect. 6, an example of construction of closed Engel manifolds is given.

2 Engel structures and prolongations of contact structures

2.1 Basic definitions

An Engel structure is a maximally non-integrable distribution of rank two on a 4-dimensional manifold. Generally, it is defined as follows. Let M be a 4-dimensional manifold, and D a distribution, or a subbundle of the tangent bundle TM, of rank 2. We can regard D as a locally free sheaf of vector fields on M. Let $[X,Y]$ denote a sheaf of vector fields generated by all Lie brackets $[X,Y]$ of vector fields X, Y which are cross-sections of D. Set $D^2 := D + [D,D]$, and $D^3 := D^2 + [D^2, D^2]$. Then, an *Engel structure* on M is defined as a distribution $D \subset TM$ of rank 2 which satisfies the following conditions:

$$\operatorname{rank} D_p^2 = 3, \qquad \operatorname{rank} D_p^3 = 4 \tag{2.1}$$

at any point $p \in M$.

An Engel structure has a characteristic direction. Let D be an Engel structure on a 4-dimensional manifold M. From this Engel structure D, a line field is defined as follows: $L(D) := \{X \in D \mid [X, D^2] \subset D^2\}$. The line field $L(D)$ is called the *Engel line field.* It is known that a contact structure is induced from an even-contact structure D^2 on an embedded manifold $N \subset M$ which is transverse to the Engel line field $L(D)$. The contact structure is obtained as $D^2 \cap TN$. Such a procedure is called a *deprolongation* (see [M], [BCG3]).

In this paper, we work just in the standard Engel space $(\mathbb{R}^4, E)$, that is, an ordinary 4-dimensional space $\mathbb{R}^4$ endowed with the standard Engel structure. The *standard Engel structure* on $\mathbb{R}^4$ is defined as a kernel of the following pair ω_1, ω_2 of 1-forms:

$$\omega_1 = dy - z dx, \qquad \omega_2 = dz - w dx, \tag{2.2}$$

where $(x,y,z,w) \in \mathbb{R}^4$ are coordinates. Let E denotes the standard Engel structure on $\mathbb{R}^4$:

$$E := \{\omega_1 = 0,\ \omega_2 = 0\} = \operatorname{Span}\left\{\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}, \frac{\partial}{\partial w}\right\}.$$

We call the 4-dimensional space $(\mathbb{R}^4, E)$ endowed with the standard Engel structure the *standard Engel space.* It is clear that the standard Engel structure E actually satisfies the condition (2.1) of the definition. In this case, the Engel line fields is $L(E) = \operatorname{Span}\{\partial/\partial w\}$. With respect to the standard Engel structure on $\mathbb{R}^4$, the induced contact structure on $\mathbb{R}^3 \subset \mathbb{R}^4$, the (x,y,z)-space, is $C = \{\omega_1 = dy - z dx = 0\}$. It is also called the *standard contact structure* on $\mathbb{R}^3$.

2.2 Prolongations

A certain Engel manifold is constructed from a 3-dimensional contact manifold. A contact structure is a completely non-integrable distribution of corank one on an odd-dimensional manifold. Let E be a contact structure on a 3-dimensional manifold N. By taking fibrewise porjectivization of the contact structure E, we obtain a new 4-dimensional manifold $\mathbb{P}E = \cup_{x \in N} \mathbb{P}(E_x)$. On the 4-dimensional manifold $\mathbb{P}E$, an Engel structure $D(E)$ is defined as $D(E)_q := (d\pi)^{-1} l$, where $\pi\colon \mathbb{P}E \to M$ is a canonical projection, $q = (p, l) \in \mathbb{P}E$ is a point, and $l \in T_p M$ is a line (see [M]). Such a procedure is called a *Cartan prolongation* (see [BCG3], [M], [Ad1]).

3 Open book structure and Contact structure

Let us begin with ordinary open book structure. Let M be a manifold. An *open book structure* on M is a pair (Σ, h) of a submanifold $\Sigma \subset M$ of codimension one with non-empty boundary and a diffeomorphism $h\colon \Sigma \to \Sigma$ which is identity on $\partial\Sigma$ that satisfies the following property: Setting $\Sigma(h) := (\Sigma \times [0,1])/_\sim$, where $(x, 0) \sim (h(x), 1)$ for any $x \in \Sigma$, M is diffeomorphic to $\Sigma(h) \cup_{\mathrm{id}} (\partial\Sigma \times D^2)$. The submanifold Σ is called a *page*, the diffeomorphism $h\colon \Sigma \to \Sigma$ is called the *monodoromy mapping*, and the submanifold $B \subset M$ corresponding to $\partial\Sigma \times \{0\} \subset \partial\Sigma \times D^2$ is called the *binding*. The decomposition of M into bindings and pages is called an *open book decomposition* of M. The existence of an open book structure on any closed orientable 3-dimensional manifold was proved by Alexander [Al]. Furthermore, number of connected components of binding was studied by Myers [M]:

Theorem 3.1 (Myers). *On any closed orientable* 3*-manifold, there is an open book structure with one connected binding.*

We should remark that open book structure has important relation with contact structure. It largely contributes to recent developments in contact topology (see [G], [H]).

4 Round handle

Let us begin with the definition of ordinary handle. Let W be a compact manifold of dimension n with boundary. A *handle* of dimension n and index k attached to W is defined as a pair $h_k = \left(D^k \times D^{n-k}, f\right)$ of an n-disk with corner and an embedding $f\colon \partial_- \left(D^k \times D^{n-k}\right) \to \partial W$, where $\partial_- \left(D^k \times D^{n-k}\right) = \partial D^k \times D^{n-k}$. Let $W \cup_f h_k$ or $W + h_k$ denote the manifold obtained from W and $D^k \times D^{n-k}$ by the attaching mapping f. Sometimes, h_k also denotes $D^k \times D^{k-n}$ itself and the corresponding subset in $W \cup_f h_k$.

Round handle is introduced by Asimov [As] to study the non-singular Morse-Smale flow. A *round handle* of dimension n and index k attached to W is defined as a pair $R_k = \left(D^k \times D^{n-k-1} \times S^1, \psi\right)$ of a manifold with corner and an embedding $\psi\colon \partial_- \left(D^k \times D^{n-k-1} \times S^1\right) \to \partial W$, where $\partial_- \left(D^k \times D^{n-k-1} \times S^1\right) = \partial D^k \times D^{n-k-1} \times S^1$. Let $W \cup_\psi R_k$ or $W + R_k$ denote the round handle body. Sometimes, R_k also denotes $D^k \times D^{k-n-1} \times S^1$ itself and the corresponding subset in $W \cup_\psi R_k$.

The decomposability of manifold into round handles was studied by Asimov [As]. He studied the decomposability of flow manifolds which are defined as follows. A *flow manifold* is a pair $(W, \partial_- W)$ of a compact connected manifold W and some specified union $\partial_- W$ of connected components of ∂W which satisfies that there is a non-singular vector field on W looking inward on $\partial_- W$ and outward on $\partial_+ W := \partial W \setminus \partial_- W$. The following theorem is proved in [As].

Theorem 4.1 (Asimov). *Let W be a flow manifold whose dimension is greater than* 3. *Then, W has a round handle decomposition.*

A generalization of this theorem is Theorem B in this paper.

One of the main tool to prove Theorem 4.1 is the following.

Lemma 4.2 (Asimov). *Let W be a manifold with non-empty boundary $\partial W \neq \emptyset$, and $\partial_1 W \subset \partial W$ a connected component. Assume that two handles h_k and h_{k+1} of index k and $k+1$ respectively, $k \geq 1$, are attached to $\partial_1 W$ independently. Then, $W + h_k + h_{k+1}$ is diffeomorphic to $W + R_k$, where R_k is a round handle of index k.*

A version of this lemma for round handles is one of important tools to prove Theorem B. In order to prove such a version, we need ideas in the proof of Lemma 4.2. Then, we roughly review the proof of Lemma 4.2.

Rough sketch of the proof of Lemma 4.2. The idea is to slide h_{k+1} onto h_k so that the union $h_{k+1} \cup h_k$ can be regarded as a round handle R_k.

The isotopy is constructed as an isotopy of the attaching sphere of h_{k+1} as follows. Let $h_k = \left(D^k \times D^{n-k}, f_k\right)$ and $h_{k+1} = \left(D^{k+1} \times D^{n-k-1}, f_{k+1}\right)$ be the given two handles. First, we can take an embedded path $a\colon [0.1] \to \partial\left(W + h_k + h_{k+1}\right)$ connecting the attaching sphere $f_{k+1}\left(\partial D^{k+1} \times \{0\}\right)$ of h_{k+1} and the corner $f_k\left((\partial D^k \times \partial D^{k-1}\right)$ of h_k which satisfies the following conditions (see Figure 1):

- $a(0) \in f_{k+1}\left(\partial D^{k+1} \times \{0\}\right)$,
- $a(1/2), a(1) \in f_k\left(\partial D^k \times \partial D^{n-k}\right)$,
- $a((0,1/2)) \cap \left(f_{k+1}(\partial D^{k+1} \times \{0\}) \cup f_k(\partial D^k \times \partial D^{n-k})\right) = \emptyset$,
- $a([1/2,1]) \subset D^k \times \partial D^{n-k} = \partial_+ h_k$,
- $a([1/2,1])$ intersects with $\{0\} \times \partial D^{n-k}$ once transversely.

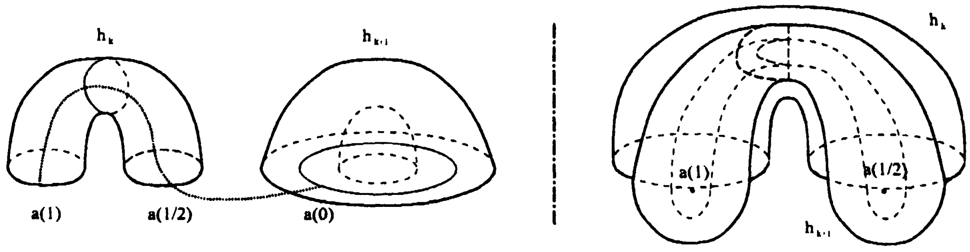

Figure 1: isotopy of h_{k+1}

Let $\varphi_t \colon \partial_- h_k \longrightarrow \partial_+ (W + h_k)$ be an isotopy pulling $N(a(0)) \times D^{n-k-1}$ along the path a and keeping the rest fix, where $N(a(0)) \subset f_{k+1}\left(\partial D^{k+1} \times \{0\}\right)$ is a neighborhood of $a(0)$ in the attaching sphere of h_{k+1}. Furthermore, we make h_k shrink to a neighborhood of the transverse disk $\{0\} \times D^{n-k} \subset D^n \times D^{n-k}$ of h_k. After applying the isotopies above, we still use the same notations $h_k = \left(D^k \times D^{n-k}, f_k\right)$, $h_{k+1} = \left(D^{k+1} \times D^{n-k-1}, f_{k+1}\right)$ (see Figure 1).

The obtained handle body $W + h_k + h_{k+1}$ is regarded as $W + R_k$ as follows. According to proper coordinates, the handles are regarded as $h_k = D^k \times \left(D^1 \times D^{n-k-1}\right)$ and $h_{k+1} = \left(D^k \times D^1\right) \times D^{n-k-1}$. These coordinates are also taken so that the intersection of two handles are written down as $h_k \cap h_{k+1} = D^k \times \left(\partial D^1 \times D^{n-k-1}\right)$ from the view point of h_k, and $h_k \cap h_{k+1} = \left(D^k \times \partial D^1\right) \times D^{n-k-1}$ from that of k_{k+1}. Then, the union of handles is

$$h_k \cup h_{k+1} = D^k \times \left(\partial D^1 \cup \partial D^1\right) \times D^{n-k-1} = D^k \times S^1 \times D^{n-k-1} = R_k,$$

and the subset of $(h_k \cup h_{k+1})$ attached to W is

$$\begin{aligned}(h_k \cup h_{k+1}) \cap W &= \partial D^k \times \left(\partial D^1 \cup \partial D^1\right) \times D^{n-k-1} \\ &= \partial D^k \times S^1 \times D^{n-k-1} = \partial_+ R_k.\end{aligned}$$

Thus, $W + h_k + h_{k+1}$ is considered as $W + R_k$ (see Figure 2). $\square$

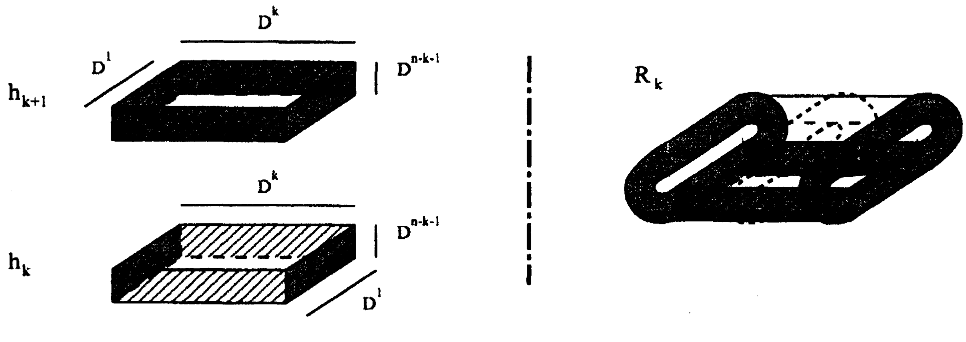

Figure 2: union of handles

Round surgery was introduced by Asimov [As] to study flow manifolds. Let M be an n-dimensional manifold. A *round surgery* of index k on M

is the procedure that embeds $\partial D^k \times D^{n-k} \times S^1$ and removes $\partial D^k \times \mathrm{int}(D^{n-k}) \times S^1$ and glues $D^k \times \partial D^{n-k} \times S^1$ by the identity mapping of $\partial D^k \times D^{n-k} \times S^1$. Similarly to ordinary handles, a kind of cobordisms is defined for round handles. Two closed manifolds M_1, M_2 of dimension N are said to be *round cobordant* if M_2 is obtained from M_1 by a finite sequence of round surgeries. Round surgeries of 3-manifolds were studied by Asimov [As] by using Theorem 4.1.

Theorem 4.3 (Asimov). *Any closed 3-manifold is obtained from S^3 by a finite sequence of round surgeries of index 1 and 2.*

5 Torus round handle

Multi-round handle is defined as a generalization of round handle.

Definition. An i-th *multi-round handle* of dimension n and index k attached to W is defied as a pair

$$Q_k^{(i)} = \left(D^k \times D^{n-k-i} \times \overbrace{S^1 \times \cdots \times S^1}^{i}, \varphi\right)$$

of a manifold with corner and an embedding

$$\varphi\colon \partial_- \left(D^k \times D^{n-k-i} \times S^1 \times \cdots \times S^1\right) \to \partial W,$$

where $\partial_- \left(D^k \times D^{n-k-i} \times S^1 \times \cdots \times S^1\right) = \partial D^k \times D^{n-k-i} \times S^1 \times \cdots \times S^1$. Let $W \cup_\varphi Q_k^{(i)}$ or $W + Q_k^{(i)}$ denote the multi-round handle body.

Sometimes, $Q_k^{(i)}$ also denotes $D^k \times D^{k-n-i} \times S^1 \times \cdots \times S^1$ itself and the corresponding subset in $W \cup_\varphi Q_k^{(i)}$. Note that $Q_k^{(0)}$ is an ordinary k-handle, and $Q_k^{(1)}$ is a round k-handle.

Using multi-round handles instead of ordinary handles, we can define multi-round handle decomposition.

Now, we define multi-round surgery. Let M be an n-dimensional flow manifold, and $f\colon \partial D^k \times D^{n-k-i+1} \times \overbrace{S^1 \times \cdots \times S^1}^{i} \to M$ an embedding.

Definition. The manifold obtained from M by an i-th *round surgery* of index k by f is $M \backslash f(\partial D^k \times D^{n-k-i+1} \times S^1 \times \cdots \times S^1)$ and $D^k \times \partial D^{n-k-i+1} \times S^1 \times \cdots \times S^1$ glued by the identity mapping of $\partial D^k \times \partial D^{n-k-i+1} \times S^1 \times \cdots \times S^1$. Especially, we call a second round surgery a *dual round surgery.*

A kind of cobordisms is defined in a way similar to round cobordism in [As].

Definition. Let M_1, M_2 be n-dimensional manifolds without boundaries. They are said to be i-th *round cobordant* if M_2 is obtained from M_1 by a finite sequence of i-th round surgeries. Especially, we call a second round cobordant a *dual round cobordant.*

Then, by similar arguments to round handle, we obtain the following ([Ad2]):

Theorem B. *Let M be a compact flow manifold. Assume that the dimension of M is greater than four, and that the $(n-1)$st Stiefel-Whitney class vanishes: $w_{n-1}(M)$. Then M can be decomposed into T^2-round handles.*

6 A construction of some closed Engel manifolds

We construct Engel structures on certain closed 4-dimensional manifolds. As a simple closed 4-manifold obtained by T^2-surgeries, we deal with

$$M = (\Sigma \times S^1 \times S^1) \cup (\partial\Sigma \times S^1 \times D^2),$$

where Σ is a compact orientable surface with boundary.

6.1 A construction of an Engel structure on $\Sigma \times S^1 \times S^1$

We construct an Engel structure on $\Sigma \times S^1 \times S^1$. Two well-known methods are applied to the construction. One is that of Thurston and Winkelnkemper [TW], the other is the prolongation of a contact structure (see in Sect. 2.2).

First, we construct a contact form on $\Sigma \times S^1$, where Σ is a compact orientable surface with boundary. Thurston and Winkelnkemper constructed in [TW] a contact form on $\Sigma \times S^1$ as follows:

$$\omega := K \cdot d\theta + \alpha,$$

where θ is coordinate of S^1, α is a 1-form on Σ whose derivative $d\alpha$ is a volume form, and $K > 0$ is a sufficiently large constant.

Next, we prolong the contact 3-manifold $(\Sigma \times S^1, \omega)$. In the prolongation procedure, a contact framing should be chosen carefully. Since $d\omega$ is a volume form on Σ, the contact framing depends on the Euler characteristic of the surface Σ. In other words, we can count how many times the framing rotates on the boundary of Σ. This information is important to glue. Then, by perturbing, we obtain an Engel structure on $\Sigma \times S^1 \times S^1$ whose characteristic line field is transverse to the boundary.

Note that the boundary of $\Sigma \times S^1 \times S^1$ is the disjoint union of 3-tori on which contact structures are induced.

6.2 A construction of an Engel structure on $S^1 \times S^1 \times D^2$

Now, we construct Engel structures on $S^1 \times S^1 \times D^2$ which induce all tight contact structures on T^3. Tight contact structures on T^3 are classified by Kanda [K].

Let $(s, t, (x_1, x_2))$ be coordinates of $S^1 \times S^1 \times D^2$. Setting

$$C_1 := \frac{\partial}{\partial x_2}, \qquad C_2 := \frac{\partial}{\partial s} - x_2 \frac{\partial}{\partial x_1},$$

we obtain a contact framing of $S^1 \times D^2$. That is, $\mathrm{Span}\{C_1, C_2\}$ is a contact structure on $S^1 \times D^2$. In addition, set $W := \partial/\partial t - x_1(\partial/\partial x_1) - x_2(\partial/\partial x_2)$. Then the plane field

$$E_k := \{W, \cos(kt)C_1 + \sin(kt)C_2\}$$

is an Engel structure on $S^1 \times S^1 \times D^2$. A diffeomorphism $\Theta\colon S^1 \times S^1 \times D^2 \to S^1 \times S^1 \times D^2$ defined by $\Theta(s, t, (r, \theta)) := (s, t, (r, \theta + t))$ changes the isotopy class of E_k. Now, setting

$$E_{k,m} := (\Theta^m)_* E_k,$$

we obtain Engel structures on $S^1 \times S^1 \times D^2$. It is easy to check that the characteristic line field of ${E_{m,k}}^2$ is W, which is transverse to T^3 as the boundary $\partial(S^1 \times S^1 \times D^2)$. Then the even contact structures ${E_{m,k}}^2$ induce contact structures on $T^3 = \partial(S^1 \times S^1 \times D^2)$, which correspond to all Kanda's model for all $m \in \mathbb{Z}$.

6.3 Gluing two Engel manifolds

We glue two Engel manifolds with the same boundary by the method due to Montgomery [M]. On account of his method, two Engel manifolds (W_1, D_1) and (W_2, D_2) can be glued if they satisfy the following conditions:

- $\partial W_1 \cong \partial W_2$,
- $\mathrm{Char} D_i^2 \pitchfork \partial W_i$,
- the induced contact structures on ∂W_i are equivalent.

Then by choosing a suitable Engel structures $E_{m,k}$ on $S^1 \times S^1 \times D^2$ for each components of $\partial(\Sigma \times S^1 \times S^1)$, we can glue $\Sigma \times S^1 \times S^1$ and some $S^1 \times S^1 \times D^2$ with Engel structures.

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