The reduced length of a polynomial with complex or real coefficients

by

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Let for a polynomial $P \in \mathbb{C}[x]$, $P(x) = \sum_{i=0}^{d} a_i x^{d-i} = a_0 \prod_{i=1}^{d} (x-\alpha_i)$, $P^*(x) = \sum_{i=0}^{d} a_i x^{d-i}$, $L(P) = \sum_{i=0}^{d} |a_i|$, $M(P) = |a_0| \prod_{i=1}^{d} \max\{1, |a_i|\}$ and let $\mathbb{C}[x]^1$, $\mathbb{R}[x]^1$ denote the set of monic polynomials over $\mathbb{C}$ or $\mathbb{R}$, respectively.

$L(P)$ is called the length of $P$. Following A. Dubickas [1] we consider $l(P)$, the reduced length of $P$ defined by the formula

$$l(P) = \inf_{G \in \mathbb{C}[x]^1} L(PG),$$

which for $P \in \mathbb{R}[x]$ reduces to

$$(1) \quad l(P) = \inf_{G \in \mathbb{R}[x]^1} L(PG).$$

Actually Dubickas considered only the case $P \in \mathbb{R}[x]$ and called the reduced length of $P$ the quantity $\min\{l(P), l(P^*)\}$. For $P \in \mathbb{R}[x]$ some of the following results of [6] are due to him.

**Proposition 1.** Suppose that $\omega, \eta, \psi \in \mathbb{C}$, $|\omega| \geq 1$, $|\eta| < 1$, then for every $Q \in \mathbb{C}[x]$

(i) $l(\psi Q) = |\psi| l(Q),$

(ii) $l(x + \omega) = 1 + |\omega|,$

(iii) if $T(x) = Q(x)(x - \eta)$, then $l(T) = l(Q),$

(iv) $l(\overline{Q}) = l(Q)$, where $\overline{Q}$ denotes the complex conjugate of $Q.$
Proposition 2. For all \( P, Q \) in \( \mathbb{C}[x]^1 \), all \( \eta \in \mathbb{C} \) with \(|\eta| = 1\) and all positive integers \( k \)

(i) \( \max\{l(P), l(Q)\} \leq l(PQ) \leq l(P)l(Q) \),
(ii) \( M(P) \leq l(P) \),
(iii) \( l(P(\eta x)) = l(P(x)) \),
(iv) \( l(P(x^k)) = l(P(x)) \).

The main problem consists in finding an algorithm of computing \( l(P) \) for a given \( P \). An apparently similar problem in which \( P \) and \( G \) in formula (1) are restricted to polynomials with integer coefficients has been considered in [2] and [3], however the restriction makes a great difference. Coming back to our problem Proposition 1 (iii) shows that it is enough to consider \( P \) with no zeros inside the unit circle. The case of zeros on the unit circle is treated in the following two theorems.

Theorem 1. Let \( P \in \mathbb{C}[x], Q \in \mathbb{C}[x]^1 \) and \( Q \) have all zeros on the unit circle. Then for all \( m \in \mathbb{N} \)

\[ l(PQ^m) = l(PQ). \]

Theorem 2. If \( P \in \mathbb{C}[x]^1 \setminus \mathbb{C} \) has all zeros on the unit circle, then \( l(P) = 2 \) with \( l(P) \) attained, if all zeros are roots of unity and simple \( l(P) \) is attained means that \( l(P) = L(Q) \), where \( Q/P \in \mathbb{C}[x]^1 \).

Proofs for \( P \in \mathbb{R}[x] \) are given in [4], proofs for \( P \in \mathbb{C}[x] \) are essentially the same. We have further (see [6]).

Theorem 3. Let \( P = P_0P_1 \), where \( P_0 \in \mathbb{C}[x], \ P_1 \in \mathbb{C}[x]^1, \ L(P_0) \leq 2|P_0(0)| \). Then

\[ l(P) \geq L(P_0) + (2|P_0(0)| - L(P_0))(l(P_1) - 1). \]

Corollary 1. If \( P \in \mathbb{C}[x] \) and \( L(P) \leq 2|P(0)| \), then

\[ l(P) = L(P). \]

Conversely, if \( l(P) = L(P) \) and all coefficients of \( P \) are real and positive, then \( L(P) \leq 2P(0) \).
Corollary 2. If \( P(x) = (x - \alpha)(x - \beta) \), where \( |\alpha| \geq |\beta| \geq 1 \), then
\[
l(P) \geq 1 + |\alpha| - |\beta| + |\alpha \beta|
\]
with equality if \( \alpha/\beta \in \mathbb{R} \) and either \( \alpha/\beta < 0 \) or \( |\beta| = 1 \).

Corollary 3. Let \( P = P_0P_1 \), where \( P_\nu \in \mathbb{C}[x] \ (\nu = 0, 1) \), \( \deg P_1 \geq 1 \) and all zeros \( z \) of \( P_\nu \) satisfy \( |z| > 1 \) for \( \nu = 0 \), \( |z| = 1 \) for \( \nu = 1 \). If
\[
l (P_0) = L (P_0),
\]
then
\[
l (P) \geq 2M (P).
\]

It remains a problem, whether (3) holds without the assumption (2). The following results of [6] point towards an affirmative answer.

Theorem 4. If \( P \in \mathbb{C}[x] \setminus \{0\} \) has a zero \( z \) with \( |z| = 1 \), then
\[
L(P) > \sqrt{2}M(P), \quad l(P) \geq \sqrt{2}M(P).
\]

Theorem 5. If \( P(x) = (x - \alpha)(x - \beta)(x - 1) \), where \( \alpha, \beta \) are real and at least one of them is positive, then (3) holds.

The question of validity of (3) for all polynomials \( P \) on \( \mathbb{C} \) is equivalent to the following

Problem 1. Is it true that for all polynomials \( P \) in \( \mathbb{C}[x] \) with a zero on the unit circle \( L(P) \geq 2M(P) \)?

The following theorems like Theorem 5 concern \( P \) in \( \mathbb{R}[x] \).

Theorem 6 ([4], Theorem 1). If \( P \in \mathbb{R}[x] \) is of degree \( d \) with \( P(0) \neq 0 \), then \( l(P) = \inf_{Q \in S_d(P)} L(Q) \), where \( S_d(P) \) is the set of all polynomials in \( \mathbb{R}[x] \) divisible by \( P \) with \( Q(0) \neq 0 \) and with at most \( d + 1 \) non-zero coefficients, all belonging to the field \( K(P) \), generated by the coefficients of \( P \).

Theorem 7 ([4], Theorem 2). If \( P \in \mathbb{R}[x] \) has all zeros outside the unit circle, then \( l(P) \) is attained and effectively computable, moreover \( l(P) \in K(P) \).
Theorem 8 ([5], Theorem 1). Let $P(x) = \prod_{i=1}^{3}(x - \alpha_{i})$, where $\alpha_{i}$ distinct, $|\alpha_{1}| \geq |\alpha_{2}| > |\alpha_{3}| = 1$. Then $l(P)$ is effectively computable.

Theorem 9 ([5], Theorem 2). Let $P(x) = (x - \alpha)(x^{2} - \varepsilon)$, where $|\alpha| > 1$, $\varepsilon = \pm 1$. Then

$$l(P) = 2(|\alpha| + 1 - |\alpha|^{-1}).$$

The following problem remains open

Problem 2. How to compute $l(2x^{3} + 3x^{2} + 4)$?

References


