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ACTIONS OF MAASS' OPERATORS ON THE ASYMPTOTIC EXPANSIONS OF NON-HOLOMORPHIC EISENSTEIN SERIES

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ABSTRACT. The present article announces the results in our forthcoming paper [KN]. Let k be an arbitrary even integer, and $E_k(s;z)$ denote the non-holomorphic Eisenstein series (of weight k attached to $SL_2(\mathbb{Z})$) defined by (1.1) below. We show here several effects of the actions of Maass' differential operators in (1.4) on non-holomorphic Eisenstein series; this first leads us to establish a complete asymptotic expansion of $E_k(s;z)$ in the descending order of $y=\operatorname{Im} z$ as $y\to+\infty$ (Theorem 1), upon transferring from the previously obtained asymptotic expansion of $E_0(s;z)$ (due to the first author [Ka7]), through successive use of Maass' operators. Theorem 1 yields various consequences on $E_k(s;z)$, including its functional properties (Corollaries 1.1–1.3), its relevant specific values (Corollaries 1.4–1.7), and its asymptotic aspects as $z\to 0$ (Corollary 1.8). We shall then apply the non-Euclidian Laplacian $\Delta_{H,k}$ (of weight k attached to the upper half-plane) to the resulting expansion of $E_k(s;z)$ (Theorem 2) in order to justify the eigenfunction equation for $E_k(s;z)$, where the justification could be made uniformly in the whole s-plane.

1. INTRODUCTION

Throughout the following, $s=\sigma+it$ denotes a complex variable, $z=x+iy$ a complex parameter in the upper-half plane, and k an arbitrary even integer. The non-holomorphic Eisenstein series $E_k(s;z)$ (of weight k attached to $SL_2(\mathbb{Z})$) is defined by

$$E_k(s;z)=\frac{1}{2}\sum_{\substack{c,d=-\infty\\(c,d)=1}}^{\infty}(cz+d)^{-k}|cz+d|^{-2s}\qquad(\operatorname{Re}s>1-k/2),\tag{1.1}$$

and its meromorphic continuation over the whole s-plane. It is readily seen when $k=0$ that the relation

$$E_0(s;z)=\zeta_{\mathbf{Z}^2}(s;z)/2\zeta(2s)\tag{1.2}$$

holds with the Riemann zeta-function $\zeta(s)$ and the Epstein zeta-function $\zeta_{\mathbf{Z}^2}(s;z)$, defined by

$$\zeta_{\mathbf{Z}^2}(s;z)={\sum_{m,n=-\infty}^{\infty}}'|m+nz|^{-2s}\qquad(\operatorname{Re}s>1)$$

with its meromorphic continuation over the whole s-plane (cf. [Si, Chap.1]), where (and in the sequel) primed summation symbols indicate omission of singular terms.

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Let

$$\frac{\partial}{\partial z}=\frac{1}{2}\Big(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y}\Big) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}}=\frac{1}{2}\Big(\frac{\partial}{\partial x}-\frac{1}{i}\frac{\partial}{\partial y}\Big) \tag{1.3}$$

be (complex) partial differentiations. Then it is in fact possible to transfer from $E_0(s;z)$ to $E_k(s;z)$ by Maass' differential operators (cf. [Maa, Chap.4(12)(13)])

$$\delta_s=\frac{1}{2\pi i}\Big(\frac{\partial}{\partial z}+\frac{s}{2iy}\Big) \quad \text{and} \quad \epsilon_s=\frac{1}{2\pi i}\Big(\frac{\partial}{\partial \bar{z}}-\frac{s}{2iy}\Big), \tag{1.4}$$

which for instance assert on setting $\widehat{\delta}_s=(-4\pi y/s)\delta_s$ and $\widehat{\epsilon}_s=(4\pi y/s)\epsilon_s$ that

$$\begin{aligned} \widehat{\delta}_{s+j}E_{2j}(s-j;z) &=E_{2j+2}(s-j-1;z) \\ \widehat{\epsilon}_{s+j}E_{-2j}(s+j;z) &=E_{-2j-2}(s+j+1;z) \end{aligned} \tag{1.5}$$

for $j=0,1,\ldots$. It is further known that $f_k(s;z)=y^sE_k(s;z)$ satisfies the eigenfunction equation

$$\varDelta_{H,k}f_k(s;z)=s(1-s-k)f_k(s;z), \tag{1.6}$$

where

$$\varDelta_{H,k}=-4y^2\Big(\frac{\partial}{\partial z}+\frac{k}{2iy}\Big)\frac{\partial}{\partial \bar{z}}=(4\pi y)^2\delta_k\epsilon_0, \tag{1.7}$$

by (1.4), denotes the non-Euclidian Laplacian (of weight k) attached to the upper half-plane. The most direct and standard way of justifying (1.6) is to apply $\varDelta_{H,k}$ to each term (multiplied by y^s) of the series in (1.1), and this gives

$$\varDelta_{H,k}\{y^s(cz+d)^{-k}|cz+d|^{-2s}\}=s(1-s-k)y^s(cz+d)^{-k}|cz+d|^{-2s},$$

which immediately implies (1.6) by analytic continuation. This method, however, can not clarify the key ingredients by which the eigenfunction equation (1.6) is to be valid especially in the region $\operatorname{Re}s<1-k/2$, where the series representation in (1.1) diverges.

It is the aim of this article to present several effects of the actions of Maass' differential operators on non-holomorphic Eisenstein series; this first leads us to establish a complete asymptotic expansion of $E_k(s;z)$ in the descending order of $y=\operatorname{Im}z$ as $y\to+\infty$ (Theorem 1) with the explicit t-estimates for the remainder terms (see (2.8) and (2.9)), upon transferring from the previously derived asymptotic expansions of $E_0(s;z)$ (due to the first author [Ka7]) to that of $E_k(s;z)$ through successive use of $\widehat{\delta}_s$ and $\widehat{\epsilon}_s$. Our main formula (2.3) can then be applied to justify the eigenfunction equation (1.6) uniformly in the whole s-plane (Theorem 2 and Corollary 2.1).

Theorem 1 at first implies several known functional properties of non-holomorphic Eisenstein series (Corollaries 1.1–1.3); the proof of Theorem 1 particularly clarifies the key ingredients by which the functional equation of $E_k(s;z)$ is to be valid (see (2.1), (2.14) and Corollary 1.3). Our main formula (2.3) naturally reduces when $k=0$ to the classical Kronecker limit formula for $E_0(s;z)$ as $s\to1$, also to its variants for $(\partial/\partial s)E_0(s;z)$ at $s=0$, and further for the particular values of $E_0(s;z)$ at other integer points (Corollary 1.4). Moreover, the cases $k=\pm2$ of (2.3) show that $E_{\pm2}(s;z)$ have no singularities on the real line, while the real simple poles of $E_k(s;z)$ appear when $|k|\geq4$ at $s=n\in\mathbb{Z}$ either with $-k+1\leq n\leq-k/2-1$ if $k\geq4$, or with $1\leq n\leq-k/2-1$ if $k\leq-4$ (Corollary 1.1). We can deduce (similarly to the case $k=0$) various explicit formulae for specific values of $E_k(s;z)$ associated with the cases above (Corollaries 1.5 and 1.7);

these in particular yield the classical Lambert series expressions of $E_k(0;z)$ ($k \geq 2$) with the base $q = e^{2\pi i z}$ (Corollary 1.6). Furthermore, our main formula (2.3) gives a complete asymptotic expansion of $E_k(s;z)$ in the ascending order of z as $z \to 0$ (Corollary 1.8), through the quasi-modularity of $E_k(s;z)$. The proof of Theorem 2 clarifies, on the other hand, that the key ingredients, by which the eigenfunction equation (1.6) is to be valid, are the differential relations in [KN, Lemma 4].

It is emphasized that the study of asymptotic aspects of $\zeta_{\mathbb{Z}^2}(s;z)$ and further of $E_k(s;z)$ when $y = \operatorname{Im} z$ becomes small or large is of importance from both theoretical and applicable point of views (cf. [CS1][CS2][Maa]). The first author has established a complete asymptotic expansion of $\zeta_{\mathbb{Z}^2}(s;z)$ in the descending order of y as $y \to +\infty$ ([Ka7, Theorem 1]); the method of its proof was further elaborated to show that a similar asymptotic series still exists for the Laplace-Mellin transform of $\zeta_{\mathbb{Z}^2}(s;z)$ with respect to y ([Ka7, Theorem 2]), where the crucial rôles in the proofs were played by Mellin-Barnes type integrals. On the other hand, certain bounded growth conditions for $E_k(s;z)$ as $y \to +0$ and $y \to +\infty$ have recently been applied to determine the region of s in which $E_k(s;z)$ is orthogonal to the space of cusp forms, by the second author ([No2, Theorem 1(I)]), who further proved the related orthogonality (in a local sense) by directly showing that the projection coefficients of $E_k(s;z)$ to the space of cusp forms vanish identically ([No2, Theorem 1(II)]). Here the relevant coefficients are expressed by means of Laplace-Mellin transforms of confluent hypergeometric functions; these transforms were again manipulated with Mellin-Barnes type integrals. It is worth while noting that the integrals of this type have advantage over heuristic treatments in studying certain asymptotic aspects and transformation properties of zeta and theta functions (see also [Ka1–Ka6]).

As for the results related to Theorem 1, an asymptotic formula for $E_0(s;z)$ when $t \to +\infty$ on the line $\sigma = 1/2$ was studied by the second author [No1], while Matsumoto [Mat, (1.6) and (1.7)] obtained asymptotic expansions (with respect to z) of the *holomorphic* Eisenstein series $F(s;z)$ defined by $F(s;z) = \sum'^{\infty}_{m,n=-\infty}(mz+n)^{-s}$, where the branch (of each term) is to be chosen as $-\pi \leq \arg(mz+n) < \pi$. Note that the results in [Mat] above can be regarded as counterparts of our asymptotic expansions (2.3) and (2.35).

The next section is devoted to state our main results (Theorems 1, 2 and their corollaries), whose complete proofs will be given in [KN]. We therefore content ourselves only with the presentation of a formula (in Section 3) which is fundamental in proving Theorem 1.

2. Statement of results

We write $\sigma_w(l) = \sum_{0<d|l} d^w$ and use the notation $e(z) = e^{2\pi i z}$ hereafter. Then Ramanujan [Ram] (see also [Be]) first introduced and studied the function

$$\Phi_{s_1,s_2}(e(z)) = \sum_{l_1,l_2=1}^{\infty} l_1^{s_1} l_2^{s_2} e(l_1 l_2 z) = \sum_{l=1}^{\infty} \sigma_{s_1-s_2}(l) l^{s_2} e(lz), \tag{2.1}$$

where the series converges absolutely for all $(s_1,s_2) \in \mathbb{C}^2$ and defines there an entire function. Ramanujan's main concern there was to supply various evaluations of (2.1) in terms of the holomorphic Eisenstein series $E_k(0;z)$ with $k = 2,4,6$ (see also Corollary 1.6 below). Next let $\Gamma(s)$ be the gamma function, $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any integer n Pochhammer's symbol, and write $\Gamma\binom{\alpha_1,\dots,\alpha_m}{\beta_1,\dots,\beta_n} = \prod_{i=1}^m \Gamma(\alpha_i)/\prod_{j=1}^n \Gamma(\beta_j)$ for complex parameters α_i and β_j ($i = 1,\dots,m$; $j = 1,\dots,n$). Further, let $U(\alpha;\gamma;Z)$ denote the confluent hypergeometric function defined by

$$(2.2)\qquad U(\alpha;\gamma;Z)=\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-wZ}w^{\alpha-1}(1+w)^{\gamma-\alpha-1}dw$$

for $\mathrm{Re}\,\alpha>0$ and $|\arg Z|<\pi/2$ (cf. [Sl, p.5, 1.3(1.3.5)]).

Our first main result can be stated as

Theorem 1. *Let $E_k(s;z)$ be defined by (1.1) with an arbitrary even integer weight k. Then for any integer $N\geq |k|/2$ the formula*

$$(2.3)\qquad \begin{aligned} E_k(s;z) &= 1+(-1)^{k/2}2\pi\Gamma\binom{2s+k-1}{s,s+k}\frac{\zeta(2s+k-1)}{\zeta(2s+k)}(2y)^{1-2s-k}\\ &+\frac{(-1)^{k/2}(2\pi)^{2s+k}}{\zeta(2s+k)\Gamma(s+k)}\{S_{N+k/2}(s,2s+k;z)+R_{N+k/2}(s,2s+k;z)\}\\ &+\frac{(-1)^{k/2}(2\pi)^{2s+k}}{\zeta(2s+k)\Gamma(s)}\{S_{N-k/2}(s+k,2s+k;-\bar z)+R_{N-k/2}(s+k,2s+k;-\bar z)\}\end{aligned}$$

holds in the region $-N-k/2<\sigma<N-k/2+1$ except at the complex zeros of $\zeta(2s+k)$ and at the real poles of $E_k(s;z)$ (described in Corollary 1.1 below). Here $S_{N\pm k/2}$ are defined by (3.2), which (in the present case) reduces to

$$(2.4)\qquad \begin{aligned} S_{N+k/2}(s,2s+k;z) = \sum_{n=0}^{N+k/2-1}&\frac{(-1)^n(s)_n(1-s-k)_n}{n!}\\ &\times\Phi_{s+k-n-1,-s-n}(e(z))(4\pi y)^{-s-n},\end{aligned}$$

$$(2.5)\qquad \begin{aligned} S_{N-k/2}(s+k,2s+k;-\bar z) = \sum_{n=0}^{N-k/2-1}&\frac{(-1)^n(s+k)_n(1-s)_n}{n!}\\ &\times\Phi_{s-n-1,-s-k-n}(e(-\bar z))(4\pi y)^{-s-k-n},\end{aligned}$$

both giving the asymptotic series in the descending order of y as $y\to+\infty$. Also $R_{N\pm k/2}$ are the remainder terms expressed by (3.5), which (in the present case) reduces to

$$(2.6)\qquad \begin{aligned} &R_{N+k/2}(s,2s+k;z)\\ &=\frac{(-1)^{N+k/2}(s)_{N+k/2}(1-s-k)_{N+k/2}}{(N+k/2-1)!}\sum_{l_1,l_2=1}^\infty l_1^{2s+k-1}e(l_1l_2z)\\ &\times\int_0^1\xi^{-s-k/2-N}(1-\xi)^{N+k/2-1}U(s+k/2+N;2s+k;4\pi l_1l_2y/\xi)d\xi,\end{aligned}$$

$$(2.7)\qquad \begin{aligned} &R_{N-k/2}(s+k,2s+k;-\bar z)\\ &=\frac{(-1)^{N-k/2}(s+k)_{N-k/2}(1-s)_{N-k/2}}{(N-k/2-1)!}\sum_{l_1,l_2=1}^\infty l_1^{2s+k-1}e(-l_1l_2\bar z)\\ &\times\int_0^1\xi^{-s-k/2-N}(1-\xi)^{N-k/2-1}U(s+k/2+N;2s+k;4\pi l_1l_2y/\xi)d\xi,\end{aligned}$$

(the cases $N \pm k/2 = 0$ should read without the factor $(-1)!$ and the ξ-integration), satisfying the estiamtes

$$R_{N+k/2}(s; 2s+k; z) = O\{(|t|+1)^{2N+k}e^{-2\pi y}y^{-\sigma-N-k/2}\}, \tag{2.8}$$

$$R_{N-k/2}(s+k; 2s+k; -\bar{z}) = O\{(|t|+1)^{2N-k}e^{-2\pi y}y^{-\sigma-N-k/2}\} \tag{2.9}$$

for any $y \geq y_0 > 0$ in the same region of s above, where the implied O-constants depend at most on k, N and y_0.

Remark 1. Since $\sigma_w(l) = O\{l^{\max(\mathrm{Re}\,w,0)+\varepsilon}\}$ as $l \to +\infty$ for any $\varepsilon > 0$, we see from (2.1) that

$$\Phi_{s_1,s_2}(e(z)) = e(z) + O(e^{-4\pi y}) \qquad \text{as } y \to +\infty. \tag{2.10}$$

Hence the terms with the index n on the right sides of (2.4) and (2.5) are of order $\asymp (|t|+1)^{2n}e^{-2\pi y}y^{-\sigma-n}$ $(0 \leq n < N + k/2)$ and of order $\asymp (|t|+1)^{2n}e^{-2\pi y}y^{-\sigma-n-k}$ $(0 \leq n < N - k/2)$ respectively; the presence of the bounds in (2.8) and (2.9) are therefore reasonable.

Remark 2. It is possible to reformulate Theorem 1 with $E_k(s+k/2; z)$, which further clarifies the symmetry of our main formula; however, the present formulation is rather convenient for practical applications.

We first mention the location of the singularities of $E_k(s; z)$.

Corollary 1.1. *The real singularities of $E_k(s; z)$ are all simple poles, which are located on*

$$s = \begin{cases} 1 & \text{if } k = 0; \\ -k+1, -k+2, \ldots, -k/2-1 & \text{if } k \geq 4; \\ 1, 2, \ldots, -k/2-1 & \text{if } k \leq -4. \end{cases} \tag{2.11}$$

Remark. The appearance of this corollary differs from those of the statements, for e.g., in [Mi, p.286, Chap.7, Corollary 7.2.11] or [Sh, p.64, Chap.9, Theorem 9.7]; this is because our formulation of the non-holomorphic Eisenstein series contains the extra factor $\zeta(2s)$ in the denominator.

A slight modification of the proof of Theorem 1 yields the Fourier series expansion of $E_k(s; z)$ (cf. [Mi, p.284, Chap.7, Theorem 7.2.9]).

Corollary 1.2. *Let $E_k(s; z)$ be defined by (1.1) with an arbitrary even integer k. Then the formula*

$$\begin{aligned} E_k(s; z) = 1 &+ (-1)^{k/2}2\pi\Gamma\binom{2s+k-1}{s, s+k}\frac{\zeta(2s+k-1)}{\zeta(2s+k)}(2y)^{1-2s-k} \\ &+ \frac{(-1)^{k/2}(2\pi)^{2s+k}}{\zeta(2s+k)\Gamma(s)}\sum_{l=1}^{\infty} e(lx)\sigma_{2s+k-1}(l)e^{-2\pi ly}U(s; 2s+k; 4\pi ly) \\ &+ \frac{(-1)^{k/2}(2\pi)^{2s+k}}{\zeta(2s+k)\Gamma(s+k)}\sum_{l=1}^{\infty} e(-lx)\sigma_{2s+k-1}(l)e^{-2\pi ly}U(s+k; 2s+k; 4\pi ly) \end{aligned} \tag{2.12}$$

holds for all s except at the complex zeros of $\zeta(2s+k)$ and at the real poles of $E_k(s; z)$ given in (2.11).

We next define the function $E_k^*(s;z)$ by

$$E_k(s;z) = 1 + (-1)^{k/2} 2\pi \Gamma\binom{2s+k-1}{s,s+k} \frac{\zeta(2s+k-1)}{\zeta(2s+k)} (2y)^{1-2s-k} + E_k^*(s;z), \tag{2.13}$$

and set

$$\widetilde{E}_k(s;z) = \zeta(2s+k)E_k(s;z) \qquad \text{and} \qquad \widetilde{E}_k^*(s;z) = \zeta(2s+k)E_k^*(s;z).$$

Then the Mellin-Barnes type integral formula (3.3) below shows that the validity of the following functional equation of $\widetilde{E}_k^*(s;z)$ reduces eventually to the primary symmetry

$$\Phi_{s_1,s_2}(e(z)) = \Phi_{s_2,s_1}(e(z)). \tag{2.14}$$

We can prove

Corollary 1.3. *For any real x, y with $y > 0$ the functional equation*

$$(y/\pi)^s \Gamma(s) \widetilde{E}_k^*(s;z) = (y/\pi)^{1-s-k} \Gamma(1-s-k) \widetilde{E}_k^*(1-s-k;z) \tag{2.15}$$

holds, and this with that of $\zeta(s)$ implies

$$(y/\pi)^s \Gamma(s) \widetilde{E}_k(s;z) = (y/\pi)^{1-s-k} \Gamma(1-s-k) \widetilde{E}_k(1-s-k;z).$$

Remark. The functional equation of $E_k(s;z)$ itself can be found, for e.g., in [Sh, p.64, Theorem 9.7].

We next proceed to state the explicit formulae for various specific values associated with $E_k(s;z)$. Let $\eta(z) = e(z/24)\prod_{l=1}^{\infty}\{1-e(lz)\}$ be the Dedekind eta-function, and $\gamma_0 = -\Gamma'(1)$ the $0th$ Euler constant (cf. [Er, p.34, 1.12(17)]). We write $q = e(z)$ and so $\overline{q} = e(-\overline{z})$ for brevity. Then the case $k = 0$ of (2.3) gives

Corollary 1.4. *The following formulae hold for $E_0(s;z)$:*

i) *For any integer $m \geq 2$,*

$$\begin{aligned} E_0(m;z) = 1 &+ \frac{2(-1)^{m+1}(2m-2)!(2m)!\zeta(2m-1)}{\{(m-1)!\}^2 B_{2m}} (4\pi y)^{1-2m} \\ &+ \frac{2(-1)^{m+1}(2m)!}{\{(m-1)!\}^2 B_{2m}} \sum_{n=0}^{m-1} \binom{m-1}{n} (m+n-1)! \\ &\times \{\Phi_{m-n-1,-m-n}(q) + \Phi_{m-n-1,-m-n}(\overline{q})\}(4\pi y)^{-m-n}; \end{aligned} \tag{2.16}$$

ii) *As for $s \to 1$,*

$$\begin{aligned} &\lim_{s\to 1}\left\{E_0(s;z) - \frac{3}{\pi y}\frac{1}{s-1}\right\} \\ &= 1 + \frac{6}{\pi y}\left\{\gamma_0 - \frac{\zeta'}{\zeta}(2) - \log(2y) + \Phi_{0,-1}(q) + \Phi_{0,-1}(\overline{q})\right\} \\ &= \frac{6}{\pi y}\left\{\gamma_0 - \frac{\zeta'}{\zeta}(2) - 2\log(\sqrt{2y}|\eta(z)|)\right\}; \end{aligned} \tag{2.17}$$

iii) *Upon the notation $E_0'(s;z) = (\partial/\partial s)E_0(s;z)$,*

$$E_0'(0;z) = -\frac{1}{3}\pi y - 2\Phi_{-1,0}(q) - 2\Phi_{-1,0}(\overline{q}) = 4\log|\eta(z)|; \tag{2.18}$$

iv) *For any integer* $m \geq 1$,

$$E_0(-m;z) = 1 + \frac{(-1)^m (m!)^2 B_{2m+2}}{2(2m+2)!(2m)!\zeta(2m+1)}(4\pi y)^{2m+1} + \frac{1}{(2m)!\zeta(2m+1)} \sum_{n=0}^{m} \binom{m}{n}(m+n)! \times \{\Phi_{-m-n-1,m-n}(q) + \Phi_{-m-n-1,m-n}(\overline{q})\}(4\pi y)^{m-n}. \tag{2.19}$$

Let $\operatorname{Res}_{s=s_0} E_k(s;z)$ denote the residue of $E_k(s;z)$ at $s = s_0$. Then the case $k \geq 2$ of (2.3) further yields

Corollary 1.5. *The following formulae hold for* $E_k(s;z)$ *with any* $k \geq 2$:

i) *For any integer* $m \geq 1$,

$$\begin{aligned} &E_k(m;z) \\ &= 1 + \frac{2(-1)^{m+1}(2m+k-2)!(2m+k)!\zeta(2m+k-1)}{(m-1)!(m+k-1)!B_{2m+k}}(4\pi y)^{1-2m-k} \\ &+ \frac{2(-1)^{m+1}(2m+k)!}{(m-1)!(m+k-1)!B_{2m+k}} \left\{ \sum_{n=0}^{m+k-1} \binom{m+k-1}{n} \right. \\ &\times (m+n-1)!\Phi_{m+k-n-1,-m-n}(q)(4\pi y)^{-m-n} \\ &+ \sum_{n=0}^{m-1} \binom{m-1}{n} (m+k+n-1)! \\ &\left. \times \Phi_{m-n-1,-m-k-n}(\overline{q})(4\pi y)^{-m-k-n} \right\}; \end{aligned} \tag{2.20}$$

ii) *For any integer* m *with* $0 \leq m \leq k/2 - 2$,

$$E_k(-m;z) = 1 + \frac{2(-1)^{m+1}(k-2m)!}{B_{k-2m}} \sum_{n=0}^{m} \binom{m}{n} \times \frac{(-1)^n}{(k-m-n-1)!}\Phi_{-m+k-n-1,m-n}(q)(4\pi y)^{m-n}; \tag{2.21}$$

iii) *As for* $s = 1 - k/2$,

$$E_k(1-k/2;z) = 1 - \frac{6}{k\pi y} + (-1)^{k/2} 24 \sum_{n=0}^{k/2-1} \binom{k/2-1}{n} \times \frac{(-1)^n}{(k/2-n)!}\Phi_{k/2-n,k/2-n-1}(q)(4\pi y)^{k/2-n-1}; \tag{2.22}$$

iv) *As for* $s = -k/2$,

$$E_k(-k/2;z) = 1 - \frac{1}{6}k\pi y + 2(-1)^{k/2-1} \sum_{n=0}^{k/2} \binom{k/2}{n} \times \frac{(-1)^n}{(k/2-n-1)!}\Phi_{k/2-n-1,k/2-n}(q)(4\pi y)^{k/2-n}; \tag{2.23}$$

v) *For any integer m with $k/2+1 \leq m \leq k-1$,*

$$\text{Res}_{s=-m} E_k(s;z) \tag{2.24}$$
$$= \frac{m! B_{2m-k+2}}{2(2m-k)!(2m-k+2)!(k-m-1)!\zeta(2m-k+1)}(4\pi y)^{2m-k+1}$$
$$+ \frac{(-1)^m}{(2m-k)!\zeta(2m-k+1)} \sum_{n=0}^{m} \binom{m}{n} \frac{(-1)^n}{(k-m-n-1)!}$$
$$\times \varPhi_{-m+k-n-1,m-n}(q)(4\pi y)^{m-n};$$

vi) *For any integer $m \geq k$,*

$$E_k(-m;z) = 1 + \frac{(-1)^m m!(m-k)! B_{2m-k+2}}{2(2m-k+2)!(2m-k)!\zeta(2m-k+1)}(4\pi y)^{2m-k+1} \tag{2.25}$$
$$+ \frac{1}{(2m-k)!\zeta(2m-k+1)} \Big\{ \sum_{n=0}^{m} \binom{m}{n} (m-k+n)!$$
$$\times \varPhi_{-m+k-n-1,m-n}(q)(4\pi y)^{m-n} + \sum_{n=0}^{m-k} \binom{m-k}{n}$$
$$\times (m+n)! \varPhi_{-m-n-1,m-k-n}(\overline{q})(4\pi y)^{m-k-n} \Big\}.$$

Remark. The cases ii) and v) of this corollary become null if $k=2$.

The cases $m=0$ of (2.21) and $k=2$ of (2.22) further reduce respectively to the Lambert series expressions of the holomorphic Eisenstein series $E_k(0;z)$ for $k \geq 4$ and the nearly-holomorphic Eisenstein series $E_2(s;z)$.

Corollary 1.6. *The following expressions are valid for $E_k(0;z)$:*

$$E_2(0;z) = 1 - \frac{3}{\pi y} - 24\varPhi_{1,0}(q) = 1 - \frac{3}{\pi y} - 24 \sum_{l=1}^{\infty} \frac{l q^l}{1-q^l}; \tag{2.26}$$

and for any $k \geq 4$,

$$E_k(0;z) = 1 - \frac{2k}{B_k}\varPhi_{k-1,0}(q) = 1 - \frac{2k}{B_k} \sum_{l=1}^{\infty} \frac{l^{k-1} q^l}{1-q^l}. \tag{2.27}$$

Remark 1. Formulae (2.26) and (2.27) are classic; these can be found for e.g., in [Ran].

We next state explicit formulae for specific values associated with the non-holomorphic Eisenstein series with negative weights.

Corollary 1.7. *The following formulae hold for $E_{-k}(s;z)$ with any $k \geq 2$:*

i) *For any integer* $m \geq k+1$,

$$E_{-k}(m;z) \tag{2.28}$$

$$= 1 + \frac{2(-1)^{m-k+1}(2m-k-2)!(2m-k)!\zeta(2m-k-1)}{(m-k-1)!(m-1)!B_{2m-k}}(4\pi y)^{1-2m+k}$$

$$+ \frac{2(-1)^{m-k+1}(2m-k)!}{(m-k-1)!(m-1)!B_{2m-k}}\left\{\sum_{n=0}^{m-k-1}\binom{m-k-1}{n}\right.$$

$$\times (m+n-1)!\Phi_{m-k-n-1,-m-n}(q)(4\pi y)^{-m-n}$$

$$+\sum_{n=0}^{m-1}\binom{m-1}{n}(m-k+n-1)!$$

$$\left.\times \Phi_{m-n-1,-m+k-n}(\overline{q})(4\pi y)^{-m+k-n}\right\};$$

ii) *For any integer* m *with* $k/2+2 \leq m \leq k$,

$$E_{-k}(m;z) = 1 + \frac{2(-1)^{k-m-1}(2m-k)!}{B_{2m-k}}\sum_{n=0}^{k-m}\binom{k-m}{n} \times \frac{(-1)^n}{(m-n-1)!}\Phi_{m-n-1,k-m-n}(\overline{q})(4\pi y)^{k-m-n}; \tag{2.29}$$

iii) *As for* $s = 1+k/2$,

$$E_{-k}(1+k/2;z) = 1 - \frac{6}{k\pi y} + (-1)^{k/2}24\sum_{n=0}^{k/2-1}\binom{k/2-1}{n} \times \frac{(-1)^n}{(k/2-n)!}\Phi_{k/2-n,k/2-n-1}(\overline{q})(4\pi y)^{k/2-n-1}; \tag{2.30}$$

iv) *As for* $s = k/2$,

$$E_{-k}(k/2;z) = 1 - \frac{1}{6}k\pi y + 2(-1)^{k/2-1}\sum_{n=0}^{k/2}\binom{k/2}{n} \times \frac{(-1)^n}{(k/2-n-1)!}\Phi_{k/2-n-1,k/2-n}(\overline{q})(4\pi y)^{k/2-n}; \tag{2.31}$$

v) *For any integer* m *with* $1 \leq m \leq k/2-1$,

$$\operatorname{Res}_{s=m} E_{-k}(s;z) \tag{2.32}$$

$$= \frac{(k-m)!B_{k-2m+2}}{2(k-2m)!(k-2m+2)!(m-1)!\zeta(k-2m+1)}(4\pi y)^{k-2m+1}$$

$$+ \frac{(-1)^{k-m}}{(k-2m)!\zeta(k-2m+1)}\sum_{n=0}^{k-m}\binom{k-m}{n}$$

$$\times \frac{(-1)^n}{(m-n-1)!}\Phi_{m-n-1,k-m-n}(\overline{q})(4\pi y)^{k-m-n};$$

vi) *For any integer* $m \geq 0$,

$$E_{-k}(-m;z) = 1 + \frac{(-1)^{m+k}(m+k)!m!B_{2m+k+2}}{2(2m+k+2)!(2m+k)!\zeta(2m+2k+1)}(4\pi y)^{2m+k+1} + \frac{1}{(2m+k)!\zeta(2m+k+1)}\left\{\sum_{n=0}^{m}\binom{m}{n}(m+k+n)! \times \varPhi_{-m-k-n-1,m-n}(q)(4\pi y)^{m-n} + \sum_{n=0}^{m+k}\binom{m+k}{n}(m+n)! \times \varPhi_{-m-n,m+k-n}(\bar{q})(4\pi y)^{m+k-n}\right\}. \tag{2.33}$$

Remark. The cases ii) and v) of this corollary become null if $k=2$.

We next mention that the asymptotic expansion of $E_k(s;z)$ as $z \to 0$ is deducible from Theorem 1 by applying the quasi-modularity of $E_k(s;z)$. For this it is convenient to set $z = i\tau$ with $|\arg\tau| < \pi/2$. Then one can see from (1.1) that the relation $E_k(s;-1/z) = z^k|z|^{2s}E_k(s;z)$ holds, and hence

$$E_k(s;i\tau) = (-1)^{k/2}|\tau|^{-2s-k}e^{-ik\arg\tau}E_k(s;i/\tau), \tag{2.34}$$

by which Formula (2.3) can be switched to

Corollary 1.8. *For any complex* τ *with* $|\arg\tau| < \pi/2$ *and any integer* $N \geq |k|/2$ *the formula*

$$E_k(s;i\tau) = \frac{(-1)^{k/2}e^{-ik\arg\tau}}{|\tau|^{2s+k}} + \frac{2\pi e^{-ik\arg\tau}}{|\tau|\{2\cos(\arg\tau)\}^{2s+k-1}}\Gamma\binom{2s+k-1}{s,s+k}\frac{\zeta(2s+k-1)}{\zeta(2s+k)} + \frac{(2\pi/|\tau|)^{2s+k}e^{-ik\arg\tau}}{\zeta(2s+k)\Gamma(s+k)} \times \{S_{N+k/2}(s,2s+k;i/\tau) + R_{N+k/2}(s,2s+k;i/\tau)\} + \frac{(2\pi/|\tau|)^{2s+k}e^{-ik\arg\tau}}{\zeta(2s+k)\Gamma(s)} \times \{S_{N-k/2}(s+k,2s+k;i/\bar{\tau}) + R_{N-k/2}(s+k,2s+k;i/\bar{\tau})\} \tag{2.35}$$

holds in the region $-N-k/2 < \sigma < N+k/2+1$ *except at the complex zeros of* $\zeta(2s+k)$ *and the real poles of* $E_k(s;i\tau)$. *Here* $S_{N\pm k/2}$ *are of the form*

$$\begin{aligned} S_{N+k/2}(s,2s+k;i/\tau) &= \sum_{n=0}^{N+k/2-1}\frac{(-1)^n(s)_n(1-s-k)_n}{n!} \\ &\quad\times \varPhi_{s+k-n-1,-s-n}(e^{-2\pi/\tau})\{|\tau|/4\pi\cos(\arg\tau)\}^{s+n}, \\ S_{N-k/2}(s+k,2s+k;i/\bar{\tau}) &= \sum_{n=0}^{N-k/2-1}\frac{(-1)^n(s+k)_n(1-s)_n}{n!} \\ &\quad\times \varPhi_{s-n-1,-s-k-n}(e^{-2\pi/\bar{\tau}})\{|\tau|/4\pi\cos(\arg\tau)\}^{s+k+n} \end{aligned} \tag{2.36}$$

respectively, both giving the asymptotic series in the ascending order of τ *as* $\tau \to 0$ *through the sector* $|\arg\tau| < \pi/2$. *Also* $R_{N\pm k/2}$ *are expressed by* (2.6) *and* (2.7) *with* (y,z) *replaced by* $(\cos(\arg\tau)/|\tau|, i/\tau)$ *respectively, satisfying the estimates*

$$\begin{aligned} R_{N+k/2}(s, 2s+k; i/\tau) &= O(|\tau|^{\sigma+N+k/2}), \\ R_{N-k/2}(s+k, 2s+k; i/\overline{\tau}) &= O(|\tau|^{\sigma+N+k/2}) \end{aligned} \tag{2.37}$$

as $\tau \to 0$ *through the sector* $|\arg\tau| \leq \pi/2 - \delta$ *with any small* $\delta > 0$, *where the implied* O*-constants depend at most on* k, N, s *and* δ.

We lastly proceed to state our second main result.

Theorem 2. *For any integer* $N \geq |k|/2$ *the actions of* $\Delta_{H,k}$ *upon* $S_{N\pm k/2}$ *and* $R_{N\pm k/2}$ *in* (2.3) (*multiplied by* y^s) *are explicitly given by*

$$\begin{aligned} &\Delta_{H,k}\{y^s S_{N+k/2}(s, 2s+k; z)\} \\ &\quad = y^s\Big\{ s(1-s-k) S_{N+k/2}(s, 2s+k; z) \\ &\qquad - \frac{(-1)^{N+k/2-1}(s)_{N+k/2}(1-s-k)_{N+k/2}}{(N+k/2-1)!} \\ &\qquad \times \Phi_{s+k/2-N, -s-N-k/2+1}(e(z))(4\pi y)^{-s-N-k/2+1}\Big\}, \end{aligned} \tag{2.38}$$

$$\begin{aligned} &\Delta_{H,k}\{y^s R_{N+k/2}(s, 2s+k; z)\} \\ &\quad = y^s\Big\{ \frac{(-1)^{N+k/2-1}(s)_{N+k/2}(1-k-s)_{N+k/2}}{(N+k/2-1)!} \\ &\qquad \times \Phi_{s+k/2-N, -s-N-k/2+1}(e(z))(4\pi y)^{-s-N-k/2+1} \\ &\qquad + s(1-s-k) R_{N+k/2}(s, 2s+k; z)\Big\}, \end{aligned} \tag{2.39}$$

$$\begin{aligned} &\Delta_{H,k}\{y^s S_{N-k/2}(s+k, 2s+k; -\overline{z})\} \\ &\quad = y^s\Big\{ s(1-s-k) S_{N-k/2}(s+k, 2s+k; -\overline{z}) \\ &\qquad - \frac{(-1)^{N-k/2-1}(s+k)_{N-k/2}(1-s)_{N-k/2}}{(N-k/2-1)!} \\ &\qquad \times \Phi_{s+k/2-N, -s-N-k/2+1}(e(-\overline{z}))(4\pi y)^{-s-N-k/2+1}\Big\}, \end{aligned} \tag{2.40}$$

$$\begin{aligned} &\Delta_{H,k}\{y^s R_{N-k/2}(s+k, 2s+k; -\overline{z})\} \\ &\quad = y^s\Big\{ \frac{(-1)^{N-k/2-1}(s+k)_{N-k/2}(1-s)_{N-k/2}}{(N-k/2-1)!} \\ &\qquad \times \Phi_{s+k/2-N, -s-N-k/2+1}(e(-\overline{z}))(4\pi y)^{-s-N-k/2+1} \\ &\qquad + s(1-s-k) R_{N-k/2}(s+k, 2s+k; -\overline{z})\Big\} \end{aligned} \tag{2.41}$$

in the region $-N - k/2 < \sigma < N - k/2 + 1$ except at the complex zeros of $\zeta(2s+k)$ and the real poles of $E_k(s;z)$.

It is observed upon combining (2.38) with (2.39), and also (2.40) with (2.41) that the common factor $s(1-s-k)y^s$ can be extracted from these two combinations; this together with the fact that $\Delta_{H,k}y^w = w(1-w-k)y^w$ shows

Corollary 2.1. *Formula* (2.3) *with the relations* (2.38)–(2.41) *justifies the eigenfunction equation* (1.6) *throughout the s-plane.*

3. A FUNDAMENTAL FORMULA

The aim of this section is to prepare the formula which is fundamental in proving Theorem 1.

Let N be an arbitrary nonnegative integer, and $(s_1; s_2)$ in the region

$$\mathrm{Re}\, s_1 = \sigma_1 > -N \qquad \text{and} \qquad \mathrm{Re}\, s_2 = \sigma_2 < \sigma_1 + N + 1. \tag{3.1}$$

In order to reformulate our previous results on $\zeta_{\mathbb{Z}^2}(s;z)$ (in [Ka7, Theorem 1]) to $E_0(s;z)$, we introduce

$$S_N(s_1, s_2; z) = \sum_{n=0}^{N-1} \frac{(-1)^n (s_1)_n (s_1 - s_2 + 1)_n}{n!} \times \Phi_{s_2 - s_1 - n - 1, -s_1 - n}(e(z))(4\pi y)^{-s_1 - n}, \tag{3.2}$$

$$R_N(s_1, s_2; z) = \frac{1}{2\pi i} \int_{(c_N)} \Gamma\binom{s_1 + w, -w, 1 - s_2 - w}{s_1, s_1 - s_2 + 1} \times \Phi_{s_2 - 1 + w, w}(e(z))(4\pi y)^w dw, \tag{3.3}$$

where $c_N = c_N(\sigma_1, \sigma_2)$ is a constant satisfying

$$-\sigma_1 - N < c_N < \min(-\sigma_1 - N + 1, 0, 1 - \sigma_2), \tag{3.4}$$

and (c_N) denotes the vertical straight line from $c_N - i\infty$ to $c_N + i\infty$. Note that the parameter z may be replaced by $-\overline{z}$ (with $y = \mathrm{Im}\, z = \mathrm{Im}(-\overline{z})$) in (3.2) and (3.3). Here the conditions (3.1) and (3.4) ensure that the path (c_N) separates the poles of the integrand at $w = -s_1 - n$ $(n = N, N+1, \dots)$ from those at $w = -s_1 - n$ $(n = 0, 1, \dots, N-1)$ and at $w = n$, $1 - s_2 + n$ $(n = 0, 1, \dots)$; the integral in (3.3) converges uniformly on any compact set in the region (3.1), and defines there a holomorphic function of (s_1, s_2), since the integrand is of order $O\{|\mathrm{Im}\, w|^C e^{-3\pi|\mathrm{Im}\, w|/2}\}$ as $\mathrm{Im}\, w \to \pm\infty$ with some constant $C = C(\mathrm{Im}\, z, \mathrm{Re}\, w, \sigma_1, \sigma_2)$ (see (2.10) and [Iv, p.492, A.7(A.34)]. It is in fact possible to transform the Mellin-Barnes type integral in (3.3) as

$$R_N(s_1, s_2; z) = \frac{(-1)^N (s_1)_N (s_1 - s_2 + 1)_N}{(N-1)!} \sum_{l_1, l_2 = 1}^{\infty} e(l_1 l_2 z) l_1^{s_2 - 1} \times \int_0^1 \xi^{-s_1 - N}(1 - \xi)^{N-1} U(s_1 + N; s_2; 4\pi l_1 l_2 y/\xi) d\xi. \tag{3.5}$$

Then Formula (2.5) with (2.6) and (4.4) in [Ka7] readily yields (upon splitting the asymptotic expansion into the parts corresponding to z and $-\overline{z}$)

Proposition 1. *Let $E_0(s;z)$ be defined by* (1.1) *with $k=0$. Then for any integer $N \geq 0$ the formula*

$$\begin{aligned} E_0(s;z) = 1 + 2\pi\Gamma\binom{2s-1}{s,s}\frac{\zeta(2s-1)}{\zeta(2s)}(2y)^{1-2s} \\ + \frac{(2\pi)^{2s}}{\Gamma(s)\zeta(2s)}\{S_N(s,2s;z) + R_N(s,2s;z) \\ + S_N(s,2s;-\overline{z}) + R_N(s,2s;-\overline{z})\} \end{aligned}$$

holds in the region $-N < \sigma < N+1$ except at $s=1$ and the complex zeros of $\zeta(2s)$.

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