Convergence of some truncated Riesz transforms on predual of generalized Campanato spaces and its application to a uniqueness theorem for nondecaying solutions of Navier-Stokes equations.

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1. INTRODUCTION

This is an announcement of our recent work [8]. In [6] the first author introduced predual of generalized Campanato spaces. In this report, we show convergence of some truncated Riesz transforms on the function spaces and its application to a uniqueness theorem for nondecaying solutions of Navier-Stokes equations. Our uniqueness theorem is an extension of Kato's [3].

2. GENERALIZED CAMPANATO SPACE $\mathcal{L}_{p,\phi}(\mathbb{R}^{n})$

Let $1 \leq p < \infty$ and $\phi : (0, \infty) \rightarrow (0, \infty)$. For a ball $B = B(x, r)$, we shall write $\phi(B)$ in place of $\phi(r)$. The function spaces $\mathcal{L}_{p,\phi} = \mathcal{L}_{p,\phi}(\mathbb{R}^{n})$ is defined to be the sets of all $f$ such that $\|f\|_{\mathcal{L}_{p,\phi}} < \infty$, where

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_{B} \frac{1}{\phi(B)} \left( \frac{1}{|B|} \int_{B} |f(x) - f_{B}|^{p} \, dx \right)^{1/p},$$

$$f_{B} = \frac{1}{|B|} \int_{B} f(x) \, dx.$$

Then $\mathcal{L}_{p,\phi}$ is a Banach space modulo constants with the norm $\|f\|_{\mathcal{L}_{p,\phi}}$. If $p = 1$ and $\phi \equiv 1$, then $\mathcal{L}_{1,\phi} = \text{BMO}$. It is known that if $\phi(r) = r^{\alpha}$, $0 < \alpha \leq 1$, then $\mathcal{L}_{p,\phi} = \text{Lip}_{\alpha}$, and, if $\phi(r) = r^{-n/p}$, $1 \leq p < \infty$, then $\mathcal{L}_{p,\phi} = L^{p}$.
A function $\phi : (0, \infty) \to (0, \infty)$ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that
\[
C^{-1} \leq \frac{\phi(r)}{\phi(s)} \leq C \quad \text{for} \quad \frac{1}{2} \leq \frac{r}{s} \leq 2.
\]
A function $\phi : (0, \infty) \to (0, \infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that
\[
\phi(r) \leq C\phi(s) \quad (\phi(r) \geq C\phi(s)) \quad \text{for} \quad r \leq s.
\]

**Lemma 2.1.** Assume that $\phi(r)r^{n/p}$ is almost increasing and that $\phi(r)/r$ is almost decreasing. Then $\phi$ satisfies the doubling condition and
\[
\|f\|_{\mathcal{L}_{p,\phi}} \leq C \left( \|(1 + |x|^{n+1})f\|_{\infty} + \|
abla f\|_{\infty} \right).
\]
That is $S \subset \mathcal{L}_{p,\phi}$.

**Proof.** Let $B = B(z, r)$.

**Case 1:** $r < 1$: In this case $r \lesssim \phi(r)$. Then
\[
|f(x) - f(y)| \lesssim r\|\nabla f\|_{\infty} \lesssim \phi(r)\|\nabla f\|_{\infty}, \quad x, y \in B.
\]
\[
\left( \frac{1}{|B|} \int_{B} |f(x) - f_{B}|^{p} \, dx \right)^{1/p} \lesssim \sup_{x, y \in B} |f(x) - f(y)| \lesssim \phi(r)\|\nabla f\|_{\infty}.
\]

**Case 2:** $1 \leq r$: In this case $1 \lesssim \phi(r)r^{n/p}$ and
\[
|f(x)| \leq \frac{\|(1 + |x|^{n+1})f\|_{\infty}}{1 + |x|^{n+1}}, \quad \left( \int |f(x)|^{p} \, dx \right)^{1/p} \lesssim \|(1 + |x|^{n+1})f\|_{\infty}.
\]

Then
\[
\left( \frac{1}{|B|} \int_{B} |f(x) - f_{B}|^{p} \, dx \right)^{1/p} \leq 2 \left( \frac{1}{|B|} \int_{B} |f(x)|^{p} \, dx \right)^{1/p} \lesssim \frac{\|(1 + |x|^{n+1})f\|_{\infty}}{|B|^{1/p}} \lesssim \phi(r)\|(1 + |x|^{n+1})f\|_{\infty}.
\]

3. $H_{I}^{[\phi,\infty]}(\mathbb{R}^{n})$, PREDual of $\mathcal{L}_{1,\phi}(\mathbb{R}^{n})$

The space $H_{U}^{[\phi,q]}$ was introduced in [6], which is a generalization of Hardy space. The duality $\left( H_{U}^{[\phi,q]} \right)^{*} = \mathcal{L}_{q',\phi}$ also proved in [6].

In this talk we recall the definition of $H_{I}^{[\phi,\infty]}(\mathbb{R}^{n})$, which is a special case of $H_{U}^{[\phi,\phi]}$.

In what follows, we always assume that $\phi(r)r^{n}$ is almost increasing and that $\phi(r)/r$ is almost decreasing.
Definition 3.1 ([\(\phi, \infty\)]-atom). A function \(a\) on \(\mathbb{R}^n\) is called a \([\phi, \infty]\)-atom if there exists a ball \(B\) such that

(i) \(\text{supp } a \subset B\),
(ii) \(\|a\|_\infty \leq \frac{1}{|B|\phi(B)}\),
(iii) \(\int_{\mathbb{R}^n} a(x) \, dx = 0\).

where \(\|a\|_\infty\) is the \(L^\infty\) norm of \(a\). We denote by \(A[\phi, \infty]\) the set of all \([\phi, \infty]\)-atoms.

If \(a\) is a \([\phi, \infty]\)-atom and a ball \(B\) satisfies (i)–(iii), then, for \(g \in \mathcal{L}_{1,\phi}\),

\[
\left| \int_{\mathbb{R}^n} a(x)g(x) \, dx \right| = \left| \int_B a(x)(g(x) - g_B) \, dx \right|
\leq \|a\|_\infty \int_B |g(x) - g_B| \, dx
\leq \frac{1}{\phi(B)} \frac{1}{|B|} \int_B |g(x) - g_B| \, dx
\leq \|g\|_{\mathcal{L}_{1,\phi}}.
\]

That is, the mapping \(g \mapsto \int_{\mathbb{R}^n} ag \, dx\) is a bounded linear functional on \(\mathcal{L}_{1,\phi}\) with norm not exceeding 1. Hence \(a\) is also in \(S'\), since \(S \subset \mathcal{L}_{1,\phi}\).

Definition 3.2 \((H_{I}^{\phi,\infty})\). The space \(H_{I}^{\phi,\infty} \subset (\mathcal{L}_{1,\phi})^\ast\) is defined as follows:

\(f \in H_{I}^{\phi,\infty}\) if and only if there exist sequences \(\{a_j\} \subset A[\phi, \infty]\) and positive numbers \(\{\lambda_j\}\) such that

\[
(3.1) \quad f = \sum_j \lambda_j a_j \quad \text{in } (\mathcal{L}_{1,\phi})^\ast \quad \text{and} \quad \sum_j \lambda_j < \infty.
\]

In general, the expression (3.1) is not unique. Let

\[
\|f\|_{H_{I}^{\phi,\infty}} = \inf \left\{ \sum_j \lambda_j \right\},
\]

where the infimum is taken over all expressions as in (3.1). Then \(H_{I}^{\phi,\infty}\) is a Banach space equipped with the norm \(\|f\|_{H_{I}^{\phi,\infty}}\) and \((H_{I}^{\phi,\infty})^\ast = \mathcal{L}_{1,\phi}\).

4. Truncated Riesz transforms on \(H_{I}^{\phi,\infty}(\mathbb{R}^n)\) and main result

The Riesz transforms of \(f\) are defined by

\[
R_j f(x) = c_n \text{ p.v.} \int \frac{y_j}{|y|^{n+1}} f(x - y) \, dy, \quad j = 1, \cdots, n,
\]
where
\[ c_n = \Gamma((n+1)/2)\pi^{-(n+1)/2}. \]

Let
\[
k(x) = \begin{cases} 
C_n \frac{1}{|x|^{n-2}} & \text{if } n \geq 3, \\
C_2 \log \frac{1}{|x|} & \text{if } n = 2,
\end{cases}
\]
where
\[ C_n = \Gamma(n/2)(2(n-2)\pi^{n/2})^{-1}, \quad C_2 = (2\pi)^{-1}. \]

Then
\[-\Delta k = \delta.\]

It is known that
\[ R_{j}R_{k}f(x) = \text{p.v.} \int (\partial_{j}\partial_{k}k)(y)f(x-y)dy - \delta_{j,k}\frac{1}{r\iota}f(x), \]
for \( j, k = 1, \cdots, n \), and
\[ \sum_{j} R_{j}^{2}f = -f. \]

Let \( \psi \in C^\infty(\mathbb{R}^n) \) be a radial function with \( 0 \leq \psi \leq 1 \), \( \psi(x) = 0 \) for \( |x| \leq 1 \), and \( \psi(x) = 1 \) for \( |x| \geq 2 \). We set \( \lambda = 1 - \psi \), and \( \lambda(\epsilon x) = \lambda(x) \frac{1}{\epsilon} \) so that \( \text{supp } k_{\epsilon} \subset \{ \xi: \epsilon \leq |x| \leq 2/\epsilon \} \).

**Definition 4.1** \((R_{i,j}^\epsilon)\). Let \( 1 \leq i, j \leq n \). For \( 0 < \epsilon < 1/4 \), the operators \( R_{i,j}^\epsilon \) are defined by
\[ R_{i,j}^\epsilon f = \partial_{i}\partial_{j}k_{\epsilon} \ast f \]
for \( f \in S' \).

We consider the following condition.
\[
(4.1) \quad \begin{cases} 
\int_{1}^{\infty} \frac{\phi(t)}{t^2} \, dt < \infty, & \text{if } n \geq 3, \\
\int_{1}^{\infty} \frac{\phi(t)\log(1+t)}{t^2} \, dt < \infty, & \text{if } n = 2.
\end{cases}
\]

**Theorem 4.1.** Assume that \( \phi \) satisfies (4.1). If \( \varphi \in S \) and \( \int \varphi = 0 \), then
\[ \lim_{\epsilon\rightarrow 0} R_{i,j}^\epsilon \varphi = R_{i}R_{j}\varphi \quad \text{in } H_{I}^{[\phi,\infty]}.
\]
In particular,
\[ \lim_{\epsilon\rightarrow 0} (-\Delta)k_{\epsilon} \ast \varphi = \varphi \quad \text{in } H_{I}^{[\phi,\infty]}.
\]

Using the duality \((H_{I}^{[\phi,\infty]})^* = \mathcal{L}_{1,\phi}\) and the equality
\[
\lim_{\epsilon\rightarrow 0} \left\langle \sum_{j=1}^{n} R_{i,j}^\epsilon \partial_{j}f, \varphi \right\rangle = \lim_{\epsilon\rightarrow 0} \langle f, (-\Delta)k_{\epsilon} \ast \partial_{i}\varphi \rangle = \langle f, \partial_{i}\varphi \rangle
\]
for all \( \varphi \in S \), we have the following.
**Corollary 4.2.** Assume that $\phi$ satisfies (4.1). For $f \in \mathcal{L}_{1,\phi}$, 
\[
\lim_{\epsilon \to 0} \sum_{j=1}^{n} R_{i,j}^{\epsilon} \partial_j f = -\partial_i f \quad \text{in} \quad S'.
\]

5. **Proof of the Main Result**

To prove Theorem 4.1 we state two lemmas.

**Lemma 5.1.** Let $\ell$ be a continuous decreasing function from $[0, \infty)$ to $(0, \infty)$ such that $\ell(r)r^\theta$ is almost increasing for some $\theta < 1$ and that
\[
\int_1^\infty \frac{\phi(t)}{t^2 \ell(t)} \, dt < \infty.
\]
Define
\[
w(x) = (1 + |x|)^{n+1} \ell(|x|) \quad \text{for} \quad x \in \mathbb{R}^n.
\]
If a function $f$ satisfies
\[
w f \in L^\infty \quad \text{and} \quad \int f = 0,
\]
then $f \in H^{[\phi,\infty]}_{1}$. Moreover, there exist a constant $C > 0$ such that
\[
\|f\|_{H^{[\phi,\infty]}_{1}} \leq C \|w f\|_{\infty},
\]
where $C$ is independent of $f$.

**Lemma 5.2.** Let $\ell$ be a continuous decreasing function from $[0, \infty)$ to $(0, \infty)$ such that $\ell(r) \geq (1 + r)^{-n-1}$ and that
\[
\lim_{r \to \infty} \ell(r) = 0 \text{ if } n \geq 3, \quad \lim_{r \to \infty} \ell(r) \log r = 0 \text{ if } n = 2.
\]
Define
\[
w(x) = (1 + |x|)^{n+1} \ell(|x|) \quad \text{for} \quad x \in \mathbb{R}^n.
\]
If $\varphi \in S$ and $\int \varphi = 0$, then
\[
\lim_{\epsilon \to 0} \|(R_{i,j}^{\epsilon} \varphi - R_i R_j \varphi) w\|_{\infty} = 0.
\]

**Proof of Theorem 4.1.** If (4.1) holds, then there exists a continuous decreasing function $m$ such that $\lim_{r \to \infty} m(r) = 0$ and that
\[
\begin{cases}
\int_1^\infty \frac{\phi(t)}{t^2 m(t)} \, dt < \infty, & \text{if } n \geq 3, \\
\int_1^\infty \frac{\phi(t) \log(1 + t)}{t^2 m(t)} \, dt < \infty, & \text{if } n = 2.
\end{cases}
\]
Actually, if \( \int_{1}^{\infty} F(t) \, dt < \infty \), \( F(t) = \phi(t)/t^{2} \) or \( \phi(t) \log(1+r)/t^{2} \), then we can take a positive increasing sequence \( \{r_{j}\} \) and a continuous decreasing function \( m \) such that
\[
\int_{r_{j}}^{\infty} F(t) \, dt \leq \frac{1}{j^{3}}, \quad \text{for} \quad j = 1, 2, \cdots,
\]
and
\[
m(t) \geq \frac{1}{j} \quad \text{for} \quad r_{j} \leq t \leq r_{j+1}.
\]
Then
\[
\int_{r_{1}}^{\infty} \frac{F(t)}{m(t)} \, dt = \sum_{j=1}^{\infty} \int_{r_{j}}^{r_{j+1}} \frac{F(t)}{m(t)} \, dt \leq \sum_{j=1}^{\infty} \frac{1}{j^{2}} < \infty.
\]
We may assume that \( m(r)r^{\nu} \) is almost increasing for some small \( \nu > 0 \). Let \( \ell \) be a continuous decreasing function from \([0, \infty)\) to \((0, \infty)\) such that, for \( r \geq 1 \),
\[
\ell(r) = \begin{cases} 
    m(r), & \text{if } n \geq 3, \\
    m(r)/\log(1+r), & \text{if } n = 2.
\end{cases}
\]
Then \( \ell \) satisfies the assumption of both Lemmas 5.1 and 5.2.

Using the following relations,
\[
wf \in L^\infty \quad \text{and} \quad \int f = 0, \quad \text{Lemma4.1} \quad \|f\|_{H_{f}^{[\phi, \infty]}} \leq C\|wf\|_{\infty};
\]
\[
\varphi \in S \quad \text{and} \quad \int \varphi = 0 \quad \text{Lemma4.2} \quad \lim_{\epsilon \to 0} \|(R_{i,j}^{\epsilon}\varphi - R_{i}R_{j}\varphi)w\|_{\infty} = 0;
\]
we have that, if \( \varphi \in S \) and \( \int \varphi = 0 \), then
\[
\|R_{i,j}^{\epsilon}\varphi - R_{i}R_{j}\varphi\|_{H_{f}^{[\phi, \infty]}} \leq C\|(R_{i,j}^{\epsilon}\varphi - R_{i}R_{j}\varphi)w\|_{\infty} \to 0,
\]
as \( \epsilon \to 0 \).

6. Application

Let \( n \geq 2 \). We are concerned with the uniqueness of solutions for the Navier-Stokes equation,
\[
(6.1) \quad u_t - \Delta u + (u, \nabla)u + \nabla p = 0 \quad \text{in} \quad (0, T) \times \mathbb{R}^{n},
\]
\[
(6.2) \quad \text{div} \ u = 0 \quad \text{in} \quad (0, T) \times \mathbb{R}^{n},
\]
with initial data \( u|_{t=0} = u_{0} \), where \( u = u(t, x) = (u_{1}(t, x), \cdots, u_{n}(t, x)) \) and \( p = p(t, x) \) stand for the unknown velocity vector field of the fluid and its pressure field respectively, while \( u_{0} = u_{0}(x) = (u_{0}^{1}(x), \cdots, u_{0}^{n}(x)) \) is the given initial velocity vector field.
It is well known (see [2]) that for initial data $u_0 \in L^\infty(\mathbb{R}^n)$ the equations (6.1), (6.2) admit a unique time-local (regular) solution $u$ with

$$ p = \sum_{i,j=1}^{n} R_i R_j u_i u_j. $$

In this report, following J. Kato [3], by "a solution in the distribution sense" we mean a weak solution in the following sense.

**Definition 6.1.** We call $(u, p)$ the solution of the Navier-Stokes equations (6.1), (6.2) on $(0, T) \times \mathbb{R}^n$ with initial data $u_0$ in the distribution sense if $(u, p)$ satisfy

$$ \text{div} u = 0 $$

in $S'$ for a.e. $t$ and

$$ \int_{0}^{T} \left\{ \langle u(s), \partial_s \Phi(s) \rangle + \langle u(s), \Delta \Phi(s) \rangle + \langle (u \times u)(s), \nabla \Phi(s) \rangle + \langle p(s), \text{div} \Phi(.9) \rangle \right\} ds = -\langle u_0, \Phi(0) \rangle $$

for $\Phi \in C^1([0, T] \times \mathbb{R}^n)$ satisfying $\Phi(s, \cdot) \in S(\mathbb{R}^n)$ for $0 \leq s \leq T$, and $\Phi(T, \cdot) \equiv 0$, where $\langle (u \times u), \nabla \Phi \rangle = \sum_{i,j=1}^{n} \langle u_i u_j, \partial_i \Phi_j \rangle$. Here $S$ denotes the space of rapidly decreasing functions in $\mathbb{R}^n$ and $S'$ denotes the space of tempered distributions in the sense of Schwartz. The space $S'$ is the topological dual of $S$ and its canonical pairing is denoted by $\langle , \rangle$.

J. Kato [3] proved the following uniqueness theorem.

**Theorem 6.1 (J. Kato [3]).** Let $u_0 \in L^\infty$ with $\text{div} u_0 = 0$. Suppose that $(u, p)$ is the solution in the distribution sense satisfying

$$ u \in L^\infty((0, T) \times \mathbb{R}^n), \quad p \in L_{1loc}^{1}((0, T); \text{BMO}). $$

Then $(u, \nabla p)$ is uniquely determined by the initial data $u_0$. Moreover, $\nabla p = \sum_{i,j=1}^{n} \nabla R_i R_j u^i u^j$ in $S'$ for a.e. $t$.

On the other hand, Galdi and Maremonti [1] showed that if $u$ and $\nabla u$ are bounded in $(0, T) \times \mathbb{R}^3$, then the uniqueness of classical solutions holds provided that for some $C > 0$ and some $\epsilon > 0$ the inequality

$$ |p(t, x)| \leq C(1 + |x|)^{1-\epsilon} $$

holds. See also [9] and [4]. The assumption (6.4) does not imply (6.5).
To prove Theorem 6.1, Kato [3] used the duality \((H^1)^* = \text{BMO}\) and the following fact: If \(\varphi \in \mathcal{S}\) and \(\int \varphi = 0\), then
\[
\lim_{\epsilon \to 0} R^{\epsilon}_{i,j} \varphi = R_i R_j \varphi \quad \text{in } H^1.
\]
The duality \((H^1_{I}^{[\phi,\infty]} = \mathcal{L}_{1,\phi}\) is known and we have proved in Theorem 4.1 that if \(\varphi \in \mathcal{S}\) and \(\int \varphi = 0\), then
\[
\lim_{\epsilon \to 0} R^{\epsilon}_{i,j} \varphi = R_i R_j \varphi \quad \text{in } H^1_{I}^{[\phi,\infty]}.
\]
Then we have the following.

**Theorem 6.2.** Assume that \(\phi \in \mathcal{G}\) satisfies (4.1). Let \(u_0 \in L^\infty\) with \(\text{div } u_0 = 0\). Suppose that \((u,p)\) is the solution of (6.1), (6.2) in the distribution sense satisfying
\[
u \in L^\infty(0, T) \times \mathbb{R}^n, \quad p \in L^1_{\text{loc}}((0, T); \mathcal{L}_{1,\phi}).
\]
Then \((u, \nabla p)\) is uniquely determined by the initial data \(u_0\). Moreover, \(\nabla p = \sum_{i,j=1}^{n} \nabla R_i R_j u^i u^j\) in \(\mathcal{S}'\) for a.e. \(t\).

For example, let
\[
\phi(r) = \begin{cases} 
     r^{-n} & \text{for } 0 < r < 1, \\
     r (\log (1 + r))^{-\beta} & \text{for } r \geq 1,
\end{cases}
\]
where \(\beta > 1\) if \(n \geq 3\) and \(\beta > 2\) if \(n = 2\). In this case
\[
\mathcal{L}_{1,\phi} \supset L^1 \cup \text{BMO}
\]
and \(\mathcal{L}_{1,\phi}\) contains functions \(f\) such that
\[
|f(x)| \leq C \phi(1 + |x|) = C (1 + |x|)(\log (2 + |x|))^{-\beta} \quad \text{for } x \in \mathbb{R}^n.
\]
Therefore, our result is an extension of both Kato’s theorem and the result of Galdi and Maremonti. Note that, if \(\beta = 0\), then the uniqueness fails (see [2]).

**References**


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