Some recent results on James and von Neumann-Jordan constants

岡山県立大学情報工学部 高橋泰嗣（Yasuji Takahashi）
九州工業大学工学研究院 加藤健雄（Mikio Kato）

Among various geometric constants of a Banach space $X$ the von Neumann-Jordan constant $C_{NJ}(X)$ and James constant $J(X)$ have been treated most widely. Since Kato, Maligranda and Takahashi [3] showed a relation (inequalities) between these constants, several authors have improved their result. Recently covering all the previous results, the present authors showed that the quite simple inequality $C_{NJ}(X) \leq J(X)$ holds for all Banach spaces $X$. In this expository short note we shall present a concise description of these developments.

Let $X$ be a real Banach space with $\dim X \geq 2$. The closed unit ball and unit sphere of $X$ are denoted by $B_X$ and $S_X$, respectively. The James (non-square) constant of $X$ is

$$J(X) = \sup \{ \min(\|x+y\|, \|x-y\|) : x, y \in S_X \},$$

(1)

and the von Neumann-Jordan constant of $X$ is

$$C_{NJ}(X) = \sup \left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X \right\}.$$  

(2)

The modified von Neumann-Jordan constant $C_{NJ}'(X)$ is defined by taking the supremum over all $x, y \in S_X$ in (2). Recall that a Banach space $X$ is uniformly non-square provided $J(X) < 2$ or equivalently $C_{NJ}(X) < 2$. 
1. The relation between $J(X)$ and $C_{NJ}(X)$ was first investigated by Kato-Maligranda-Takahashi [3] who proved that

$$\frac{J(X)^2}{2} \leq C_{NJ}(X) \leq \frac{J(X)^2}{1 + (J(X) - 1)^2},$$

(3)

where all the terms coincide if $J(X) = 2$, i.e., $X$ is not uniformly non-square. The first inequality attains equality with many uniformly non-square spaces, while the second is strict for all uniformly non-square spaces.


$$C_{NJ}(X) \leq \frac{J(X)^2}{4} + 1 + \frac{J(X)}{4} \left[ \sqrt{J(X)^2 - 4J(X) + 8} - 2 \right]$$

(4)

(see also Maligranda et al. [4], Takahashi [7]). Maligranda formulated the following conjecture: For all Banach space $X$

$$C_{NJ}(X) \leq \frac{J(X)^2}{4} + 1.$$  

(5)

3. In 2008 Alonso et al. [1] proved that

$$C_{NJ}(X) \leq 2 \left[ 1 + J(X) - \sqrt{2J(X)} \right],$$

(6)

which was obtained by combining the inequalities

$$C'_{NJ}(X) \leq J(X)$$

(7)

and

$$C_{NJ}(X) \leq 2 \left[ 1 + C'_{NJ}(X) - \sqrt{2C'_{NJ}(X)} \right].$$

(8)

Note that

$$2 \left[ 1 + J(X) - \sqrt{2J(X)} \right] \leq J(X)^2/4 + 1,$$

(9)

where equality holds only when $J(X) = 2$. This indicates that the inequality (6) is sharper than (5), a fortiori, (4). Thus they answered Maligranda’s conjecture (5) affirmatively.
4. A slight improvement of the inequality (6) was obtained by Wang-Pang [10] in 2009:

\[ C_{NJ}(X) \leq J(X) + \sqrt{J(X)} - 1 \left[ \sqrt{1 + (1 - \sqrt{J(X)} - 1)^2} - 1 \right]. \] (10)

To prove (10) they used the inequalities

\[ \rho_X(1) \leq \sqrt{J(X)} - 1 \] (11)

and

\[ C_{NJ}'(X) \leq J(X), \] (12)

where \( \rho_X(\tau) \) is the modulus of smoothness of \( X \).

5. Covering all the previous results, we proved in [9] that the inequality

\[ C_{NJ}(X) \leq J(X) \] (13)

holds true for all Banach spaces \( X \), where equality holds only when \( X \) is not uniformly non-square. This answered affirmatively the question mentioned in Alonso et al. [1, Question 1]. More precisely, we first improved the inequality (11) by Wang-Pang [10] as

\[ \rho_X(1) \leq 2 \left\{ 1 - \frac{1}{J(X)} \right\} \leq \sqrt{J(X)} - 1. \] (14)

The first inequality of (14) is equivalent to

\[ \frac{2}{2 - \rho_X(1)} \leq J(X). \] (15)

Secondly we showed that

\[ C_{NJ}(X) \leq 1 + \rho_X(1) \left[ \sqrt{\{1 - \rho_X(1)\}^2 + 1} - \{1 - \rho_X(1)\} \right]. \] (16)

It is easy to see that

\[ 1 + \rho_X(1) \left[ \sqrt{\{1 - \rho_X(1)\}^2 + 1} - \{1 - \rho_X(1)\} \right] \leq \frac{2}{2 - \rho_X(1)}. \]
Therefore by combining (16) and (15), we obtained that $C_{NJ}(X) \leq J(X)$.

6. Recently we considered the inequality $C_{NJ}(X) \leq J(X)$ by another approach with the modified von Neumann-Jordan constant $C'_{NJ}(X)$ and obtained another proof of the above inequality and some other results, which will appear elsewhere.

References


Yasuji Takahashi
Department of System Engineering
Okayama Prefectural University
Soja 719-1197, Japan
e-mail: takahasi@cse.oka-pu.ac.jp

Department of Basic Sciences
Kyushu Institute of Technology
Kitakyushu 804-8550, Japan
e-mail: katom@tobata.isc.kyutech.ac.jp