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Integrability of maximal functions in Orlicz spaces of variable exponent

大同大学・教養部　二村 俊英 (Toshihide Futamura)
School of Liberal Arts and Sciences,
Daido University
広島大学大学院・理学研究科　水田 義弘 (Yoshihiro Mizuta)
Graduate School of Science,
Hiroshima University
広島大学大学院・教育学研究科　下村 哲 (Tetsu Shimomura)
Graduate School of Education,
Hiroshima University

1 Introduction

Let $\mathbf{R}^n$ denote the n-dimensinonal Euclidean space. We denote by $B(x,r)$ the open ball centered at x of radius r. For a locally integrable function f on $\mathbf{R}^n$, we consider the maximal function Mf defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

where $|B(x,r)|$ denotes the volume of $B(x,r)$.

In classical (constant exponent) Lebesgue spaces, we know the following basic facts about the maximal operator (see the book by Stein [29, Chapter 1]):

(i) If $q > 1$, then

$$\|Mf\|_q \le C\|f\|_q \qquad \text{for all } f \in L^q(\Omega).$$

(ii) If Ω is bounded, then

$$\|Mf\|_1 \le C\|f\|_{L\log L} \qquad \text{for all } f \in L\log L(\Omega).$$

Following Orlicz [25] and Kováčik and Rákosník [21], we consider a positive continuous function $p(\cdot)$ on $\mathbf{R}^n$ and the space of all measurable functions f on $\mathbf{R}^n$ satisfying

$$\int \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy < \infty$$

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for some $\lambda > 0$. We define the norm on this space by

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \le 1\right\}.$$

In connection with these classical results, a natural question arises about conditions on $p(\cdot)$ implying the inequality

$$\|Mf\|_{p(\cdot)} \le C\|f\|_{p(\cdot)}$$

for $f \in L^{p(\cdot)}(\Omega)$. Diening [6] is the first who treated the local boundedness of the maximal operator, and Cruz-Uribe, Fiorenza and Neugebauer [5] showed that this remains true for $\mathbf{R}^n$ when $p(\cdot)$ satisfies a log-Hölder condition on $\mathbf{R}^n$ including the point at infinity. In fact, they showed the following result.

THEOREM A. Let Ω be an open set, and let $p(\cdot)$ be a variable exponent in Ω satisfying $1 < \inf_\Omega p(x) \le \sup_\Omega p(x) < \infty$,

$$|p(x) - p(y)| \le \frac{C}{\log(1/|x-y|)}, \qquad x,\ y \in \Omega,\ |x-y| < \frac{1}{2}$$

and

$$|p(x) - p(y)| \le \frac{C}{\log|x|}, \qquad x,\ y \in \Omega,\ |y| > |x| > e.$$

Then the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$, that is,

$$\|Mf\|_{p(\cdot)} \le C\|f\|_{p(\cdot)} \qquad \text{for all } f \in L^{p(\cdot)}(\Omega).$$

In this paper we aim to extend their results and the authors [10].

We say that a positive nondecreasing function φ on the interval $[0,\infty)$ satisfies (P) if there exist $\varepsilon_0 > 0$ and $0 < r_0 < 1/e$ such that

$$\text{(P)} \qquad (\log(1/r))^{-\varepsilon_0}\varphi(1/r) \qquad \text{is nondecreasing on } (0, r_0).$$

For positive nondecreasing functions φ and ψ satisfying (P), let us assume that our variable exponent $p(\cdot)$ is a positive continuous function on $\mathbf{R}^n$ satisfying :

(p1) $1 < p^- = \inf_{\mathbf{R}^n} p(x) \le \sup_{\mathbf{R}^n} p(x) = p^+ < \infty$;

(p2) $|p(x) - p(y)| \le \dfrac{\log\varphi(1/|x-y|)}{\log(1/|x-y|)}$ whenever $|x-y| < 1/e$;

(p3) $|p(x) - p(y)| \le \dfrac{\log\psi(|x|)}{\log|x|}$ whenever $|y| > |x|/2 > e/2$.

Condition (p3) implies that $p(\cdot)$ has a finite limit p_∞ at infinity and

$$\text{(p4)} \qquad |p(x) - p_\infty| \le \frac{\log \psi(|x|)}{\log |x|} \qquad \text{whenever } |x| > e.$$

If $f \in L^{p(\cdot)}(\mathbf{R}^n)$, then we find for $B_0 = B(x_0, r_0)$ with $0 < r_0 < 1/e$

$$\int_{B_0} |f(y)|^{p(x_0)} |f(y)|^{\frac{\log \varphi(1/|x_0-y|)}{\log(1/|x_0-y|)}} dy < \infty \Rightarrow \int_{B_0} |f(y)|^{p(y)} dy < \infty$$

$$\Rightarrow \int_{B_0} |f(y)|^{p(x_0)} |f(y)|^{\frac{-\log \varphi(1/|x_0-y|)}{\log(1/|x_0-y|)}} dy < \infty.$$

Since the left and right hand sides are considered to be Orlicz-type conditions, the class $L^{p(\cdot)}(\mathbf{R}^n)$ is related to certain Orlicz spaces. More precisely, see Remarks 2.9 – 2.11 below.

Now we set

$$\Phi_A(x,t) = t^{p(x)} \varphi(t)^{-A/p(x)},$$
$$\Psi_A(x,t) = t^{p(x)} \psi(t^{-1})^{-A/p(x)}$$

and

$$\mathcal{P}_A(x,t) = \min\{\Phi_A(x,t), \Psi_A(x,t)\}.$$

In view of Lemma 2.1 (ii) below, we see that $\Phi_A(x,\cdot)$, $\Psi_A(x,\cdot)$ and $\mathcal{P}_A(x,\cdot)$ are quasi-increasing on $(0,\infty)$; for example, there exists $C > 1$ such that

$$\Phi_A(x,s) \le C\Phi_A(x,t) \qquad \text{whenever } 0 < s < t \text{ and } x \in \mathbf{R}^n. \tag{1.1}$$

We define the quasi-norm

$$\|f\|_{\mathcal{P}_A(\cdot,\cdot)} = \inf\left\{\lambda > 0 : \int \mathcal{P}_A(x, |f(x)|/\lambda) dx \le 1\right\}$$

and denote by $L^{\mathcal{P}_A(\cdot,\cdot)}(\mathbf{R}^n)$ the family of all functions f on $\mathbf{R}^n$ such that $\|f\|_{\mathcal{P}_A(\cdot,\cdot)} < \infty$.

It is well known (see for example Cianchi [3]) that the maximal operator is bounded in the Orlicz space consisting of functions f satisfying

$$\int_{\mathbf{R}^n} \Phi(|f(y)|) dy < \infty,$$

where Φ is a convex function on the interval $[0,\infty)$ such that $\Phi(r)/r^p$ is nondecreasing for some $p > 1$. As an extension of this fact to the variable exponent case, we first aim to establish the following result concerning the boundedness of maximal operators.

THEOREM 1.1 *The maximal operator M is bounded from $L^{p(\cdot)}(\mathbf{R}^n)$ to $L^{\mathcal{P}_A(\cdot,\cdot)}(\mathbf{R}^n)$ when $A > n$.*

If φ and ψ are constants, then we can take $A = 0$. Hence our theorem extends the results by D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer [5]. In Theorem 1.1, we can not take $A < n$ in general, as will be seen from Remark 2.11 below.

In his paper [12], P. Hästö studied local integrability of maximal functions for the exponent

$$p(x) = 1 + a\frac{\log(e + \log(e + \delta_K(x)^{-1}))}{\log(e + \delta_K(x)^{-1})},$$

where $\delta_K(x)$ denotes the distance of x from the compact set K in $\mathbf{R}^n$. Further, P. Harjulehto and P. Hästö [13] showed continuity of Sobolev functions for exponents of the form

$$p(x) = p_0 + a\frac{\log(e + \log(e + \delta_K(x)^{-1}))}{\log(e + \delta_K(x)^{-1})},$$

which can be seen as an extension of the fact : if $u \in W^{1,n}_{loc}(\mathbf{R}^n)$ satisfies

$$\int_{\mathbf{R}^n} |\nabla u(x)|^n (\log(1 + |\nabla u(x)|))^a dx < \infty$$

with $a > n - 1$, then u is continuous on $\mathbf{R}^n$. For further related results, see [9] and [23].

If G is a bounded open set in $\mathbf{R}^n$, then the conclusion of our theorem implies

$$\int_G |Mf(x)|^{p(x)} \varphi(Mf(x))^{-A/p(x)} dx < \infty$$

for $f \in L^{p(\cdot)}(\mathbf{R}^n)$, which gives the Orlicz-type condition

$$\int_{B(x_0,r_0)} |Mf(x)|^{p(x_0)} \{|Mf(x)|^{-\frac{\log \varphi(1/|x_0-x|)}{\log(1/|x_0-x|)}} \varphi(Mf(x))^{-A/p(x_0)}\} dx < \infty$$

for small r_0.

To show Theorem 1.1, different from the bounded domain case, we need to discuss a boundedness property for the Hardy operator defined by

$$Hf(x) = \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} |f(y)| dy.$$

As applications of Theorem 1.1, we discuss Sobolev's type inequality for Riesz potentials of functions in Orlicz spaces of variable exponent by use of the so called Hedberg trick (see [19]). For the case of variable exponents satisfying the so called log-Hölder condition, there are many papers, e.g, Almeida-Samko [1], Capone-Cruz-Uribe-Fiorenza [2], Cruz-Uribe-Fiorenza-Martell-Pérez [4], Diening [7], Edmunds-Rákosník [8], Futamura-Mizuta [9], Futamura-Mizuta-Shimomura [10, 11], Mizuta-Shimomura [24], Harjulehto-Hästö [13], Harjulehto-Hästö-Koskenoja [14, 15], Harjulehto-Hästö-Koskenoja-Varonen [16], Harjulehto-Hästö-Latvala [17], Harjulehto-Hästö-Pere [18], Kokilashvili-Samko [20], Samko-Shargorodsky-Vakulov [27] and Samko-Vakulov [28].

2 Proof of Theorem 1.1

Throughout this paper, let C denote various constants independent of the variables in question.

First we note the following result, which can be derived by condition (P).

LEMMA 2.1 ([22], [23, Lemma 2.1]).

(i) *$\varphi(r)$ is of log-type, that is, there exists $C > 0$ such that*

$$C^{-1}\varphi(r) \le \varphi(r^2) \le C\varphi(r) \qquad \text{whenever } r > 0.$$

(ii) *For $\delta > 0$, $r^{-\delta}\varphi(r)$ is almost decreasing, that is, there exists $C > 0$ such that*

$$r_2^{-\delta}\varphi(r_2) \le Cr_1^{-\delta}\varphi(r_1) \qquad \text{whenever } r_2 > r_1 > 0.$$

(iii) *There exists $0 < r_0 < 1/e$ such that $\omega_1(r) = \log\varphi(1/r)/\log(1/r)$ is nondecreasing on $(0, r_0]$; set $\omega_1(r) = \omega_1(r_0)$ for $r > r_0$.*

(iv) *There exists $R_0 > e$ such that $\omega_2(r) = \log\psi(r)/\log r$ is nonincreasing on $[R_0, \infty)$; set $\omega_2(r) = \omega_2(R_0)$ for $0 < r < R_0$.*

In view of (i) we see that

(i′) for each $\gamma > 0$ there exists $C > 0$ such that

$$C^{-1}\varphi(r) \le \varphi(r^\gamma) \le C\varphi(r) \qquad \text{whenever } r > 0.$$

Recall

$$\Phi_A(x,t) = t^{p(x)}\varphi(t)^{-A/p(x)}$$

for $A > n$. Setting

$$\|f\|_{\Phi_A(\cdot,\cdot)} = \inf\left\{\lambda > 0 : \int \Phi_A(x, |f(x)|/\lambda)dx \le 1\right\},$$

we denote by $L^{\Phi_A(\cdot,\cdot)}(\mathbf{R}^n)$ the family of all functions f on $\mathbf{R}^n$ such that $\|f\|_{\Phi_A(\cdot,\cdot)} < \infty$. Then we see that $\|\cdot\|_{\Phi_A(\cdot,\cdot)}$ is a quasi-norm, that is,

(i) $\|f\|_{\Phi_A(\cdot,\cdot)} = 0$ if and only if $f = 0$,

(ii) $\|kf\|_{\Phi_A(\cdot,\cdot)} = |k|\|f\|_{\Phi_A(\cdot,\cdot)}$,

(iii) $\|f + g\|_{\Phi_A(\cdot,\cdot)} \le C\left(\|f\|_{\Phi_A(\cdot,\cdot)} + \|g\|_{\Phi_A(\cdot,\cdot)}\right)$

for $f, g \in L^{\Phi_A(\cdot,\cdot)}(\mathbf{R}^n)$ and a real number k. The same is true for $\|\cdot\|_{\Psi_A(\cdot,\cdot)}$ as well as $\|\cdot\|_{\Phi_A(\cdot,\cdot)}$.

EXAMPLE 2.2 (1) Our typical example of φ is

$$\varphi(r) = a(\log r)^b(\log(\log r))^c \qquad \text{for } r \geq R_0$$

and $\varphi(r) = \varphi(R_0)$ for $0 \leq r < R_0$ if the numbers $R_0 > e$, $a > 0$, $b \geq 0$ and c are chosen so that $\varphi(r)$ is nondecreasing on $(0, \infty)$.

(2) For a positive nondecreasing function φ satisfying (P), set

$$\omega(r) = \frac{\log \varphi(1/r)}{\log(1/r)} \qquad (0 < r \leq r_0 = 1/R_0).$$

Then we see that

$$|\omega(s) - \omega(t)| \leq \omega(|s-t|) \qquad \text{for all } 0 < s, t \leq r_0.$$

For this, we have only to see that

$$\omega(s+t) \;\leq\; \log \varphi(1/(s+t)) \left\{ \frac{1}{\log(1/s)} + \frac{1}{\log(1/t)} \right\} \leq \omega(s) + \omega(t)$$

for $s, t > 0$ with $s + t \leq r_0$.

(3) Let K be a compact set in $\mathbf{R}^n$ and denote the distance of x from K by $\delta_K(x)$. For φ as in the introduction and $p_0 > 1$,

$$p(x) = p_0 + \frac{\log \varphi(1/\delta_K(x))}{\log(1/\delta_K(x))} \qquad \text{for } x \text{ near } K$$

can be extended to an exponent satisfying conditions (p1) and (p2).

(4) For $p_0 > 1$ and $\delta > 0$,

$$p(x) = p_0 + \left(\frac{1}{\log(e + \log(e + |x|))} \right)^{\delta}$$

satisfies (p1) – (p4) with φ and ψ replaced by suitable constants.

For a proof of Theorem 1.1, we need the following result. For this purpose, it is worth to see that

(ω_1) $$r^{-\omega_1(r)} \leq C\varphi(1/r)$$

and

(ω_2) $$r^{\omega_2(r)} \leq C\psi(r)$$

whenever $r > 0$.

LEMMA 2.3 *Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{p(\cdot)} \le 1$ such that $f(x) \ge 1$ or $f(x) = 0$ for each $x \in \mathbf{R}^n$. Set*

$$F = F(x,r,f) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy$$

and

$$G = G(x,r,f) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)^{p(y)}dy.$$

Then

$$F \le CG^{1/p(x)} \varphi(G)^{n/p(x)^2}.$$

PROOF. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{p(\cdot)} \le 1$ such that $f(x) \ge 1$ or $f(x) = 0$ for each $x \in \mathbf{R}^n$. First consider the case when $G \ge 1$. Note by (ω_1) that

$$G^{\omega_1(CG^{-1/n})} \le C\varphi(G)^n$$

and

$$\varphi(G)^{\omega_1(CG^{-1/n})} \le C.$$

Since $\|f\|_{p(\cdot)} \le 1$ by our assumption, we find

$$\int f(y)^{p(y)}dy \le 1,$$

so that $G \le 1/|B(x,r)|$. Hence we have for $y \in B(x,r)$,

$$\begin{aligned}
\left\{G^{1/p(x)}\varphi(G)^{n/p(x)^2}\right\}^{-p(y)} &\le \left\{CG^{1/p(x)}\varphi(G)^{n/p(x)^2}\right\}^{-p(x)+\omega_1(r)} \\
&\le \left\{CG^{1/p(x)}\varphi(G)^{n/p(x)^2}\right\}^{-p(x)+\omega_1(CG^{-1/n})} \le CG^{-1},
\end{aligned}$$

so that

$$\begin{aligned}
F &\le G^{1/p(x)}\varphi(G)^{n/p(x)^2} + \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y) \left\{\frac{f(y)}{G^{1/p(x)}\varphi(G)^{n/p(x)^2}}\right\}^{p(y)-1} dy \\
&\le CG^{1/p(x)}\varphi(G)^{n/p(x)^2}.
\end{aligned}$$

In the case $G \le 1$, noting that $f(y) \le f(y)^{p(y)}$ for $y \in \mathbf{R}^n$, we find

$$F \le G \le CG^{1/p(x)} \le CG^{1/p(x)}\varphi(G)^{n/p(x)^2}$$

since $\varphi(0) > 0$. Now the result follows. □

PROPOSITION 2.4 *Let $0 < R < \infty$. Then the maximal operator M is bounded from $L^{p(\cdot)}(B(0,R))$ to $L^{\Phi_A(\cdot,\cdot)}(\mathbf{R}^n)$ when $A > n$.*

PROOF. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{p(\cdot)} \le 1$ such that $f = 0$ outside $B(0, R)$. We write

$$f = f\chi_{\{y: f(y) \ge 1\}} + f\chi_{\{y: f(y) < 1\}} = f_1 + f_2,$$

where χ_E denotes the characteristic function of E.

Now take p_0 such that $1 < p_0 < p^-$, and set $p_0(x) = p(x)/p_0$. Then we see that

$$\int_{B(0,R)} f_1(y)^{p_0(y)} dy \le \int_{B(0,R)} f(y)^{p(y)} dy \le 1,$$

so that $\|f_1\|_{p_0(\cdot)} \le 1$. Applying Lemma 2.3 with $p(x)$ and $\varphi(r)$ replaced by $p_0(x)$ and $\varphi(r)^{1/p_0}$ respectively, we find

$$Mf_1(x) \le C\{Mg_0(x)\}^{1/p_0(x)} \varphi(Mg_0(x))^{n/\{p_0 p_0(x)^2\}}$$

for $x \in B(0, 2R)$, where $g_0(y) = f(y)^{p_0(y)}$. Since $Mf_2(x) \le 1$, we establish

$$Mf(x) \le C\{Mg_0(x)\}^{1/p_0(x)} \varphi(Mg_0(x))^{n/\{p_0 p_0(x)^2\}} + C,$$

so that Lemma 2.1 gives

$$\{Mf(x)\}^{p(x)} \varphi(Mf(x))^{-np_0/p(x)} \le C\left(Mg_0(x) + 1\right)^{p_0}.$$

Thus it follows that

$$\Phi_A(x, Mf(x)) \le C + C\{Mg_0(x)\}^{p_0}$$

with $A = np_0$. Hence, by the well-known boundedness of the maximal operator, we insist that

$$\int_{B(0,2R)} \Phi_A(x, Mf(x)) dx \le C.$$

If $|x| \ge 2R$, then

$$Mf(x) \le C|x|^{-n} \int_{B(0,R)} \{1 + f(y)^{p(y)}\} dy \le C|x|^{-n},$$

which proves

$$\int_{\mathbf{R}^n \setminus B(0,2R)} \Phi_A(x, Mf(x)) dx \le C.$$

Thus the required result is proved. □

LEMMA 2.5 *Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $f = 0$ on $B(0, R_0)$ and $f < 1$ on $\mathbf{R}^n$. Then*

$$F \le C\{G\psi(G^{-1})^{n/p(x)}\}^{1/p(x)} + C\gamma(x) + CHf(x)$$

whenever $|x| \ge e$, where $\gamma(x) = |x|^{-n/p(x)} \psi(|x|)^{n/p_\infty^2}$ and

$$Hf(x) = \frac{1}{|B(0, |x|)|} \int_{B(0,|x|)} |f(y)| dy.$$

PROOF. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $f = 0$ on $B(0, R_0)$ and $f < 1$ on $\mathbf{R}^n$. Then note that

$$G = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)^{p(y)}dy < 1.$$

Let $|x| \geq e$. In the case $G \geq |x|^{-n}$, we have by (p3) and (ω_2)

$$\begin{aligned}\left\{G^{1/p(x)}\psi(G^{-1})^{n/p(x)^2}\right\}^{-p(y)} &\leq \left\{CG^{1/p(x)}\psi(G^{-1})^{n/p(x)^2}\right\}^{-p(x)-\omega_2(|x|)} \\ &\leq \left\{CG^{1/p(x)}\psi(G^{-1})^{n/p(x)^2}\right\}^{-p(x)-\omega_2(G^{-1/n})} \\ &\leq CG^{-1}\end{aligned}$$

for $|y| > |x|/2$. Hence we find

$$\begin{aligned}&\frac{1}{|B(x,r)|}\int_{B(x,r)\setminus B(0,|x|/2)} f(y)dy \\ \leq\ & G^{1/p(x)}\psi(G^{-1})^{n/p(x)^2} \\ &+\frac{1}{|B(x,r)|}\int_{B(x,r)\setminus B(0,|x|/2)} f(y)\left\{\frac{f(y)}{G^{1/p(x)}\psi(G^{-1})^{n/p(x)^2}}\right\}^{p(y)-1}dy \\ \leq\ & CG^{1/p(x)}\psi(G^{-1})^{n/p(x)^2} \\ \leq\ & CG^{1/p(x)}\psi(G^{-1})^{n/p_\infty^2}.\end{aligned}$$

In the case $G \leq |x|^{-n}$, we see that

$$\begin{aligned}&\frac{1}{|B(x,r)|}\int_{B(x,r)\setminus B(0,|x|/2)} f(y)dy \\ \leq\ & |x|^{-n/p(x)}\psi(|x|)^{n/p(x)^2} \\ &+\frac{1}{|B(x,r)|}\int_{B(x,r)\setminus B(0,|x|/2)} f(y)\left\{\frac{f(y)}{|x|^{-n/p(x)}\psi(|x|)^{n/p(x)^2}}\right\}^{p(y)-1}dy \\ \leq\ & C|x|^{-n/p(x)}\psi(|x|)^{n/p(x)^2} \leq C\gamma(x).\end{aligned}$$

Finally we obtain

$$\frac{1}{|B(x,r)|}\int_{B(x,r)\cap B(0,|x|/2)} f(y)dy \leq CHf(x),$$

which completes the proof. □

LEMMA 2.6 *Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $f = 0$ on $B(0, R_0)$, $f < 1$ on $\mathbf{R}^n$ and*

$$G_0 = \frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} f(y)^{p(y)}dy \leq C|x|^{-\delta} \tag{2.1}$$

for some $C > 0$ and $\delta > 0$ independent of x and f. If $0 < \beta < n$, then

$$Hf(x) \le C\left\{G_0\psi(G_0^{-1})^{\beta/p(x)}\right\}^{1/p(x)} + C|x|^{-\beta/p(x)}$$

for $|x| \ge R_0$.

PROOF. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying $f = 0$ on $B(0, R_0)$, $f < 1$ on $\mathbf{R}^n$ and (2.1). For $|x| \ge R_0$, we have by Hölder's inequality

$$\begin{aligned} Hf(x)^{p(x)} &\le \frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} f(y)^{p(x)}dy \\ &= \frac{1}{|B(0,|x|)|}\int_{B(0,|x|)\cap E} f(y)^{p(x)}dy + \frac{1}{|B(0,|x|)|}\int_{B(0,|x|)\setminus E} f(y)^{p(x)}dy \\ &= H_1 + H_2, \end{aligned}$$

where $E = \{y \in \mathbf{R}^n \setminus B(0, R_0) : |y|^{-\beta/p(x)} \le f(y) < 1\}$. Note that

$$H_2 \le C|x|^{-\beta}.$$

If $y \in B(0, |x|) \cap E$, then

$$f(y)^{p(x)} \le f(y)^{p(y)-\omega_2(|y|)} \le f(y)^{p(y)}\psi(|y|)^{\beta/p(x)} \le f(y)^{p(y)}\psi(|x|)^{\beta/p(x)},$$

so that

$$H_1 \le \psi(|x|)^{\beta/p(x)}G_0,$$

which together with (2.1) gives

$$H_1 \le C\psi(G_0^{-1})^{\beta/p(x)}G_0,$$

as required. □

Applying Hardy's inequality, we can prove the following result.

LEMMA 2.7 *For $1 < p_0 < \infty$,*

$$\|Hg_0\|_{p_0} \le C\|g_0\|_{p_0}$$

for all functions $g_0 \in L^{p_0}(\mathbf{R}^n)$.

Now we are ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{p(\cdot)} \le 1$. Write

$$f = f\chi_{\{y: f(y)\ge 1\}} + f\chi_{\{y: f(y)<1\}} = f_1 + f_2.$$

We have by Lemma 2.3,

$$Mf_1(x) \le C\{Mg(x)\}^{1/p(x)}\varphi(Mg(x))^{n/p(x)^2},$$

where $g(y) = f(y)^{p(y)}$, so that

$$\Phi_n(x, Mf_1(x)) \le CMg(x). \tag{2.2}$$

Hence, in view of the proof of Proposition 2.4, we see that

$$\int_{\mathbf{R}^n} \Phi_A(x, Mf_1(x))dx \le C$$

when $A > n$. Since $Mf_2 \le 1$ on $\mathbf{R}^n$, we have

$$\int_{B(0,e)} \Phi_A(x, Mf_2(x))dx \le C.$$

Further we find by Proposition 2.4

$$\int_{\mathbf{R}^n} \Phi_A(x, Mf_2'(x))dx \le C,$$

where $f_2'(y) = f_2(y)\chi_{B(0,e)}(y)$. Therefore it suffices to prove

$$\int_{\mathbf{R}^n \setminus B(0,e)} \Psi_A(x, Mf_2''(x))dx \le C, \tag{2.3}$$

where $f_2'' = f_2 - f_2'$.

Thus we may assume that $0 \le f < 1$ on $\mathbf{R}^n$ and $f = 0$ on $B(0,e)$. In this case, by Lemmas 2.5 and 2.6, we have for $0 < \beta < n$

$$\begin{aligned} Mf(x) &\le C\{Mg(x)\psi(Mg(x)^{-1})^{n/p(x)}\}^{1/p(x)} + C\gamma(x) + CHf(x) \\ &\le C\{Mg(x)\psi(Mg(x)^{-1})^{n/p(x)}\}^{1/p(x)} + C|x|^{-\beta/p(x)} \\ &\quad + C\{Hg(x)\psi(Hg(x)^{-1})^{n/p(x)}\}^{1/p(x)}, \end{aligned}$$

so that

$$\Psi_n(x, Mf(x)) \le CMg(x) + CHg(x) + C|x|^{-\beta} \tag{2.4}$$

for $|x| \ge e$. Let $1 < p_0 < p^-$. Applying (2.4) with $p(x)$ and $\psi(r)$ replaced by $p_0(x) = p(x)/p_0$ and $\psi(r)^{1/p_0}$ respectively, we find

$$\Psi_A(x, Mf(x))^{1/p_0} \le CMg_0(x) + CHg_0(x) + C|x|^{-\beta},$$

where $A = np_0$ and $g_0(y) = f(y)^{p_0(y)} = g(y)^{1/p_0}$. Hence, letting $\beta p_0 > n$, by Lemma 2.7 and the boundedness of maximal operator on L^{p_0}, we derive

$$\int_{\mathbf{R}^n \setminus B(0,e)} \Psi_A(x, Mf(x))dx \le C.$$

Thus the proof is completed. □

REMARK 2.8 In Theorem 1.1, we can replace $\mathcal{P}_A(x,t)$ by

$$\min\{t^{p(x)}\varphi(t)^{-A/p(x)}, t^{p(x)}\psi(t^{-1})^{-A/p_\infty}\}$$

or

$$\left[\min\{t\varphi(t)^{-A/p(x)^2}, t\psi(t^{-1})^{-A/p_\infty^2}\}\right]^{p(x)}.$$

REMARK 2.9 Let $p(\cdot)$ be the variable exponent such that

$$p(x) = p_0 + a\frac{\log\log(c_0/|x|)}{\log(c_0/|x|)}$$

for $x \in \mathbf{B} = B(0,1)$, where $a > 0$ and $c_0 > e$ are chosen so that $p(x) \geq p_0$ on $\mathbf{B}$ and $p(x)$ satisfies (p2) with $\varphi(r) = (\log(e+r))^a$. If f is a nonnegative measurable function in $L^{p(\cdot)}(\mathbf{B})$, then

$$\int_{\mathbf{B}} f(y)^{p_0}(\log(e+f(y)))^{an/p_0}dy < \infty.$$

In fact, letting $E = \{y \in \mathbf{B} : f(y) \leq |y|^{-n/p_0}(\log(e+|y|^{-1}))^{-\lambda}\}$ with $\lambda > (an/p_0 + 1)/p_0$, then

$$\begin{aligned}
&\int_{\mathbf{B}} f(y)^{p_0}(\log(e+f(y)))^{an/p_0}dy \\
\leq\ & C\int_E |y|^{-n}(\log(e+|y|^{-1}))^{an/p_0-\lambda p_0}dy + C\int_{\mathbf{B}\setminus E} f(y)^{p_0}(\log(e+f(y)))^{an/p_0}dy \\
\leq\ & C + C\int_{\mathbf{B}\setminus E} f(y)^{p(y)}dy < \infty.
\end{aligned}$$

REMARK 2.10 We next consider the converse part of Remark 2.10. Let $p(\cdot)$ be the variable exponent such that

$$p(x) = p_0 - a\frac{\log\log(c_0/|x|)}{\log(c_0/|x|)}$$

for $x \in \mathbf{B}$, where $a > 0$ and $c_0 > e$ are chosen so that $p(x) > 1$ on $\mathbf{B}$ and $p(x)$ satisfies (p2) with $\varphi(r) = (\log(e+r))^a$. If f is a nonnegative measurable function on $\mathbf{B}$ satisfying

$$\int_{\mathbf{B}} f(y)^{p_0}(\log(e+f(y)))^{-an/p_0}dy < \infty,$$

then $f \in L^{p(\cdot)}(\mathbf{B})$.

REMARK 2.11 Consider the variable exponent

$$p(x) = \begin{cases} p_0 + a\frac{\log(e+\log(e+x_n^{-1}))}{\log(e+x_n^{-1})} & (x_n > 0) \\ p_0 & (x_n \leq 0) \end{cases}$$

for $x = (x_1, ..., x_n) \in \mathbf{B}$, where $a > 0$. Let

$$f(y) = \chi_{\mathbf{B}}(y) \times \begin{cases} |y|^{-n/p_0}(\log(e + |y|^{-1}))^{-1/p_0}(\log\log(e + |y|^{-1}))^{-\beta} & (y_n < 0) \\ 0 & (y_n \geq 0) \end{cases}$$

for $\beta p_0 > 1$. Then $f \in L^{p(\cdot)}(\mathbf{B})$. Noting that

$$Mf(x) \geq C|x|^{-n/p_0}(\log(e + |x|^{-1}))^{-1/p_0}(\log\log(e + |x|^{-1}))^{-\beta},$$

we have

$$\begin{aligned} & \int_{\mathbf{B}} Mf(x)^{p(x)}(\log(1 + Mf(x)))^{-K} dx \\ \geq \; & C \int_{\Gamma} |x|^{-n}(\log(e + |x|^{-1}))^{-1+an/p_0-K}(\log\log(e + |x|^{-1}))^{-\beta p_0} dx \end{aligned}$$

where $\Gamma = \{x = (x_1, ..., x_n) \in \mathbf{B} : x_n > |x|/2\}$. Hence

$$\int_{\mathbf{B}} Mf(x)^{p(x)}(\log(1 + Mf(x)))^{-K} dx = \infty$$

if $-K + an/p_0 > 0$. This implies that we can not take $A < n$ in Theorem 1.1, generally.

References

[1] A. Almeida and S. Samko, Characterization of Riesz and Bessel potentials on variable Lebesgue spaces, J. Funct. Spaces Appl. **4** (2006), 113–144.

[2] C. Capone, D. Cruz-Uribe and A. Fiorenza, The fractional maximal operator on variable L^p spaces, Rev. Mat. Iberoamericana **23** (2007), no. 3, 743-770.

[3] A. Cianchi, Strong and weak type inequalities for some classical operators in Orlicz spaces, J. London Math. Soc. **60** (1999), 187–202.

[4] D. Cruz-Uribe, A. Fiorenza, J. M. Martell and C. Pérez, The boundedness of classical operators on variable L^p spaces, Ann. Acad. Sci. Fenn. Math. **31** (2006), 239-264..

[5] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, The maximal function on variable L^p spaces, Ann. Acad. Sci. Fenn. Math. **28** (2003), 223-238, **29** (2004), 247-249.

[6] L. Diening, Maximal functions on generalized $L^{p(\cdot)}$ spaces, Math. Inequal. Appl. **7**(2) (2004), 245–253.

[7] L. Diening, Riesz potentials and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$, Math. Nachr. **263**(1) (2004), 31–43.

[8] D. E. Edmunds and J. Rákosník, Sobolev embedding with variable exponent, II, Math. Nachr. **246-247** (2002), 53–67.

[9] T. Futamura and Y. Mizuta, Continuity properties of Riesz potentials for functions in $L^{p(\cdot)}$ of variable exponent, Math. Inequal. Appl. **8** (2005), 619–631.

[10] T.Futamura, Y. Mizuta and T. Shimomura, Sobolev embeddings for Riesz potential space of variable exponent, Math. Nachr. **279** (2006), 1463-1473.

[11] T. Futamura, Y. Mizuta and T. Shimomura, Sobolev embeddings for variable exponent Riesz potentials on metric spaces, Ann. Acad. Sci. Fenn. Math. **31** (2006), 495–522.

[12] P. Hästö, The maximal operator in Lebesgue spaces with variable exponent near 1, Math. Nachr. **280** (2007), 74–82.

[13] P. Harjulehto and P. Hästö, A capacity approach to the Poincaré inequality and Sobolev imbeddings in variable exponent Sobolev spaces, Rev. Mat. Complut. **17** (2004), no. 1, 129–146.

[14] P. Harjulehto, P. Hästö and M. Koskenoja, Hardy's inequality in a variable exponent Sobolev space, Georgian Math. J. **12** (2005), no. 3, 431–442.

[15] P. Harjulehto, P. Hästö and M. Koskenoja, Properties of capacities in variable exponent Sobolev spaces, J. Anal. Appl. **5** (2007), no. 2, 71–92.

[16] P. Harjulehto, P. Hästö, M. Koskenoja and S. Varonen, Sobolev capacity on the space $W^{1,p(\cdot)}(\mathbb{R}^n)$, J. Funct. Spaces Appl. **1** (2003), no. 1, 17–33.

[17] P. Harjulehto, P. Hästö and V. Latvala, Sobolev embeddings in metric measure spaces with variable dimension, Math. Z. **254** (2006), no. 3, 591–609.

[18] P. Harjulehto, P. Hästö and M. Pere, Variable exponent Sobolev spaces on metric measure spaces, Funct. Approx. Comment. Math. **36** (2006), 79–94.

[19] L. I. Hedberg, On certain convolution inequalities, Proc. Amer. Math. Soc. **36** (1972), 505–510.

[20] V. Kokilashvili and S. Samko, On Sobolev theorem for Riesz-type potentials in Lebesgue spaces with variable exponent, Z. Anal. Anwendungen. **22** (2003), 899-910.

[21] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. **41** (1991), 592–618.

[22] Y. Mizuta, Potential theory in Euclidean spaces, Gakkōtosho, Tokyo, 1996.

[23] Y. Mizuta, T. Ohno and T. Shimomura, Integrability of maximal functions for generalized Lebesgue spaces with variable exponent, Math. Nachr. **281** (2008), 386-395.

[24] Y. Mizuta and T. Shimomura, Sobolev's inequality for Riesz potentials with variable exponent satisfying a log-Hölder condition at infinity, J. Math. Anal. Appl. **311** (2005), 268-288.

[25] W. Orlicz, Über konjugierte Exponentenfolgen, Studia Math. **3** (1931), 200–211.

[26] M. Růžička, Electrorheological fluids : modeling and Mathematical theory, Lecture Notes in Math. **1748**, Springer, 2000.

[27] S. Samko, E. Shargorodsky and B. Vakulov, Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators. II, J. Math. Anal. Appl. **325** (2007), 745–751.

[28] S. Samko and B. Vakulov, Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators, J. Math. Anal. Appl. **310** (2005), 229–246.

[29] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, 1970.