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Limits at infinity of superharmonic functions and solutions of semilinear elliptic equations of Matukuma type

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1 Motivation and Question

In 1930, Matukuma introduced the following semilinear elliptic equation

$$-\Delta u(x)=\frac{u(x)^{p}}{1+\|x\|^{2}} \quad \text{in } \mathbb{R}^{3}$$

to study a gravitational potential u of a globular cluster of stars. Here Δ is the Laplacian, $\|x\|$ the Euclidean norm of a point x, and $p>1$ is a constant. This equation was deduced from Poisson's equation under several hypotheses in astrophysics. For details, see [8].

For the last several decades, many mathematicians have studied the existence of positive solutions of semilinear elliptic equations of the form

$$-\Delta u(x)=V(x)u(x)^{p} \quad \text{in } \Omega, \tag{1.1}$$

where V is a measurable function on a domain Ω in $\mathbb{R}^{n}$ with appropriate properties and the equation is understood in the sense of distributions. There are a great number of papers, but we mention only results relating to this talk.

- Kenig and Ni [4] studied (1.1) in the case of $\Omega=\mathbb{R}^{n}$ $(n\geq 3)$. Indeed, they proved that if V is a measurable function on $\mathbb{R}^{n}$ such that

$$|V(x)|\leq\frac{A}{(1+\|x\|^{2})^{1+\varepsilon}}$$

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for some $\varepsilon > 0$ and $A > 0$, then (1.1) has bounded positive solutions.

- Zhao [7] generalized their result as follows. Let Ω be an unbounded domain in $\mathbb{R}^n$ $(n \geq 3)$ with a compact Lipschitz boundary or $\Omega = \mathbb{R}^n$. If V is a Green-tight function on Ω and $\alpha > 0$ is sufficiently small, then there are positive solutions u of (1.1) satisfying

$$\lim_{x \to \infty} u(x) = \alpha.$$

- The corresponding result for two dimensions was obtained by Ufuktepe and Zhao [6]. Let Ω be an unbounded domain in $\mathbb{R}^2$ with a compact boundary consisting of finitely many Jordan curves. If V is a Green-tight function on Ω and $\alpha > 0$ is sufficiently small, then there are positive solutions u of (1.1) satisfying

$$\lim_{x \to \infty} \frac{u(x)}{\log \|x\|} = \alpha.$$

In view of the last two results, the following question arises naturally.

Question. *Let Ω be an unbounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with a compact boundary or $\Omega = \mathbb{R}^n$ and let V be a nonnegative measurable function on Ω with appropriate properties. Does every positive solution u of* (1.1) *satisfy*

$$\lim_{x \to \infty} u(x) = \alpha \quad (n \geq 3)$$

or

$$\lim_{x \to \infty} \frac{u(x)}{\log \|x\|} = \alpha \quad (n = 2)$$

for some $\alpha \geq 0$?

Remark 1.1. When $n \geq 3$ and V is a negative function with suitable properties, there is a positive solution u of $-\Delta u = V u^p$ in $\mathbb{R}^n$ such that $u(x) \to +\infty$ as $\|x\| \to +\infty$. See [1, 5] (ODE or PDE) and [2] (potential theoretic proof), etc. Thus the above question is significant in the case that V is nonnegative.

2 Notation and Convention

In the rest of this note, we let $n \geq 2$ and suppose that **Ω is an unbounded domain in $\mathbb{R}^n$ with a compact boundary or $\Omega = \mathbb{R}^n$**. The symbol A stands for an absolute positive constant whose value is unimportant and may change from line to line. Denote by $B(x,r)$ the open ball of center x and radius r. A function $u : \Omega \to (-\infty, +\infty]$ is called *superharmonic* if

(i) $u \not\equiv +\infty$,

(ii) u is lower semicontinuous on Ω,

(iii) $u(x) \geq \dfrac{1}{\nu_n r^n} \displaystyle\int_{B(x,r)} u(y)\, dy$ whenever $\overline{B(x,r)} \subset \Omega$.

Here ν_n is the volume of the unit ball in $\mathbb{R}^n$. It is well known that for a superharmonic function u on Ω, there is a unique nonnegative measure μ_u such that

$$\int_\Omega \phi(x)\, d\mu_u(x) = \int_\Omega u(x)(-\Delta\phi(x))\, dx \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

We discuss superharmonic functions u such that **μ_u is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$**. Then the Radon-Nykodým derivative is denoted by f_u. It is obvious that $f_u = -\Delta u$ for $u \in C^2(\Omega)$.

3 Main Results

This section presents our main results (answers to the question in Section 1).

Theorem 3.1. *Let $n \geq 3$. Suppose that*

$$0 \leq p < \frac{n}{n-2}.$$

If u is a positive superharmonic function on Ω satisfying

$$f_u(x) \leq \frac{c}{\|x\|^2}\, u(x)^p \quad \textit{for a.e. } x \in \Omega \setminus B(0,R)$$

with some $c > 0$ and $R > 0$, then u has a finite limit at infinity.

As seen in the following, the bound $p < n/(n-2)$ is nearly optimal in Theorem 3.1. The case $p = n/(n-2)$ is still unsolved.

Theorem 3.2. *Let $n \geq 3$ and $c > 0$. If*

$$p > \frac{n}{n-2},$$

then for each $\beta > 0$, there is a positive function $u \in C^2(\mathbb{R}^n)$ such that

$$0 \leq -\Delta u(x) \leq \frac{c}{1+\|x\|^2} u(x)^p \quad \text{in } \mathbb{R}^n$$

and

$$\limsup_{x\to\infty} \frac{u(x)}{\|x\|^\beta} = +\infty.$$

Two dimensional result corresponding to Theorem 3.1 is as follows.

Theorem 3.3. *Let $n = 2$ and let $p \geq 0$ be arbitrary constant. If u is a positive superharmonic function on Ω satisfying*

$$f_u(x) \leq \frac{c}{\|x\|^2(\log\|x\|)^p} u(x)^p \quad \text{for a.e. } x \in \Omega \setminus B(0,R) \tag{3.1}$$

with some $c > 0$ and $R > 1$, then $u(x)/\log\|x\|$ has a finite limit at infinity.

4 Outline of proofs of Theorems 3.1 and 3.3

In this section, we give a sketch of the proof of Theorem 3.1 as well as Theorem 3.3. For details, see [3].

Lemma 4.1. *Let $\{z_i\}$ be a sequence in Ω with $z_i \to \infty$ $(i \to +\infty)$. If v is a positive superharmonic function on Ω such that*

$$f_v(x) \leq \frac{A}{\|x\|^2} \quad \text{for a.e. } x \in \bigcup_i B(z_i, \rho\|z_i\|)$$

with some $A > 0$ and $0 < \rho \leq 1/2$, then the following hold:

(i) *if $n \geq 3$, then $v(z_i)$ has a finite limit as $i \to \infty$;*

(ii) *if $n = 2$, then $v(z_i)/\log\|z_i\|$ has a finite limit as $i \to \infty$.*

Here the value of the limit is independent of $\{z_i\}$.

Indeed, this lemma is a special case $p = 0$ of Theorems 3.1 and 3.3. When $p > 0$, the following lemma plays an essential role.

Lemma 4.2. *Let* $n \geq 3$. *Suppose that*

$$0 < p < \frac{n}{n-2}.$$

Let u *be a positive superharmonic function on* Ω *satisfying*

$$f_u(x) \leq \frac{c}{\|x\|^2} u(x)^p \quad \textit{for a.e. } x \in \Omega \setminus B(0, R)$$

with some $c > 0$ *and* $R > 0$. *If* $\{z_i\}$ *is a sequence in* Ω *with* $z_i \longrightarrow \infty$ $(i \longrightarrow +\infty)$, *then there exist* $A > 0$ *and* $i_0, \ell \in \mathbb{N}$ *such that*

$$u \leq A \quad \textit{on} \bigcup_{i \geq i_0} B(z_i, 2^{-\ell-3}\|z_i\|).$$

The proof is based on arguments of minimal fine topology and nonlinear analysis. The corresponding result for two dimensions is as follows.

Lemma 4.3. *Let* $n = 2$ *and let* $p > 0$ *be arbitrary constant. Let* u *be a positive superharmonic function on* Ω *satisfying*

$$f_u(x) \leq \frac{c}{\|x\|^2(\log\|x\|)^{p-1}} u(x)^p \quad \textit{for a.e. } x \in \Omega \setminus B(0, R)$$

with some $c > 0$ *and* $R > 1$. *If* $\{z_i\}$ *is a sequence in* Ω *with* $z_i \longrightarrow \infty$ $(i \longrightarrow +\infty)$, *then there exist* $A > 0$ *and* $i_0 \in \mathbb{N}$ *such that*

$$\frac{u(x)}{\log\|x\|} \leq A \quad \textit{for } x \in \bigcup_{i \geq i_0} B(z_i, 2^{-5}\|z_i\|).$$

Now, Theorem 3.1 is proved immediately. Let $\{z_i\}$ be arbitrary sequence in Ω with $z_i \longrightarrow \infty$ $(i \longrightarrow +\infty)$. By Lemma 4.2,

$$\begin{aligned} f_u(x) &\leq \frac{c}{\|x\|^2} u(x)^p \\ &\leq \frac{A}{\|x\|^2} \quad \text{for a.e. } x \in \bigcup_{i \geq i_0} B(z_i, 2^{-\ell-3}\|z_i\|). \end{aligned}$$

By Lemma 4.1, $u(z_i)$ has a finite limit and its value is independent of $\{z_i\}$. Thus Theorem 3.1 follows.

The proof of Theorem 3.3 is similar.

5 Conjecture

In the proof of Lemma 4.2, we assumed $p < \dfrac{n}{n-2}$ to use the fact

$$\|\cdot\|^{2-n} \in L^q_{loc} \quad \text{for some } q > p.$$

I do not have other techniques, but we expect that

$$\text{Theorem 3.1 holds for } p = \frac{n}{n-2} \text{ as well.}$$

i.e.

$$f_u(x) \leq \frac{c}{\|x\|^2} u(x)^{\frac{n}{n-2}} \quad \Longrightarrow \quad \lim_{x \to \infty} u(x) \text{ exists.}$$

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