On solutions of quasilinear elliptic equations with general structure

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§1. Introduction and preliminaries

Let $G$ be an open set in $\mathbb{R}^N$ ($N \geq 2$) and $1 < p < N$. We consider quasi-linear second order elliptic differential equations of the form

$$(E_T) \quad - \text{div} \mathcal{A}(x, \nabla u) + \mathcal{B}(x, u) = T$$

in $G$. Here, $T$ is a distribution, $\mathcal{A} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and $\mathcal{B} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

(A.1) $x \mapsto \mathcal{A}(x, \xi)$ is measurable on $\mathbb{R}^N$ for every $\xi \in \mathbb{R}^N$ and $\xi \mapsto \mathcal{A}(x, \xi)$ is continuous for a.e. $x \in \mathbb{R}^N$;

(A.2) $\mathcal{A}(x, \xi) \cdot \xi \geq \alpha_1 |\xi|^p$ for all $\xi \in \mathbb{R}^N$ and a.e. $x \in \mathbb{R}^N$ with a constant $\alpha_1 > 0$;

(A.3) $|\mathcal{A}(x, \xi)| \leq \alpha_2 |\xi|^{p-1}$ for all $\xi \in \mathbb{R}^N$ and a.e. $x \in \mathbb{R}^N$ with a constant $\alpha_2 > 0$;

(B.1) $x \mapsto \mathcal{B}(x, t)$ is measurable on $\mathbb{R}^N$ for every $t \in \mathbb{R}$ and $t \mapsto \mathcal{B}(x, t)$ is continuous for a.e. $x \in \mathbb{R}^N$;

(B.2) For any bounded open set $D$ in $\mathbb{R}^N$, there is a constant $\alpha_3(D) \geq 0$ such that $|\mathcal{B}(x, t)| \leq \alpha_3(D)(|t|^{p-1} + 1)$ for all $t \in \mathbb{R}$ and a.e. $x \in D$;

A prototype of the equation $(E_T)$ is

$$- \text{div} (|\nabla u|^{p-2} \nabla u) + b|u|^{p-2}u = T$$

with a locally bounded function $b$ in $G$.

As a matter of fact, we treat the following two topics: (i) Hölder continuity of a solution of the equation of $(E_T)$ (section 2); (ii) Integrability of the gradients of a solution of the equation of $(E_T)$ (section 3).

Throughout this paper, we use some standard notation without explanation.
§2. Hölder continuity of a solution

In this section, we suppose the following monotoneity conditions on $A : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and $B : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$:

(A.4) \((A(x, \xi_1) - A(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0\) whenever \(\xi_1, \xi_2 \in \mathbb{R}^N, \xi_1 \neq \xi_2\), for a.e. \(x \in \mathbb{R}^N\);

(B.3) \(t \mapsto B(x, t)\) is nondecreasing on \(\mathbb{R}\) for a.e. \(x \in \mathbb{R}^N\).

We consider elliptic quasi-linear equations of the form

\[(E) - \text{div} \, A(x, \nabla u(x)) + B(x, u(x)) = 0\]

For an open subset \(G\) of \(\mathbb{R}^N\), we consider the Sobolev spaces \(W^{1,p}(G)\), \(W_0^{1,p}(G)\) and \(W_{1\text{loc}}^{1,p}(G)\).

Let \(G\) be an open subset of \(\mathbb{R}^N\). A function \(u \in W_{1\text{loc}}^{1,p}(G)\) is said to be a (weak) solution of \((E_0)\) in \(G\) if

\[\int_G A(x, \nabla u) \cdot \nabla \varphi \, dx + \int_G B(x, u) \varphi \, dx = 0\]

for all \(\varphi \in C_0^\infty(G)\).

A continuous solution of \((E_0)\) in an open subset \(G\) is called \((A, B)\)-harmonic in \(G\). For any \((A, B)\)-harmonic functions, the following locally Hölder continuity estimate holds ([6; Theorem 4.7] or [8; Proposition 2.1]):

**Proposition 1.** Let \(G\) be an open set. Then there are constants \(c\) and \(0 < \lambda \leq 1\) such that for \(B(x_0, R) \Subset G\) and for every \((A, B)\)-harmonic function \(h\) in \(G\) with \(|h| \leq L\) in \(B(x_0, R)\),

\[\text{osc}(h, B(x_0, r)) \leq c \left( \frac{r}{R} \right)^\lambda \left( \text{osc}(h, B(x_0, R)) + R \right),\]

whenever \(0 < r < R \leq 1\). Here \(c\) depends only on \(N, p, \alpha_1, \alpha_2, \alpha_3(G)\) and \(L\) and \(\lambda\) depends only on \(N, p, \alpha_1, \alpha_2\) and \(\alpha_3(G)\).

In the case of \(A(x, \xi) = |\xi|^{p-2} \xi\) and \(B = 0\), namely for the \(p\)-Laplace equation, we can choose \(\lambda = 1\) ([3; Lemma 2.1]).

Suppose that \(\nu\) is a signed Radon measure on \(G\). Hölder continuity of a solution to the equation of the form

\[(E_\nu) - \text{div} \, A(x, \nabla u(x)) + B(x, u(x)) = \nu\]

was investigated in [9], [2] and [3]. In [5], Kilpeläinen and Zhong showed that, for the equation

\[(1) - \text{div} \, A(x, \nabla u(x)) = \nu\]
and for the case $\nu$ is a nonnegative Radon measure, if there exist constants $M > 0$ and $0 < \beta < \lambda$ with

$$\nu(B(x_0, r)) \leq M \, r^{N-p+\beta(p-1)}$$

whenever $B(x, 3r) \subset G$, where $\lambda$ is the number in Proposition 1 above, then a solution to the equation (1) is Hölder continuous with the same exponent $\beta$. We can extend this result to the case of the equation $(E_{\nu})$ and of the signed Radon measure ([8]).

**Theorem 1.** Let $G$ be an open set and $u \in W^{1,p}_{loc}(G)$ be a solution of $(E_{\nu})$ in $G$. If $\nu$ is a signed Radon measure on $G$ such that there exist constants $M > 0$ and $0 < \beta < \lambda$, where $\lambda = \lambda(N, p, \alpha_1, \alpha_2, \alpha_3(G)) > 0$ is the number in Proposition 1 above, with

$$|\nu|(B(x, r)) \leq M \, r^{N-p+\beta(p-1)}$$

whenever $B(x, 3r) \subset G$, then $u$ is locally Hölder continuous in $G$ with the exponent $\beta$.

### §3. Global integrability of the gradient of a solution

In this section, we treat the higher integrability of the gradient of a solution of $(E_T)$ in a bounded open set $G$. In [9], Rakotoson and Ziemer showed the local integrability of the gradient of a solution of $(E_T)$ with $T \in W^{-1,p'+\delta}_{loc}(G)$ for some $\delta > 0$. In [4], Kilpeläinen and Koskela treated the global integrability of the gradient of a solution of the equation (1) in the previous section under the condition the complement of $G$ satisfies the uniformly thickness.

A set $E$ is said to be uniformly $p$-thick with constants $c_0$ and $r_0 > 0$, if

$$\text{cap}_p(\overline{B}(x_0, r) \cap E, B(x_0, 2r)) \geq c_0 \text{cap}_p(\overline{B}(x_0, r), B(x_0, 2r))$$

for all $x_0 \in E$ and for all $0 < r < r_0$. For the notion of $p$-capacity $\text{cap}_p$, we refer to [1; Chapter 2].

We can show the following global integrability of the gradient of a solution of $(E_T)$.

**Theorem 2.** Suppose that $G$ is a bounded open set, $\cap G$ is uniformly $p$-thick with constants $c_0, r_0 > 0$ and $u$ is a solution of $(E_T)$ in $G$ such that $u - \theta \in W_0^{1,p}(G)$. Then there exists $\delta_0 = \delta(N, p, \alpha_1, \alpha_2, \alpha_3(G), c_0)$ such that $|\nabla u| \in L^{p+\delta}(G)$ whenever $T \in W^{-1,p'+\delta}(G)$ and $|\nabla \theta| \in L^{p+\delta}(G)$ for $0 < \delta < \delta_0$. 
Remark 1. The uniformly thickness condition cannot be suppressed in Theorem 2. (see [4; Remark 3.3])

References


