SMITH SET FOR A NONGAP OLIVER GROUP

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1. Introduction

We study the Smith problem that two tangential representations are isomorphic or not for a smooth action on a homotopy sphere with exactly two fixed points ([11]). Two real $G$-modules $U$ and $V$ are called Smith equivalent if there exists a smooth action of $G$ on a sphere $\Sigma$ such that $S^G = \{x, y\}$ for two points $x$ and $y$ at which $T_x(\Sigma) \cong U$ and $T_y(\Sigma) \cong V$ as real $G$-modules which is a finite dimensional real vector space with a linear $G$-action. Let $Sm(G)$, called a Smith set, be the subset of the real representation ring $RO(G)$ of $G$ consisting of the differences $U - V$ of real $G$-modules $U$ and $V$ which are Smith equivalent. In many groups, Smith equivalent modules are not isomorphic. Let $P(G)$ be the set of subgroups of $G$ of prime power order, possibly 1. We also define a subset $CSm(G)$ of $Sm(G)$ consisting of the differences $U - V \in Sm(G)$ of real $G$-modules $U$ and $V$ such that for the sphere $\Sigma$ appearing in the definition of Smith equivalence of $U$ and $V$ satisfies that $\Sigma^P$ is connected for every $P \in \mathcal{P}(G)$. For any $U - V \in CSm(G)$, $G$-modules $U$ and $V$ are $\mathcal{P}(G)$-matched pair, that is,

$$\text{Res}^G_p U \cong \text{Res}^G_p V$$

for any subgroup $P$ of $G$ of prime power order, possibly 1. Let $RO(G)$ be the real representation ring and we denote by $RO(G)^{\mathcal{P}(G)}$ the subset of $RO(G)$ consisting the differences of real $\mathcal{P}(G)$-matched pairs. Then $CSm(G)$ is a subset of $RO(G)^{\mathcal{P}(G)}$.

Proposition 1.1.

$$\begin{cases} 0 \in CSm(G) & \text{if } G \text{ is not of prime power order} \\ CSm(G) = \emptyset & \text{if } G \text{ is of prime power order.} \end{cases}$$

In this paper, we discuss the Smith problem for an Oliver nongap group. Throughout this paper we assume a group is finite.

2. $RO(G)^{\mathcal{P}(G)}$ and Induced Virtual Modules

We denote by $\pi(G)$ the set of all primes dividing the order $|G|$ of $G$. For a prime $p$, we denote by $O^p(G)$, called the Dress subgroup of type $p$, the smallest normal subgroup of $G$ with index a power of $p$:

$$O^p(G) = \bigcap_{L \triangleleft G, [G:L]=p^{\geq 1}} L.$$  

Note that $O^p(G) = G$ if $p \notin \pi(G)$. Let $\mathcal{L}(G)$ be the set of subgroups of $G$ containing some Dress subgroup.

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Let
\[ LO(G) := (RO(G)_{P(G)})^{L(G)} = \bigcap_{p \in \pi(G)} \ker(\text{fix}^{O''(G)}: RO(G) \to RO(G/O^{P}(G)) \cap RO(G)_{P(G)}) \]

A group $G$ is called Oliver if there is no series of subgroups
\[ P = H \triangleleft G \]
such that $P$ and $G/H$ are of prime power order and $H/P$ is cyclic. An Oliver group can be characterized as a group having a one fixed action on a sphere ([2]). A group $G$ is called gap if there is a real $G$-module $W$ such that $V^{O''(G)} = 0$ for any prime $p$ and
\[ \dim V^{P} > 2 \dim V^{H} \]
for all pairs $(P, H)$ of subgroups of $G$ which satisfy that $P$ is of prime power order and $P < H$. If $G$ is a gap Oliver group, then $LO(G)$ is a subset of $CSm(G)$ ([8]). We remark that $CSm(G)$ is not a subset of $LO(G)$ in general (cf. [3]).

For an element not of prime power order, we call it an NPP element. We denote by $a_G$ the number of real conjugacy classes of NPP elements of $G$.

**Proposition 2.1.** $RO(G)_{P(G)}$ is a free abelian subgroup of $RO(G)$ with rank $a_G$.

For a complex $G$-module $\xi$ we denote by $\bar{\xi}$ whose character is the complex conjugate of the character of $\xi$.

**Proposition 2.2.** Let $p_1, p_2, \ldots, p_k$ be distinct primes each other and let $a_1, a_2, \ldots, a_k$ be positive integers. Put $G = C_{p_1^{a_1}} \times C_{p_2^{a_2}} \cdots \times C_{p_k^{a_k}}$, where $C_{p_j^{a_j}}$ is a cyclic group of order $p_j^{a_j}$. Then $RO(G)_{P(G)}$ is spanned by the set of virtual real $G$-modules having characters as same as
\[ \bigotimes_{j}(\mathbb{C} - \xi_j) + \bigotimes_{j}(\mathbb{C} - \bar{\xi}_j), \]
where $\xi_j$'s are irreducible complex $C_{p_j^{a_j}}$-modules or zero and two of them are nonzero at least. In particular the rank of $RO(G)_{P(G)}$ is equal to $((\prod_j p_j^{a_j} - 1) - \sum_j (p_j^{a_j} - 1))/2$.

This proposition can be extend to nilpotent groups instead of cyclic groups.

**Theorem 2.3.** Let $p_1, p_2, \ldots, p_k$ be distinct primes each other and $P_j$ a nontrivial $p_j$-group for each $j$. Put $G = P_1 \times P_2 \times \cdots \times P_k$. Then the set of virtual real $G$-modules having characters as same as
\[ \bigotimes_{j}(\dim_{\mathbb{C}}(\xi_j)\mathbb{C} - \xi_j) + \bigotimes_{j}(\dim_{\mathbb{C}}(\xi_j)\mathbb{C} - \bar{\xi}_j), \]
where $\xi_j$'s are irreducible complex $P_j$-modules or zero and two of them are nonzero at least, become a basis of $RO(G)_{P(G)}$. In particular the rank of $RO(G)_{P(G)}$ is equal to $((\prod_j q_j - 1) - \sum_j (q_j - 1))/2$, where $q_j$ is the number of irreducible complex $P_j$-modules.

**Theorem 2.4.** Let $p_1, p_2, \ldots, p_k$ be distinct primes each other, $P$ a nontrivial $p_1$-group and $C_j$ a nontrivial cyclic $p_j$-group for each $j \geq 2$. Put $G = P \times C_2 \times \cdots \times C_k$ which is an elementary group. Then $RO(G)_{P(G)}$ is spanned by the set of virtual real $G$-modules $\text{Ind}_{\xi}^{\eta}$ for subgroups $E$ and for virtual real $E$-modules $\eta$ whose character is same as one of
\[ \bigotimes_{j}(\mathbb{C} - \xi_j) + \bigotimes_{j}(\mathbb{C} - \bar{\xi}_j), \]
where $\xi_j$'s are 1-dimensional complex $p_j$-modules or zero and two of them are nonzero at least.
We denote by \(\mathfrak{B}(G)\) the set of all virtual real \(G\)-modules as in Theorem 2.4 for an elementary group \(G\).

\(CSm(G)\) is a subset of
\[
RO(G)^{(G)}_{P(G)} = \ker(\text{fix}^G: RO(G) \to RO(G/G)) \cap RO(G)^{(G)}_{P(G)}.
\]

For a nilpotent group \(G\), by fixing \(X_0 \in \mathfrak{B}(G)\), the set consisting of \(X - X_0\) for \(X \in \mathfrak{B}(G), X \neq X_0\) spans \(RO(G)^{(G)}_{P(G)}\).

Artin’s induction theorem gives the following.

**Theorem 2.5.** The set
\[
\bigcup_C \{\text{Ind}_C^G \eta \mid \eta \in \mathfrak{B}(C)\}
\]
where \(C\) runs over all representative of conjugacy classes of cyclic subgroups of \(G\) not of prime power order spans the vector space \(\mathbb{Q} \otimes_{\mathbb{Z}} RO(G)^{(G)}_{P(G)}\) over the rational number field \(\mathbb{Q}\). The set of differences of virtual modules of the above set spans \(\mathbb{Q} \otimes_{\mathbb{Z}} RO(G)^{(G)}_{P(G)}\).

The following theorem is related to Brauer’s induction theorem.

**Theorem 2.6.** An virtual \(G\)-module \(RO(G)^{(G)}_{P(G)}\) is described as a linear combination (with integer coefficients) of virtual modules of
\[
\bigcup_E \{\text{Ind}_E^G \eta \mid \eta \in \mathfrak{B}(E)\}
\]
where \(E\) runs over all representatives of conjugacy classes of elementary subgroups \(E\) of \(G\). Furthermore, \(RO(G)^{(G)}_{P(G)}\) is described as a linear combination (with integer coefficients) of differences of the above virtual modules.

Let \(\overline{\text{NPP}}(G)\) be the set of all representatives of real conjugacy classes of \(\text{NPP}\) elements of \(G\). For a normal subgroup \(N\) of \(G\) and \(gN \in G/N\) we denote by \(a_{G,N}(gN)\) the number of elements of \(f_N^{-1}(gN)\), where \(f_N: \overline{\text{NPP}}(G) \to G/N\) is a mapping induced by a canonical epimorphism \(G \to G/N\). It holds that
\[
a_G = \sum_{gN \in G/N} a_{G,N}(gN).
\]

For a normal subgroup \(N\) of \(G\) let
\[
RO(G)^{(N)}_{P(G)} = \ker(\text{fix}^N: RO(G) \to RO(G/N)) \cap RO(G)^{(N)}_{P(G)}.
\]
We denote by \(G^{\text{nil}}\) the smallest normal subgroup of \(G\) by which a quotient group of \(G\) is nilpotent:
\[
G^{\text{nil}} = \bigcap_{p \in \pi(G)} O^p(G)
\]

**Proposition 2.7.** Let \(p\) be a prime and \(N\) a normal subgroup of \(G\). The rank of \(RO(G)^{(N)}_{P(G)}\) is less than or equal to
\[
\sum_{gN \in G/N} \max(a_{G,N}(gN) - 1, 0).
\]
The rank of \(LO(G)\) is greater than or equal to
\[
\sum_{gG^{\text{nil}} \in G/G^{\text{nil}}} \max(a_{G,G^{\text{nil}}}(gG^{\text{nil}}) - 1, 0)
\]
and in particular if \(G/G^{\text{nil}}\) is a \(p\)-group then the equality holds.
Theorem 2.8 ([4, Morimoto]). Let $G$ be a finite group. $Sm(G) \subset RO(G)^{|G^2|}$ where $G^2 = \cap_{[G:L] = 2}L$ is a normal subgroup of $G$.

Therefore, if $G/G^{nil}$ is an elementary abelian 2-group then $CSm(G) \subset LO(G)$.

Theorem 2.9. Let $N$ be a normal subgroup of $G$. Then $\mathbb{Q} \otimes \mathbb{Z} RO(G)^{|N|}$ is spanned by the set of virtual modules $X - Y$ such that

\[X, Y \in \bigcup_{E} [\text{Ind}_{E}^{G} \eta | \eta \in \mathfrak{B}(C)]\]

with $\text{fix}^{N}(X - Y) = 0$ in $RO(G/N)$, where $C$ runs over all representative of conjugacy classes of cyclic subgroups of $G$ not of prime power order.

Theorem 2.10. Let $N$ be a normal subgroup of $G$. An virtual $G$-module $RO(G)^{|N|}$ is described as a linear combination (with integer coefficients) of virtual modules $X - Y$ such that

\[X, Y \in \bigcup_{E} [\text{Ind}_{E}^{G} \eta | \eta \in \mathfrak{B}(E)]\]

with $\text{fix}^{N}(X - Y) = 0$ in $RO(G/N)$, where $E$ runs over all representatives of conjugacy classes of elementary subgroups $E$ of $G$.

3. Weak gap condition

We say that a smooth $G$-manifold $X$ satisfies the weak gap condition (WGC) if the conditions (WGC1)–(WGC4) all hold (cf. [5]).

(WGC1) $\dim X^p \geq 2 \dim X^H$ for every $P < H \leq G$, $P \in \mathcal{P}(G)$.

(WGC2) If $\dim X^p = 2 \dim X^H$ for some $P < H \leq G$, $P \in \mathcal{P}(G)$, then $[H : P] = 2$, $\dim X^H > \dim X^K + 1$ for every $H < K \leq G$, and $X^H$ is connected.

(WGC3) If $\dim X^p = 2 \dim X^H$ for some $P < H \leq G$, $P \in \mathcal{P}(G)$, and $[H : P] = 2$, then $X^H$ can be oriented in such a way that the map $g: X^H \to X^H$ is orientation preserving for any $g \in N_G(H)$.

(WGC4) If $\dim X^p = 2 \dim X^H$ and $\dim X^p = 2 \dim X^{H'}$ for some $P < H, P < H'$, $P \in \mathcal{P}(G)$, then the smallest subgroup $\langle H, H' \rangle$ of $G$ containing $H$ and $H'$ is not a large subgroup of $G$.

A real $G$-module $V$ is called $\mathcal{L}(G)$-free if $\dim V^H = 0$ for each $H \in \mathcal{L}(G)$, which amounts to saying that $\dim V^{\mathfrak{O}(G)} = 0$ for each prime $p \in \pi(G)$. For a finite group $G$, we define subgroups $WLO(G)$ of the free abelian group $LO(G)$ as follows.

\[WLO(G) = \{ U - V \in LO(G) | U \text{ and } V \text{ both satisfy the weak gap condition} \}\]

A real $G$-module $W$ is called nonnegative if (WGC1) holds for $X = W$.

We denote by $V(G)$ as

\[\mathbb{R}[G]_{\mathcal{L}(G)} = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p \in \pi(G)} (\mathbb{R}[G] - \mathbb{R})^{\mathfrak{O}(G)}\]

Theorem 3.2 in [2] implies the following proposition.

Proposition 3.1. Let $W$ be a real nonnegative $G$-module. For $X = W \oplus V(G)$, (WGC2) holds if $G$ is a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ and (WGC4) holds if $G$ is an Oliver group.

Theorem 3.2. For an Oliver group $G$, it holds that $WLO(G)$ is a subset of $CSm(G)$. 
More generally we obtain

Theorem 3.3. Let \( G \) be an Oliver group and let \( V_1, \ldots, V_k \) be real \( G \)-modules satisfying that \( V_i - V_j \in WLO(G) \). Then there exist a real \( G \)-module \( W \) and a smooth action on a sphere \( \Sigma \) such that \( \Sigma^G = \{x_1, \ldots, x_4\} \) and \( V_i \oplus W \) is isomorphic to the tangential \( G \)-module \( T_x(\Sigma) \) for any \( i \).

4. \( LO(G) \) vs \( WLO(G) \)

In this section we consider the difference between \( LO(G) \) and \( WLO(G) \). Note that if \( G/G^{\text{nil}} \) is an elementary abelian 2-group then \( WLO(G) \subset CSm(G) \subset LO(G) \).

We say that \( G \) is a gap group if \( G \) admits an \( \mathcal{L}(G) \)-free positive \( G \)-module \( V \), that is, \( \dim V^{O_p(G)} = 0 \) for any prime \( p \in \pi(G) \) and \( \dim V^p > 2 \dim V^H \) for any pair \((P, H)\) of subgroups of \( G \) with \( P \in \mathcal{P}(G), P < H \).

Theorem 4.1. Let \( G \) be a group with \( \mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset \). Suppose that for each \( X \in LO(G) \) there are \( \mathcal{L}(G) \)-free nonnegative \( G \)-modules \( U \) and \( V \) such that \( X = U - V \). For each subgroup \( K \) of \( G \) with \( K > O^2(G) \), \( [K : O^2(G)] = 2 \), if all elements \( x \) of \( K \setminus O^2(G) \) of order 2 such that \( C_K(x) \) is not a 2-group are not conjugate in \( K \), then \( K \) is a gap group.

Theorem 4.2. Let \( G \) be an Oliver group. Let \( U \) and \( V \) be \( \mathcal{L}(G) \)-free nonnegative \( G \)-modules with \( U - V \in RO(G)_{P(G)} \). There are \( \mathcal{L}(G) \)-free \( G \)-modules \( X \) and \( Y \) such that they satisfy the weak gap condition and \( U - V = X - Y \).

Thus we have immediately the following theorem.

Theorem 4.3. Let \( G \) be an Oliver group. Suppose that for each subgroup \( K \) of \( G \) with \( K > O^2(G) \), \( [K : O^2(G)] = 2 \), if \( K \) is not a gap group then all elements \( x \) of \( K \setminus O^2(G) \) of order 2 such that \( C_K(x) \) is not a 2-group are conjugate in \( K \). Then \( LO(G) \subset CSm(G) \). Furthermore, if \( G/G^{\text{nil}} \) is an elementary abelian 2-group then \( LO(G) = CSm(G) \).

If \( K \) is an Oliver group with \(|K| \leq 2000 \) and \([K : O^2(K)] = 2 \), then \( K \) is a gap group or all elements \( x \) of \( K \setminus O^2(K) \) of order 2 such that \( C_K(x) \) is not a 2-group are conjugate in \( K \). We have still no example of a group \( G \) so that \( WLO(G) \neq LO(G) \).

Let \( H = D_{2^{p_1}} \times D_{2^{p_2}} \times \cdots \times D_{2^{p_r}} \) be a direct product group of dihedral groups \( D_{2^{p_i}} \), where \( p_1, \ldots, p_r \geq 1 \) are odd integers. Then \( G \times H \) is a nongap group if \( G \) is a nongap group.

Theorem 4.4. Let \( G \) be an Oliver group as in Theorem 4.3 and let \( H \) be as above. It holds that \( LO(G \times H) \) is a subset of \( CSm(G \times H) \). Furthermore if \( G/G^{\text{nil}} \) is an elementary abelian 2-group, then \( CSm(G \times H) = LO(G \times H) \).

5. Projective General Linear Groups

We note that \( PGL(2, q) \) is isomorphic to the dihedral group \( D_8 \) for \( q = 2 \), the symmetric group \( S_4 \) for \( q = 3 \), the alternating group \( A_5 \) for \( q = 4 \), the symmetric group \( S_5 \) for \( q = 5 \), and nonsolvable for \( q \geq 4 \). The group \( PGL(2, q) \) is isomorphic to \( PSL(2, q) \) if \( q \) is a power of 2. If \( q \geq 3 \) is odd, \( PGL(2, q) \) has a perfect subgroup \( PSL(2, q) \) with index 2, which implies \([PGL(2, q) : O^2(PGL(2, q))] = 2 \).

It is easy to see the rank of \( LO(PGL(2, q)) \). Note that rank \( LO(G) = \max(a_G - 1, 0) \) if \( G \) is a perfect group.
Proposition 5.1. Suppose that $q$ is odd.

$$\text{rank } LO(PGL(2, q)) = \begin{cases} 0 & q = 3, 5, 7 \\ a_{PGL(2,q)} - 1 & q = 9, 17 \\ a_{PGL(2,q)} - 2 & \text{otherwise} \end{cases}$$

Remark 5.2. Suppose that $q$ is an odd prime power integer.

1. $PGL(2, q)$ is not a gap group if and only if $q = 3, 5, 7, 9, 17$.
2. $PGL(2, q)$ is an Oliver group if and only if $q \geq 5$.
3. $PGL(2, q)$ is an Oliver group if and only if $q \geq 5$.
4. $PGL(2, q)$ is an Oliver group if and only if $q \geq 5$.

Theorem 4.3 gives $CSm(PGL(2, q)) = LO(PGL(2, q))$. Furthermore, we obtain the following.

Theorem 5.3. $Sm(PGL(2, q)) = LO(PGL(2, q))$.

6. SMALL GROUPS

In this section we discuss by viewing from the order of a Sylow 2-subgroup of an Oliver group. If $G$ is an Oliver group of odd order then $G$ is a gap group and $LO(G)$ is a subset of $CSm(G)$.

Example 6.2. Let $K$ be a finite abelian group of odd order whose rank is greater than 2. Let $h$ be an automorphism on $K$ which sends $k \in K$ to its inverse $k^{-1}$. Put $G = \langle h, K \rangle$. Then $G$ is an Oliver nongap group satisfying $CSm(G) = LO(G)$.

Theorem 6.3. Let $N$ be a normal subgroup of $G$. Suppose that $a_G \leq a_{G,N}(N) + 1$. The induction mapping $Ind_N^G: LO(N) \otimes \mathbb{Q} \rightarrow LO(G) \otimes \mathbb{Q}$ is surjective.

From now on, we suppose that $G$ is a finite Oliver group, $[G : G^{nil}] = 2$ and $a_G \geq 2$. Note that $a_{G,G^{nil}}(G \setminus G^{nil}) = a_G - a_{G,G^{nil}}(G^{nil})$. The above theorem yields the following.

Theorem 6.4. If $a_G \leq a_{G,G^{nil}}(G^{nil}) + 1$ then $LO(G) = WLO(G) = CSm(G)$.

Let $\mathcal{F}$ be the set of isomorphism classes of finite Oliver nongap groups $K$ such that $4 \mid |K|$, $[K : K^{nil}] = 2$, and $a_K,K^{nil}(K \setminus K^{nil}) \geq 2$. Note that $|G|$ is divisible by 8 if $|G|$ is divisible by 4 and less than or equal to 2000. The set of all representatives of elements in $\mathcal{F}$ consists of 5 groups $G_{648}, PGL(2, 9), G_{1296}, G_{1944a}, G_{1944b}$.

Here they are given as follows.

$$
\begin{array}{c}
G_{108} \leftarrow^* C_3^3 \quad G_{648} \leftarrow^* G_{324} \leftarrow^* G_{108} \\
\downarrow \quad \downarrow \quad \downarrow \\
C_2^2 \quad C_2 \quad C_3
\end{array}
$$
\( G_{648} \) gives the isomorphism class of the smallest group in \( \mathcal{F} \). \( G_{1296} \) has center \( C_2 \) and the quotient group by it’s center is isomorphic to \( G_{704} \). For these groups \( G \), it holds that \( CSm(G) = Sm(G) \). \( a_G = 4, 2, 10, 6, 6 \) and \( a_{G,G^\text{nil}}(G \setminus G^\text{nil}) = 3, 2, 4, 3, 3 \) respectively. There are only five groups up to order 2000. However we have the following.

**Proposition 6.5.** There are infinitely many finite groups \( G \) such that \( [G : G^\text{nil}] = 2 \) and \( a_{G,G^\text{nil}}(G \setminus G^\text{nil}) \geq 2 \).

**Problem 6.6.** Is there a finite nongap group \( G \) and involutions \( x \) and \( y \) of \( G \setminus O^2(G) \) such that \( [G : G^\text{nil}] = 2 \), \( x \) and \( y \) are not conjugate in \( G \), and \( C_G(x) \) and \( C_G(y) \) are both not 2-groups.

There is no such a group if the order is less than or equal to 2000.

**Proposition 6.7.** Suppose that there is a finite nongap group satisfying the property in the above problem. Then there are infinitely many finite nongap groups satisfying the same property.

### 7. Direct Product Gap Groups

In this section, we consider about when a direct product group is a gap group. First we remark that

**Proposition 7.1** ([6, 12]). Let \( K \) be a finite group with \( \mathcal{P}(K) \cap \mathcal{L}(K) = \varnothing \) and \( H \) be a 2-group. \( K \times H \) is a gap group if and only if so is \( K \).

We call a finite group \( G \) is a generalized dihedral group if \( [G : O^2(G)] = 2 \) and there is an involution \( h \in G \setminus O^2(G) \) such that \( hgh = g^{-1} \) for any \( g \in O^2(G) \). A generalized dihedral group is a subgroup of certain direct product group of dihedral groups.

**Proposition 7.2** ([13, Lemma 7.2]). Suppose \( [K : K^\text{nil}] = 2 \) and \( \mathcal{P}(K) \cap \mathcal{L}(K) = \varnothing \). For an odd prime \( p \) and a nontrivial \( p \)-group \( H \), \( K \times H \) is a gap group if and only if \( K \) is not a generalized dihedral group.

Moreover we have the following.

**Proposition 7.3.** Suppose that \( [K : K^\text{nil}] = 2 \) and \( \mathcal{P}(K) \cap \mathcal{L}(K) = \varnothing \). If \( |\pi(H/[H,H])| \geq 2 \), or \( |\pi(H/[H,H])| = 1 \) and \( K \) is not a generalized dihedral group then \( K \times H \) is a gap group, where \( [H,H] \) is a commutator subgroup of \( H \).
If $K$ or $H$ is a gap then so is $K \times H$. We put

$$\kappa(K) = \bigcup_{x \in K \setminus O^2(K)} \pi((x)).$$

$\kappa(K)$ is a subset of $\pi(K)$ and if $K \neq O^2(K)$ then it contains 2.

**Theorem 7.4.** Suppose that $K$ and $H$ are nongap with $[K : K^{\text{nil}}] = [H : H^{\text{nil}}] = 2$. Let $L$ be a unique subgroup of $K \times H$ with index 2 which is neither $K$ nor $H$. Further suppose that $\mathcal{P}(L) \cap \mathcal{L}(L) = \emptyset$. The following claims are equivalent.

1. $L$ is a gap group.
2. (i) $a_{K,O^2(K)}(K \setminus O^2(K)) \geq 1$ and there is a 2-element $x$ of $H \setminus O^2(H)$ with $|x| \geq 4$, or
   (ii) $a_{H,O^2(H)}(H \setminus O^2(H)) \geq 1$ and there is a 2-element $y$ of $K \setminus O^2(K)$ with $|y| \geq 4$, or
   (iii) $a_{K,O^2(K)}(K \setminus O^2(K)) \geq 1$, $a_{H,O^2(H)}(H \setminus O^2(H)) \geq 1$ and $|\kappa(K) \cup \kappa(H)| \geq 3$.

**Corollary 7.5.** Let $K$, $H$, and $L$ be groups as in Theorem 7.4. If

1. $a_{K,O^2(K)}(K \setminus O^2(K)) = a_{H,O^2(H)}(H \setminus O^2(H)) = 0$, or
2. $a_{K,O^2(K)}(K \setminus O^2(K)) \geq 1$ and $H$ is not a generalized dihedral group, or
3. $a_{H,O^2(H)}(H \setminus O^2(H)) \geq 1$ and $K$ is not a generalized dihedral group,

then $K \times H$ is a nongap group. Furthermore, the converse is also true if $\mathcal{P}(O^2(K)) \cap \mathcal{L}(O^2(K)) = \emptyset$ and $\mathcal{P}(O^2(H)) \cap \mathcal{L}(O^2(H)) = \emptyset$.

**References**

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