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STUDY OF THE SMITH SETS OF GAP OLIVER GROUPS

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Abstract. For various finite groups G, Smith equivalent pairs of real G-modules have been studied since 1960's. The Smith set of G is defined to be the subset of the real representation ring $\mathrm{RO}(G)$ consisting of all differences $[V]-[W]$ of Smith equivalent real G-modules V and W. In the present paper, we discuss the Smith sets of gap Oliver groups with small nilquotient.

1. Introduction

Let G be a finite group. In this paper, a manifold means a smooth manifold, a G-action on a manifold does a smooth G-action, a real G-module does a real G-representation space of finite dimension.

Given a family $\mathfrak{X}$ of G-actions on manifolds, two real G-modules V and W are called $\mathfrak{X}$-*related* and written with $V \sim_{\mathfrak{X}} W$ if there exists $X \in \mathfrak{X}$ such that $V \cong T_a(X)$ and $W \cong T_b(X)$ for some $a, b \in X$, where $T_a(X)$ and $T_b(X)$ are tangential G-representations at a and b, respectively. We call such a G-action X an $\mathfrak{X}$-*realization* of V and W. In order to study $\mathfrak{X}$-relation, we use the real representation ring $\mathrm{RO}(G)$ and the subset

$$\mathrm{RO}(G,\mathfrak{X}) = \{[V]-[W] \in \mathrm{RO}(G) \mid V \sim_{\mathfrak{X}} W\}.$$

As our convention, we regard that $\mathrm{RO}(G,\mathfrak{X}) = 0$ if $\mathfrak{X}$ is empty.

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In the present paper, we will deal with the following three families:

$\mathfrak{S}$ = the family of G-actions on standard spheres S such that $|S^G| = 2$,

$\mathfrak{S}_{\mathrm{ht}}$ = the family of G-actions on homotopy spheres Σ such that $|\Sigma^G| = 2$,

$\mathfrak{D}$ = the family of G-actions on disks D such that $|D^G| = 2$.

If two real G-modules V and W are $\mathfrak{S}_{\mathrm{ht}}$-related then we say that V and W are *Smith equivalent*. In this paper we discuss $\mathrm{RO}(G, \mathfrak{S})$, $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})$, $\mathrm{RO}(G, \mathfrak{D})$, and the set

$$\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) = \{[V] - [W] \in \mathrm{RO}(G) \mid V \sim_{\mathfrak{S}} W \ \text{ and } \ V \sim_{\mathfrak{D}} W\}.$$

In other papers, the set $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})$ has been called the *Smith set* of G, denoted by $Sm(G)$, and studied as the Smith Problem.

Smith Problem. Are two real G-modules V and W isomorphic to each other if they are Smith equivalent; namely $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}}) = 0$?

C. Sanchez [24] showed $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}}) = 0$ if the order of G is an odd-prime power; on the other hand, T. Petrie and S. Cappell-J. Shaneson showed $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}}) \neq 0$ if G is isomorpic to $C_n \times C_n$ with $n = p_1p_2p_3p_4$ or C_{4m} with $m \geq 2$, where C_n denotes the cyclic group of order n, and p_1, p_2, p_3, p_4 are distinct odd primes. One may immediately ask the next problem.

Problem. Do the following equalities hold:

(1) $\mathrm{RO}(G, \mathfrak{S}) = \mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})$?

(2) $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) = \mathrm{RO}(G, \mathfrak{D}) \cap \mathrm{RO}(G, \mathfrak{S})$?

There are no known finite groups G for which the equalities above fail. Let us abuse the term 'Smith set' not only for $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})$ but also for $\mathrm{RO}(G, \mathfrak{S})$. In the case where distinctive use of the term is necessary, we explain what the term actually means there.

In order to discuss further results, we use the following notation. For sets $\mathcal{F}$, $\mathcal{G}$ consisting of subgroups of G and for a subset $\mathcal{A}$ of $\mathrm{RO}(G)$, we define

$$\mathcal{A}^{\mathcal{F}} = \{[V] - [W] \in \mathcal{A} \mid V^H = 0 = W^H \quad (\forall H \in \mathcal{F})\},$$

$$\mathcal{A}_{\mathcal{G}} = \{[V] - [W] \in \mathcal{A} \mid \mathrm{res}^G_H V \cong \mathrm{res}^G_H W \quad (\forall H \in \mathcal{G})\},$$

$$\mathcal{A}^{\mathcal{F}}_{\mathcal{G}} = (\mathcal{A}^{\mathcal{F}})_{\mathcal{G}}.$$

Let us use the following notation.

$\mathcal{S}(G)$: the set of all subgroups of G,

$\mathcal{P}(G) = \{P \in \mathcal{S}(G) \mid |P| \text{ is a prime power}\}$,

$\mathcal{P}(G)_{\mathrm{odd}} = \{P \in \mathcal{S}(G) \mid |P| \text{ is an odd-prime power}\}$.

The next implication follows from Sanchez [24]:

$$\mathrm{RO}(G,\mathfrak{S}) \subset \mathrm{RO}(G,\mathfrak{S}_{\mathrm{ht}}) \subset \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)_{\mathrm{odd}}}. \tag{1.1}$$

In addition, if G does not contain elements of order 8 then $\mathrm{RO}(G,\mathfrak{S}_{\mathrm{ht}}) \subset \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)}$. We have the following improvements of (1.1). Set

$$\mathcal{N}_p(G) = \{N \trianglelefteq G \mid |G:N| = 1,\ p\},$$

for a prime p.

Theorem 1 ([12]). *The implication* $\mathrm{RO}(G,\mathfrak{S}_{\mathrm{ht}}) \subset \mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)_{\mathrm{odd}}}$ *holds for an arbitrary finite group* G.

Theorem 2 ([7]). *If a Sylow* 2*-subgroup of* G *is a normal subgroup of* G *then the implication* $\mathrm{RO}(G,\mathfrak{S}_{\mathrm{ht}}) \subset \mathrm{RO}(G)^{\mathcal{N}_2(G)\cup\mathcal{N}_3(G)}_{\mathcal{P}(G)_{\mathrm{odd}}}$ *holds.*

If G admits a G-action on a disk D with $|D^G| = 2$ then G is called an *Oliver group.* B. Oliver [18] proved

$$\mathrm{RO}(G,\mathfrak{D}) = \begin{cases} \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} & \text{(if } G \text{ is an Oliver group),} \\ 0 & \text{(otherwise)} \end{cases} \tag{1.2}$$

A finite group G is an Oliver group if and only if there never exists a normal series $P \trianglelefteq H \trianglelefteq G$ such that $|P|$ and $|G/H|$ both are prime powers and H/P is a cyclic group. Clearly, we obtain the equality

$$\mathrm{RO}(G,\mathfrak{D}) \cap \mathrm{RO}(G,\mathfrak{S}) = \mathrm{RO}(G,\mathfrak{S})_{\mathcal{P}(G)}$$

for an arbitrary Oliver group G.

Proposition 3. *For an arbitrary finite group* G, $\mathrm{RO}(G,\mathfrak{S}) \smallsetminus \mathrm{RO}(G,\mathfrak{S})_{\mathcal{P}(G)}$ *is a finite (possibly empty) set. If* G *does not contain elements of order* 8 *then* $\mathrm{RO}(G,\mathfrak{S})$ *coincides with* $\mathrm{RO}(G,\mathfrak{S})_{\mathcal{P}(G)}$.

We call $\mathrm{RO}(G,\mathfrak{S})_{\mathcal{P}(G)}$ the *primary Smith set* of G. If G is a nontrivial perfect group then by [9], the primary Smith set $\mathrm{RO}(G,\mathfrak{S})_{\mathcal{P}(G)}$ coincides with $\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)}$.

For a prime p, let $G^{\{p\}}$ (resp. G^{nil}) denote the smallest normal subgroup H such that $|G/H|$ is a power of p (resp. G/H is nilpotent). This subgroup $G^{\{p\}}$ is called the *Dress subgroup of type p of G*. It is useful to keep the next equality in mind:

$$G^{\mathrm{nil}} = \bigcap_p G^{\{p\}},$$

where p ranges over the set of all primes dividing $|G|$. The family

$$\mathcal{L}(G) = \{H \in \mathcal{S}(G) \mid H \supset G^{\{p\}} \text{ for some prime } p\}$$

plays a key role to delete or insert components of G-fixed point sets of closed G-manifolds. A finite group G is called a *gap group* if there exists a real G-module V satisfying the condition

$$\begin{cases} V^H = 0 \text{ for any } H \in \mathcal{L}(G), \\ \dim V^P > 2\dim V^H \text{ for all } P \in \mathcal{P}(G) \text{ and } H \in \mathcal{S}(G) \text{ with } H \supsetneqq P. \end{cases}$$

K. Pawałowski-R. Solomon [19] showed the implication $\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subset \mathrm{RO}(G, \mathfrak{S})$ for an arbitrary gap Oliver group G. A little further work provides the next theorem.

Theorem 4 ([14]). *If G is a gap Oliver group then the implication*

$$\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subset \mathrm{RO}(G, \mathfrak{D}\mathfrak{S})$$

holds.

Thus one may ask the problem.

Problem ([14]). Does the implication $\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subset \mathrm{RO}(G, \mathfrak{D}\mathfrak{S})$ hold for an arbitrary Oliver group G?

T. Sumi gives results related to this problem in the present issue of Kokyuroku.

Putting implications mentioned above for an Oliver group G together, we obtain the diagram:

$$\begin{tikzcd}
\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) \arrow[r, hook] & \mathrm{RO}(G, \mathfrak{S})_{\mathcal{P}(G)} \arrow[rr, hook, dotted] \arrow[d, hook, dotted, "G_2 \lhd G"] \arrow[dr] & & \mathrm{RO}(G, \mathfrak{D}) \arrow[d, equal, "\text{by [18]}"] \\
\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \arrow[u, hook, dotted, "\text{gap}"] \arrow[r, hook] & \mathrm{RO}(G)^{\mathcal{N}_2(G) \cup \mathcal{N}_3(G)}_{\mathcal{P}(G)} \arrow[r, hook] & \mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)} \arrow[r, hook] & \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)}
\end{tikzcd}$$

We have seen in [7] that if $G = SG(1176, 220)$, $SG(1176, 221)$ then

$$\mathbb{Z} \cong \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)} \neq \mathrm{RO}(G, \mathfrak{S})_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{S}) = 0.$$

Let C_n and D_{2n} denote the cyclic group of order n and the dihedral group of order $2n$, respectively. We say that G is of *type (S)* if G/G^{nil} is isomorphic to one of the following.

(1) P: $|P|$ is a power of a prime.

(2) $C_2 \times P$: $|P|$ is a power of an odd prime.

(3) $P \times C_3$: $|P|$ is a power of 2, and any element g of P is conjugate to g^{-1} in P.

According to T. Sumi [25], if G is an Oliver group satisfying $\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} \neq 0$ and $\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = 0$ then G is of type (S). Thus we are interested in the Smith sets for finite Oliver groups G of type (S).

Theorem 5. *Let G be an Oliver group. If G/G^{nil} has order 3 and G^{nil} has a subquotient group isomorphic to D_{2q} for an odd prime q then the equalities*

$$\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) = \mathrm{RO}(G, \mathfrak{S})_{\mathcal{P}(G)}$$

hold.

For integers p, $q \geq 3$, let ${D_{2q}}^p$ denote the p-fold catesian product of D_{2q}, and let

$$D(p, 2q) = {D_{2q}}^p \rtimes C_p$$

be the semidirect product, where C_p acts on ${D_{2q}}^p$ by permuting the components.

Theorem 6. *Let G be an Oliver group. If G/G^{nil} is a cyclic group of order 6 and G contains a normal subgroup $N \subset G^{\mathrm{nil}}$ such that $G/N \cong D(3, 2q)$, then the equalities*

$$\mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) = \mathrm{RO}(G, \mathfrak{S})_{\mathcal{P}(G)}$$

and

$$\mathrm{rank}\ \mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)} = \mathrm{rank}\ \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} + 1$$

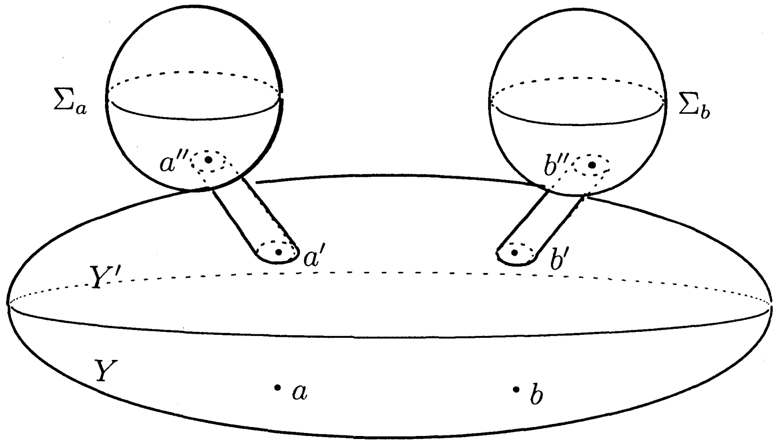
hold. In particular, the set $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) \setminus \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$ is not empty.

2. Construction of G-actions on spheres

In this section, let G be an Oliver group. Let V and W be real G-modules. Suppose there exists a G-action on a disk Y such that $Y^G = \{a, b\}$, $T_a(Y) \cong V$ and $T_b(Y) \cong W$. Then the double $D(Y) = Y \bigcup_\partial Y'$ is a sphere with $D(Y)^G = \{a, b, a', b'\}$, where Y' is a copy of Y. If there exist G-actions on spheres Σ_a and Σ_b such that $\Sigma_a^G = \{a''\}$,

$\Sigma_b^G = \{b''\}$, $T_{a''}(\Sigma_a) \cong T_{a'}(D(Y))$ and $T_{b''}(\Sigma_b) \cong T_{b'}(D(Y))$ then take the G-connected sum

$$\Sigma := D(Y) \underset{(a',a'')}{\#} \Sigma_a \underset{(b',b'')}{\#} \Sigma_b.$$



Picture of Σ

Clearly we have $\Sigma^G = \{a, b\}$ and we can conclude $V \sim_{\mathfrak{S}} W$. Thus it is useful for the study of $\mathrm{RO}(G, \mathfrak{S})$ to construct various two-fixed-point actions on disks and one-fixed-point actions on spheres.

Let us recall Oliver's construction of G-actions on disks with prescribed fixed point sets. We begin with describing necessary conditions. Now suppose a disk D with G-action has the G-fixed point set M. Since $\mathrm{res}^G_{\{e\}} T(D)$ is a product bundle, so is its restriction $\mathrm{res}^G_{\{e\}} T(D)|_M$. By the Smith theory, for each Sylow p-subgroup P of G, where p is a prime, $\mathrm{res}^G_{\{P\}} T(D)|_{D^P}{}^{\oplus m}$ and hence $\mathrm{res}^G_{\{P\}} T(D)|_M{}^{\oplus m}$ are equivariantly product bundles for some positive integer m prime to p. Thus there exists a G-vector bundle η over M satisfying

$$(2.1) \qquad \begin{cases} \eta^G = T(M) \oplus \varepsilon_M(\mathbb{R}^k) \text{ for some integer } k \geq 0, \\ [\mathrm{res}^G_{\{e\}} \eta] = 0 \text{ in } \widetilde{KO}(M), \\ [\mathrm{res}^G_P \eta] = 0 \text{ in } \widetilde{KO}_P(M)_{(p)} \text{ for all } P \in \mathcal{P}(G) \text{ and primes } p \| P|. \end{cases}$$

The converse of this is also true.

Theorem (B. Oliver). *Let G be an Oliver group, M a compact manifold (with trivial G-action) and η a real G-vector bundle over M. If M and η satisfy Condition* (2.1) *and m is a sufficiently large integer then there exists a G-action on a disk D satisfying*

$$D^G = M \quad \text{and} \quad T(D)|_M \oplus \varepsilon_M(\mathbb{R}^k) \cong \eta \oplus \varepsilon_M(\mathbb{R}[G]_G{}^{\oplus m}),$$

where here $\mathbb{R}[G]_G = \mathbb{R}[G] - \mathbb{R}$.

Applying this theorem to $M = \{a, b\}$ and $\eta = V \amalg W$, readers can easily verify the equality $\mathrm{RO}(G, \mathfrak{D}) = \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)}$.

To study the set $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S})$, since $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) \subset \mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)}$, we need modification of Oliver's method, which is studied in [15] and [16].

Theorem 7 ([14]). *Let G be an Oliver group, M a compact G-manifold and η a real G-vector bundle over M. If M and η satisfy the condition*

$$\begin{cases} \eta \supset T(M) \oplus \varepsilon_M(\mathbb{R}^k) \text{ for some integer } k \geq 0, \\ \eta^H = T(M)^H \oplus \varepsilon_{M^H}(\mathbb{R}^k) \text{ for all } H \in \mathcal{L}(G), \\ [\mathrm{res}^G_{\{e\}}\eta] = 0 \text{ in } \widetilde{KO}(\mathrm{res}^G_{\{e\}}M), [\mathrm{res}^G_P\eta] = 0 \text{ in } \widetilde{KO}(\mathrm{res}^G_P M)_{(p)} \\ \text{for all } P \in \mathcal{P}(G) \text{ and } p \,|\, |P|, \end{cases}$$

and m is a sufficiently large integer then there exists a G-action on a disk D satisfying

$$D^G = M^G \quad \text{and} \quad T(D)|_{D^G} \oplus \varepsilon_{D^G}(\mathbb{R}^k) = \eta|_{D^G} \oplus \varepsilon_{D^G}(\mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus m}),$$

where

$$\mathbb{R}[G]_{\mathcal{L}(G)} = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_p (\mathbb{R}[G/G^{\{p\}}] - \mathbb{R}).$$

When G is a gap Oliver group, the theorem above is used to construct smooth actions on disks together with the next.

Theorem 8 (Under Gap Condition, [13]). *Let G be an Oliver group and D a disk with G-action. If D satisfies the conditions*

(1) $D^G \cap \partial D = \emptyset$,
(2) $C \cap \partial D = \emptyset$ *for every connected component C of D^H, where $H \in \mathcal{L}(G)$, such that $C^G \neq \emptyset$,*
(3) $\dim D^P > 2(\dim D^H + 1)$ *for all $P \in \mathcal{P}(G)$ and $H \in \mathcal{S}(G)$ with $P \subsetneq H$,*
(4) $\pi_1(D^P)$ *is a finite group and $(|\pi_1(D^P)|, |P|) = 1$ for all $P \in \mathcal{P}(G)$,*
(5) $\dim D^{=H} \geq 3$ *for all $H \in \mathcal{S}(G)$ having $P \in \mathcal{P}(H)$ such that $P \trianglelefteq H$ and H/P is cyclic, and*
(6) $\dim D^P \geq 5$ *for all $P \in \mathcal{P}(G)$,*

then there exists a G-action on a standard sphere S satisfying

$$S^G = D^G \quad \text{and} \quad T(S)|_{S^G} = T(D)|_{D^G}.$$

In the above, $D^{=H}$ stands for the set consisting of all points in D with isotropy subgroup H.

Let V and W be real G-modules. For a prime p, we say that V and W are *p-matched* if $\mathrm{res}^G_P V \cong \mathrm{res}^G_P W$ for all $P \in \mathcal{P}(G)$ such that $|P|$ is 1 or divisible by p. Moreover, we say that V and W are *$\mathcal{P}$-matched* if V and W are p-matched for all primes p.

Corollary 9. *Let G be a gap Olievr group, and V and W real G-modules. If V and W are $\mathcal{P}$-matched and $\mathcal{L}(G)$-free, namely $V^H = 0 = W^H$ for all $H \in \mathcal{L}(G)$, U is a gap G-module, and m is a sufficiently large integer (with respect to $|G|$, V, W and U), then there exists a G-action on a standard sphere S satisfying*

$$\begin{cases} S^G = \{a, b\} \quad (a \neq b), \\ T_a(S) = V \oplus U^{\oplus \ell} \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus m}, \\ T_b(S) = W \oplus U^{\oplus \ell} \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus m}, \end{cases}$$

where $\ell = \dim V + 1$.

For a nongap group G, we can use [17, Theorem 36]. We have the next improvement due to the equivariant surgery theory of A. Bak–M. Morimoto [1] and the induction theory similar to [11].

Theorem 10 (Under Weak Gap Condition). *Let G be an Oliver group and D a disk with G-action. If D satisfies the conditions*

(1) $D^G \cap \partial D = \emptyset$,

(2) *$C \cap \partial D = \emptyset$ for every connected component C of D^H, where $H \in \mathcal{L}(G)$, such that $C^G \neq \emptyset$,*

(3) *$\dim D^P \geq 2 \dim D^H$ for all $P \in \mathcal{P}(G)$ and $H \in \mathcal{S}(G)$ with $P \subsetneq H$,*

(4) *$\pi_1(D^P)$ is simply connected for all $P \in \mathcal{P}(G)$,*

(5) *for some $P \in \mathcal{P}(G)$ and $H \in \mathcal{S}(G)$ with $P \subsetneq H$, if $\dim D^P = 2 \dim D^H$ then $|H : P| = 2$ and $D^{=H}$ is connected,*

(6) *for some $P \in \mathcal{P}(G)$ and H, $K \in \mathcal{S}(G)$ with $P \subsetneq H$ and $P \subsetneq K$, if $\dim D^P = 2 \dim D^H = 2 \dim D^K$ then the smallest subgroup of G containing $H \cup K$ does not contain any Dress subgroups $G^{\{q\}}$,*

(7) *$\dim D^{=H} \geq 3$ for all $H \in \mathcal{S}(G)$ having $P \in \mathcal{P}(H)$ such that $P \trianglelefteq H$ and H/P is cyclic, and*

(8) $\dim D^P \geq 5$ *for all* $P \in \mathcal{P}(G)$,

then there exists a G*-action on a standard sphere* S *such that*

$$S^G = D^G \quad \textit{and} \quad T(S)|_{S^G} = T(D)|_{D^G}.$$

We remark that Hypotheses (5)–(8) above can be removed if we use $D \times D(\mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus 3})$ instead of D (cf. [8], [10, Theorem 2.5]).

Theorem 11. *Let* G *be an Oliver group, and* V *and* W *real* G*-modules. If* V *and* W *are* $\mathcal{P}$*-matched,* $\mathcal{L}(G)$*-free and satisfy*

$$\dim V^P \geq 2 \dim V^H \quad \textit{and} \quad \dim W^P \geq 2 \dim W^H$$

for all $P \in \mathcal{P}(G)$ *and* $H \in \mathcal{S}(G)$ *with* $|H : P| = 2$*, and* m *is a sufficiently large integer (with respect to* $|G|$, V, W*), then there exists a* G*-action on a standard sphere* S *satisfying*

$$\begin{cases} S^G = \{a, b\} \quad (a \neq b), \\ T_a(S) = V \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus m}, \\ T_b(S) = W \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus m}. \end{cases}$$

3. Applications of $\mathcal{P}$-matched pairs of type 1

Let G be an Oliver group. This section is devoted to explaining how to construct one-fixed point G-actions on standard spheres S from given $\mathcal{P}$-matched pair (V, W) satisfying certain conditions.

A $\mathcal{P}$-matched pair (V, W) of real G-modules is called of *type 1* by B. Oliver if it satisfies

$$\dim V^G = 1 \quad \text{and} \quad \dim W^G = 0. \tag{3.1}$$

Lemma 12 (B. Oliver [18])**.** *Let* G *be a finite group not of prime-power order. There exists a* $\mathcal{P}$*-matched pair* (V, W) *of real* G*-modules of type 1 if and only if* G *has a subquotient group isomorphic to* D_{2pq}*, where* p *and* q *are distinct primes.*

Let us recall Oliver's construction of G-actions on disks with prescribed fixed point manifolds. Let (V, W) be a $\mathcal{P}$-matched pair of real G-modules of tyle 1 and M a compact manifold. Here we regard M as a G-manifold with trivial action. Let τ be a subbundle of $\varepsilon_M(\mathbb{R}^n)$, where n is a positive integer, and let ν be the complementary bundle of τ in $\varepsilon_M(\mathbb{R}^n)$, namely $\tau \oplus \nu = \varepsilon_M(\mathbb{R}^n)$. Consider the G-vector bundle

$$\eta = (\tau \otimes V) \oplus (\nu \otimes W).$$

Then η satisfies Condition 2.1. Applying Theorem 2 to these M and η, we obtain a G-action on a disk D with $D^G = M$. In order to use Theorem 8, we have to control the connected components of D^H containing G-fixed points for $H \in \mathcal{L}(G)$. For this purpose, we need to modify Lemma 12.

We call a $\mathcal{P}$-matched pair (V, W) of real G-module of *type (L1)* if it satisfies either

$$\begin{cases} V^G = V^{G\{2\}} \cong \mathbb{R}, \\ W^{G\{p\}} = 0 \ \text{ for all primes } p, \end{cases} \tag{3.2}$$

or

$$\begin{cases} V^G = V^N \cong \mathbb{R} \ \text{ for all } N \in \mathcal{N}_2(G), \\ W^{G^{\mathrm{nil}}} = 0. \end{cases} \tag{3.3}$$

Let (V, W) be a $\mathcal{P}$-matched pair of real G-modules satisfying Condition 3.3. Let $M = P(V^{G^{\mathrm{nil}}})$ denote the real projective space associated with $V^{G^{\mathrm{nil}}}$, let γ_M be the canonical line bundle over M, and let $\gamma_M^{\perp}$ be the complementary bundle of γ_M in $\varepsilon_M(V^{G^{\mathrm{nil}}})$. Then M has a unique fixed point, so say x_0, and the real G-vector bundle $T(M) \oplus \varepsilon_M(\mathbb{R})$ is isomorphic to $\gamma_M \otimes V^{G^{\mathrm{nil}}}$. Now consider the real G-vector bundle

$$\xi = (\gamma_M \otimes V) \oplus (\gamma_M^{\perp} \otimes W). \tag{3.4}$$

Then we obtain $[\mathrm{res}_{\{e\}}\xi] = 0$ in $\widetilde{KO}(\mathrm{res}_{\{e\}}M)$ as well as $[\mathrm{res}_P\xi] = 0$ in $\widetilde{KO}_P(\mathrm{res}_P M)_{(p)}$ for all subgroups $P \in \mathcal{P}(G)$ and primes $p \mid |P|$. Note

$$\xi^{G^{\mathrm{nil}}} = \gamma_M \otimes V^{G^{\mathrm{nil}}} \cong T(M) \oplus \varepsilon_M(\mathbb{R}).$$

Using the fact, we obtain the next theorem.

Theorem 13. *Let G be a gap Oliver group and (V, W) a $\mathcal{P}$-matched pair of real G-modules of type (L1). If m is a sufficiently large integer then there exists a G-action on a disk D satisfying*

$$\begin{cases} D^G = \{x_0\}, \\ T_{x_0}(S) = (V^{G^{\mathrm{nil}}} - V^G) \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus m}, \\ \textit{the connected components of } D^{G\{p\}} \textit{ are closed manifolds for all primes } p. \end{cases}$$

Using Theorem 8, we obtain the next theorem.

Theorem 14. *Let G be a gap Oliver group and (V_i, W_i), $i = 1, \ldots, t$, $\mathcal{P}$-matched pairs of real G-modules of type (L1). Then the implication*

$$\left(\langle \left[V_i^{G^{\mathrm{nil}}} - V_i^G\right] \mid i = 1, \ldots, t\rangle_{\mathbb{Z}} + \mathrm{RO}(G)^{\mathcal{L}(G)}\right)_{\mathcal{P}(G)} \subset \mathrm{RO}(G, \mathfrak{D}\mathfrak{S}),$$

where $\langle \left[V_i^{G^{\mathrm{nil}}} - V_i^G\right] \mid i = 1, \ldots, t\rangle_{\mathbb{Z}}$ *is the subgroup of* $\mathrm{RO}(G)$ *generated by the elements* $\left[V_i^{G^{\mathrm{nil}}} - V_i^G\right]$.

Let us consider which finite groups possess $\mathcal{P}$-matched pairs of real G-modules of type (L1).

Lemma 15. *Let G be an Oliver group such that $G^{\{2\}} = G$ and $G^{\mathrm{nil}} = \bigcap_p G^{\{p\}}$ has a subquotient group isomorphic to a dihedral group D_{2ab} with distinct primes a and b, where the order of D_{2ab} is $2ab$. Then there exists a $\mathcal{P}$-matched pair (V, W) of real G-modules satisfying*

$$V^{G^{\mathrm{nil}}} = \mathbb{R}[G/G^{\mathrm{nil}}] \quad \textit{and} \quad W^{G^{\mathrm{nil}}} = 0$$

as real G/G^{nil}-modules.

Immediately we get the next.

Theorem 16. *Let G be an Oliver group such that $G^{\{2\}} = G$ and G^{nil} has a subquotient group isomorphic to D_{2pq} with distinct primes p and q. Then the implication*

$$(\langle [\mathbb{R}[G/G^{\mathrm{nil}}] - \mathbb{R}] \rangle_{\mathbb{Z}} + \mathrm{RO}[G]^{\mathcal{L}(G)})_{\mathcal{P}(G)} \subset \mathrm{RO}(G, \mathfrak{D}\mathfrak{S})$$

holds, where $\langle [\mathbb{R}[G/G^{\mathrm{nil}}] - \mathbb{R}] \rangle_{\mathbb{Z}}$ *is the subgroup of* $\mathrm{RO}(G)$ *generated by the element* $[\mathbb{R}[G/G^{\mathrm{nil}}] - \mathbb{R}]$.

This theorem can be partially improved to Theorem 5 by using the next topological result.

Lemma 17. *Let C be a cyclic group of odd order $p \geq 3$ and U a faithful real C-module of dimension 2. Let $M = P(\mathbb{R} \oplus U)$ be the projective space associated with $\mathbb{R} \oplus U$ and let γ_M be the canonical line bundle over M. Then*

$${\gamma_M}^{\oplus 4} \cong \varepsilon_M(\mathbb{R}^4)$$

and

$$T(M)^{\oplus 4} \oplus \varepsilon_M(\mathbb{R}^4) \cong \varepsilon_M(U^{\oplus 4}) \oplus \varepsilon_M(\mathbb{R}^4)$$

as real C-vector bundles over M.

Next we consider cases where $G/G^{\{2\}} \cong C_2$.

Theorem 18. *If G is a gap Oliver group having a subquotient group D_{4q} of type* (B/N) *then*

$$(\langle [\mathbb{R}[G/G^{\mathrm{odd}}] - \mathbb{R}] \rangle_{\mathbb{Z}} + \mathrm{RO}[G]^{\mathcal{L}(G)})_{\mathcal{P}(G)} \subset \mathrm{RO}(G, \mathfrak{D}\mathfrak{S}).$$

In order to work in a slightly general setting, set

$$G^{\mathrm{odd}} = \bigcap_p G^{\{p\}},$$

where p ranges over the set of all odd primes dividing $|G|$.

Definition 19. We say that G *has a subquotient group* D_{4q} *of type (B/N)* if there is a pair (B, N) of subgroups B and N satisfying the following conditions.

(1) $B \subset G^{\mathrm{odd}}$ and $N \triangleleft B$.

(2) The quotient group B/N is isomorphic to a dihedral group

$$D_{2q}^{(1)} \times C_2^{(2)}$$

of order $4q$ for some odd integer $q \geq 3$ such that

$$D_{2q}^{(1)} = C_q^{(1)} \rtimes C_2^{(1)}.$$

Let $\pi : B \longrightarrow D_{2q} \times C_2^{(2)}$ denote the associated epimorphism.

(3) $B \cdot G^{\{2\}} = G$.

(4) $\pi(B \cap G^{\{2\}}) \supset C_2^{(2)}$.

For such a group G, we can obtain a modification of Lemma 12.

Lemma 20. *If G has a subquotient group D_{4q} of type (B/N), then there exists a $\mathcal{P}$-matched pair (V, W) of real G-modules satisfying Condition 3.3.*

Recall the group $D(p, 2q) = D_{2q}{}^p \rtimes C_p$ defined in Section 1.

Lemma 21. *If a finite group G has a normal subgroup N such that $N \subset G^{\mathrm{nil}}$ and $G/N \cong D(p, 2q)$ for some odd integers p and $q \geq 3$, then G is a gap Oliver group having a subquotient group D_{4q} of type* (B/N).

We can obtain the next result by using Lemma 20.

Theorem 22. *If G is a gap Oliver group having a subquotient group D_{4q} of type* (B/N) *then*

$$\left(\langle [\mathbb{R}[G/G^{\mathrm{odd}}] - \mathbb{R}] \rangle_{\mathbb{Z}} + \mathrm{RO}[G]^{\mathcal{L}(G)}\right)_{\mathcal{P}(G)} \subset \mathrm{RO}(G, \mathfrak{D}\mathfrak{S}).$$

If $G = D(3, 2q)$ then we obtain Theorem 6. In the special case where $G = D(3, 6)$, the next holds.

Corollary 23. *If $G = D(3,6)$ then G/G^{nil} is isomorphic to C_6 and the equalities*

$$\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) = \mathrm{RO}(G, \mathfrak{S}) = \mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}}) = \mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)}$$

hold and the rank of the last additive group is 3.

4. Problems

Let us close this paper with problems presently interested in.

Problem. Is the set $\mathrm{RO}(G, \mathfrak{S})_{\mathcal{P}(G)}$ an additive subgroup of $\mathrm{RO}(G)$?

Problem. Determine $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S})$ for all Oliver groups G of order ≤ 2000.

T. Sumi [25] gave information of Oliver groups G with $|G| \leq 2000$ for which we had not determined whether $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})$ was trivial or not. Still now, we can not answer whether $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})$ are trivial for the Oliver groups $SG(864, 4672)$, $SG(1152, 155470)$ and $SG(1152, 155859)$, where $SG(m,n)$ denotes the small group of order m, type n in the computer software GAP [6].

Problem. Determine $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S})$ for Oliver groups G such that G/G^{nil} is an elementary abelian 2-group.

We remark that if G is a gap Oliver group such that G/G^{nil} is an elementary abelian 2-group then the equality $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) = \mathrm{RO}(G)^{G\{2\}}_{\mathcal{P}(G)}$ holds.

Problem. Determine $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S})$ for Oliver groups G such that $G/G^{\mathrm{nil}} \cong C_{2p}$ for some odd prime p.

Note that if G is a gap Oliver group such that $G/G^{\mathrm{nil}} \cong C_6$ and a Sylow 2-subgroup of G is a normal subgroup of G then the equality $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) = \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$ holds.

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