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Lurie's Topological Quantum Field Theory

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1. Introduction

[CTFT] Jacob Lurie, *On the Classification of Topological Field Theories (Draft)*, May 1, 2009, 111pages,

[GMTW] math/0605249 The homotopy type of the cobordism category. Soren Galatius, Ib Madsen, Ulrike Tillmann, Michael Weiss

In [CTFT], a part of the proof [GMTW] of the Mumford Conjecture (= the Madsen-Weiss Theorem) for a closed oriented manifold Σ_g of dimension $n = 2$:

Mumford Conjecture = Madsen-Weiss Theorem

The map induced by the Miller-Morita-Mumford classes

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \to H^*(BDiff(\Sigma_g); \mathbb{Q})$$

is an isomorphism in degrees $\leq n(g)$ with $\lim_{g \mapsto \infty} n(g) = \infty$.

valid for general n, has been higher categorically reorganized and made transparent.

Relavent argument valid for general n

(1) For a general closed oriented manifold M of dimension n, construct

$$BDiff(M) \to \Omega^n |\mathbf{Bord}_n^{SO(n)}|$$

(2) (**Cobordism hypothesis, Group-Completed Version**) For a topological group G with a continuous homomorphism $\chi : G \to O(n)$,

$$|\mathbf{Bord}_n^G| \simeq (QS^0)_{hG}.$$

In particular,

$$|\mathbf{Bord}_n^{SO(n)}| \simeq \Omega^\infty \left(\Sigma^n BSO(n)^{-\zeta_n}\right),$$

where ζ_n is the universal rank n vector bundle over $BSO(n)$.

An outline of the proof of the Munford conjecture

(1) Specializing to the case $M = \Sigma_g$, factorize as

$$BDiff(\Sigma_g) \to Y_g \xrightarrow{\text{a connected component inclusion}} \Omega^2 |\mathbf{Bord}_2^{SO(2)}|$$

(2) (**The Harer stability**) $BDiff(\Sigma_g) \to Y_g$ is an $n(g)$-equivalence with $\lim_{g \mapsto \infty} n(g) = \infty$.

(3) $\Omega^2 |\mathbf{Bord}_2^{SO(2)}| \simeq \Omega^\infty \left(BSO(2)^{-\zeta_2}\right) \simeq \Omega^\infty \left(\mathbb{C}P_{-1}^\infty\right)$ is easy to understand homotopy theoretically (Galatius) .

In this paper, I shall present a short introduction to some higher categorical aspect of the cobordism hypothesis presented in [CTFT], in an OHP presentation style. I claim no originality here, but I intended to convey the readers with at least a rough outline of [CTFT]. Of course, I am entirely responsible for any possible mistakes and confusions here. Also, I hope to come back with a sequel with more details.

Fortunately, Lurie's own lecture series on this subject is available as video files on the web:

http://lab54.ma.utexas.edu:8080/video/lurie.html

So, just google "*Jacob Lurie video*" to locate this web site!

2. What is the Cobordism Hypothesis?

A very general form of the cobordism hypothesis is the following:

Cobordism Hypothesis for (X, ζ)-manifolds (Theorem 2.4.18)

- $\mathcal{C}$: a symmetric monoidal (∞, n)-category with duals;
  $\mathcal{C}^{\sim}$: its underlying ∞-groupoid ($=(\infty, 0)$-category), obtained by discarding all of the noninvertible morphisms [CTFT, 2.4.4];
  $T^{\sim}$: a topological space s.t. $\mathcal{C}^{\sim} \cong \pi_{\leq\infty} T^{\sim}$, as an ∞-groupoid;
- (X, ζ): a CW complex X and its n dimensional vector bundle with an inner product;
  $\tilde{X} \to X$: its associated princial $O(n)$-bundle of orthonormal frames in ζ;

$\Longrightarrow$ $\exists$ an equivalence of $(\infty, 0)$-categories:

$$\mathrm{Fun}^{\otimes}\left(\mathbf{Bord}_n^{(X,\zeta)}, \mathcal{C}\right) \simeq \mathrm{Hom}_{O(n)}\left(\tilde{X}, T^{\sim}\right)$$

Natual questions concerning the cobordism hypthesis

(a) What is an (∞, n)-category?
(b) What is a (symmetric monoidal) functor between (symmetric monoidal) (∞, n)-categories?
(c) What is the (∞, n)-category $\mathbf{Bord}_n^{(X,\zeta)}$?
(d) What does it mean for a symmetric monoidal (∞, n)-category to have duals?
(e) How can we deduce $|\mathbf{Bord}_n^G| \simeq (QS^0)_{hG}$ from the cobordism hypothesis?

3. What is an (∞, n)-category?

A rough definition of (∞, n)-category

An $\underline{(\infty, n)\text{-}\textit{category}}$ is a higher category, where all the "strictness" are dropped and only "up to coherent isomorphisms", in which all k-morphism are invertible for $k > n$.

A fundamental n-groupoid of a topological space$X$

For each $0 \le n \le \infty$, one can define an n-category $\pi_{\le n}X$, called the ***fundamental n-groupoid of X***:

- The objects of $\pi_{\le n}X$ are the points of X.
- Given a pair of objects $x, y \in X$, a 1-morphism in $\pi_{\le n}X$ from x to y is a path in X from x to y.
- Given a pair of objects $x, y \in X$ and a pair of 1-morphisms $f; g : x \to y$, a 2-morphism from f to g in $\pi_{\le n}X$ is a homotopy of paths in X (which is required to be fixed at the common endpoints x and y).
- $\cdots$
- An n-morphism in $\pi_{\le n}X$ is given by a homotopy between homotopies between . . . between paths between points of X. **Two such homotopies determined the same n-morphism in $\pi_{\le n}X$ if they are homotopic to one another (via a homotopy which is fixed on the common boundaries).**

A rough inductive definition of (∞, n)-category

- $(\infty, 0)$-***category*** = ∞-groupoid
  "=" topological space
- (∞, n)-***category*** consists of the following data:
  - a collection of objects $X, Y, Z, \ldots$
  - for pairs of objects $X, Y \in \mathcal{C}$, an $(\infty, n-1)$-category $\mathrm{Hom}_{\mathcal{C}}(X, Y)$
  - composition law
  - Associativity (with units) (up to coherent isomorphism)

Examples of (∞, n)-categories

- For a topological space X, its fundamental ∞-groupoid $\pi_{\le\infty}X$ is an $(\infty, 0)$-category;
- An $(\infty, n-1)$-category is an (∞, n)-category;
- An n-category is an (∞, n)-category by considering only identity k-morphisms for $k > n$;

  There is an adjunction:

$$\mathrm{Fun}_{n-\mathrm{category}}(h_n\mathcal{D}, \mathcal{C}) \cong \mathrm{Fun}_{(\infty,n)-\mathrm{category}}(\mathcal{D}, i\mathcal{C})$$

$$h_n : \{(\infty, n) - \text{category}\} \rightleftarrows \{n - \text{category}\} : i$$

$$\mathcal{D} \xmapsto{h_n} h_n\mathcal{D}$$

$$i\mathcal{C} \text{ (only identity } k\text{-morphisms for } k > n) \xleftarrow[i]{} \mathcal{C},$$

  where, for an (∞, n)-category$\mathcal{D}$, $h_n\mathcal{D}$ is the ***homotopy n-category*** of $\mathcal{D}$, given by
  - For $k < n$, the k-morphisms of $h_n\mathcal{D}$ are the k-morphisms of $\mathcal{D}$.
  - The n-morphisms of $h_n\mathcal{D}$ are given by isomorphism classes of n-morphisms in $\mathcal{D}$.

What is an $(\infty, 0)$-category supposed to be?

$(\infty, 0)$-category is a topological space, or a Kan complex $\Longrightarrow$ the object of the classical homotopy theory.

What is an $(\infty, 1)$-category supposed to be?

Must satisfy the following conditions:

- Both $(\infty, 0)$-category (i.e. a topological space or a Kan complex) and a strict 1-category (i.e. an ordinary category) are $(\infty, 1)$-categories
- For every objects X, Y, $\mathrm{Hom}(X, Y)$ is an $(\infty, 0)$-category (i.e. a topological space or a Kan complex).

Actually, there are many different approaches to define an $(\infty, 1)$-category, such as a topological category, a simplicial category, a quasi-category, a Segal category, a complete Segal space.
However, these are all essentaily the same concepts, and there is a diagram of right Quillen equivalences

$$\begin{tikzcd} Set_{\triangle} & Cat_{\triangle} \arrow[l, "N"'] \arrow[d] \\ \mathrm{Fun}(\Delta^{op}, Set_{\triangle}) \arrow[u, "G_0"] \arrow[r] & Seg_{Set_{\triangle}} \end{tikzcd}$$

where

- The category of simplicial sets $Set_{\triangle}$ has the Joyal model structure, whose fibrant objects are nothig but **quasi categories**.
- $Cat_{\triangle}$ is the category of **simplicial categories**.
- $\mathrm{Fun}(\Delta^{op}, Set_{\triangle})$ is the category of all bisimplicial sets with the **complete Segal** model structure
- $Seg_{Set_{\triangle}}$ is the category of pre**Segal categories**: i.e. bisimplicial sets $X_{\bullet,\bullet}$ with the property that the 0th column $X_{\bullet,0}$ is a constant simplicial set.

The original idea of Rezk and Barwick

Understand iductively w.r.t. n, by expressing an (∞, n)-category by $(\infty, n-1)$-categories:

They compared the following two objects:

- (∞, n)-category $\mathcal{C}$
- simplicial $(\infty, n-1)$-category $\mathcal{C}_{\bullet}$ with the *__Segal condition__*:

  For each $k \geq 0$, the canonical map

  $$\mathcal{C}_k \to \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \cdots \times_{\mathcal{C}_0} \mathcal{C}_1$$

  is an equivalence of $(\infty, n-1)$-categories.

(∞, n) $\mathcal{C}$ $\Longrightarrow$ simplicial $(\infty, n-1)$ $\mathcal{C}_\bullet$ with Segal

Given an (∞, n)-category $\mathcal{C}$, construct a simplicial $(\infty, n-1)$ $\mathcal{C}_\bullet$ with the Segal condition by

- $\mathcal{C}_0 :=$ $(\infty, 0)$-category (and so an $(\infty, n-1)$-category) extracted from $\mathcal{C}$, by discarding all of the noninvertible morphisms in $\mathcal{C}$ at all levels.
- $\mathcal{C}_1 :=$ $(\infty, n-1)$-category whose obejects are given by triples $(X \in \mathcal{C}_0, Y \in \mathcal{C}_0, f \in \mathrm{Map}_{\mathcal{C}}(X, Y))$, where, for each pair of objects $X, Y \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is an $(\infty, n-1)$-category, depending functorially on the pair $X, Y \in \mathcal{C}_0$.
- $\mathcal{C}_k :=$ $(\infty, n-1)$-category whose obejects are given by $(2k+1)$-tuples

$$(X_0 \in \mathcal{C}_0, X_1 \in \mathcal{C}_0, \cdots, X_k \in \mathcal{C}_k,$$
$$f_1 \in \mathrm{Map}_{\mathcal{C}}(X_0, X_1), \cdots, f_k \in \mathrm{Map}_{\mathcal{C}}(X_{k-1}, X_k)),$$

- The collection of $(\infty; n-1)$-categories $\{\mathcal{C}_k\}_{k \geq 0}$forms a simplicial $(\infty, n-1)$-category $\mathcal{C}_\bullet$ satisfying the Segal condition.

simplicial $(\infty, n-1)$ $\mathcal{C}_\bullet$ with Segal $\Longrightarrow$ (∞, n) $\mathcal{C}$

Given a simplicial $(\infty, n-1)$ category $\mathcal{C}_\bullet$ satisfying the Segal condition, construct an (∞, n)-category $\mathcal{C}$ by

- The objects of $\mathcal{C}$ are the objects of $\mathcal{C}_0$:
$$\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{C}_0)$$
- Given a pair of objects $X; Y \in \mathcal{C}_0$, the $(\infty, n-1)$-category of maps $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is given by the fiber product $\{X\} \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \{Y\}$:
$$\mathrm{Map}_{\mathcal{C}}(X, Y) = \{X\} \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \{Y\}$$
- Given a sequence of objects $X_0, \ldots, X_k \in \mathcal{C}_0$, the composition law
$$\mathrm{Map}_{\mathcal{C}}(X_0, X_1) \times \cdots \mathrm{Map}_{\mathcal{C}}(X_{k-1}, X_k) \to \mathrm{Map}_{\mathcal{C}}(X_0, X_k)$$
is given by the composite map
$$\begin{aligned}
&\mathrm{Map}_{\mathcal{C}}(X_0, X_1) \times \cdots \mathrm{Map}_{\mathcal{C}}(X_{k-1}, X_k) \\
&= (\{X_0\} \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \{X_1\}) \times \cdots (\{X_{k-1}\} \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \{X_k\}) \\
&\xleftarrow[\because)\text{ Segal condition}]{\simeq} \mathcal{C}_k \times_{\mathcal{C}_0 \times \cdots \mathcal{C}_0} (\{X_0\} \times \cdots \{X_k\}) \\
&\to \{X_0\} \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \{X_k\} = \mathrm{Map}_{\mathcal{C}}(X_0, X_k)
\end{aligned}$$

A bad news: a motivation of "*completeness*"

Although the composite

$$\begin{array}{ccc}
(\infty, n) \to & \text{simplicial } (\infty, n-1) & \to (\infty, n) \\
\mathcal{C} \mapsto & \mathcal{C}_\bullet & \mapsto \mathcal{C}'
\end{array}$$

is an equivalence: $\mathcal{C} \simeq \mathcal{C}'$, the composite

$$\begin{array}{ccc}
\text{simplicial } (\infty, n-1) \to & (\infty, n) & \to \text{simplicial } (\infty, n-1) \\
\mathcal{C}_\bullet \mapsto & \mathcal{C} & \mapsto \mathcal{C}'_\bullet
\end{array}$$

is <u>*NOT*</u> an equivalence $\mathcal{C}_\bullet \not\simeq \mathcal{C}'_\bullet$ in general, for $\mathcal{C}$ may have more invertaible morphisms than $\mathcal{C}_0$ and $\mathcal{C}_0 \not\simeq \mathcal{C}'_0$.

Technical preparations...

- An ***n-fold simplicial space*** is an n-fold simplicial object in the category of topological spaces and continuous maps:

$$\triangle^{op} \times \cdots \triangle^{op} \to Top$$

- We will say that a map $X \to Y$ of n-fold simplicial spaces is a ***weak homotopy equivalence*** if the induced map $X_{k_1;\ldots;k_n} \to Y_{k_1;\ldots;k_n}$ is a weak homotopy equivalence of topological spaces, for every sequence of nonnegative integers $k_1;\ldots;k_n \geq 0$.
- A commutative diagram of topological spaces

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z : \end{tikzcd}$$

is said to be a ***homotopy pullback square*** (or a ***homotopy Cartesian diagram***) if

$$W \to X \times_Z Y \to X \times_Z^R Y := X \times_Z Z^{[0,1]} \times_Z Y$$

is a weak homotopy equivalence.
- A diagram of n-fold simplicial spaces:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

is a ***homotopy pullback square*** if, for every sequence of nonnegative integers $k_1;\ldots;k_n \geq 0$, the induced square

$$\begin{tikzcd} X_{k_1;\ldots;k_n} \arrow[r] \arrow[d] & Y_{k_1;\ldots;k_n} \arrow[d] \\ X'_{k_1;\ldots;k_n} \arrow[r] & Y'_{k_1;\ldots;k_n} \end{tikzcd}$$

is a homotopy pullback square.
- We will say that an n-fold simplicial space X is ***essentially constant*** if there exists a weak homotopy equivalence of n-fold simplicial spaces $X' \to X$, where X' is a constant functor.

The case $n = 1$: the Complete Segal Space

- Let $X_\bullet$ be a simplicial space. We say that $X_\bullet$ is a ***Segal space*** if the following condition is satisfied:
  - For every pair of integers $m, n \geq 0$, the diagram

$$\begin{tikzcd} X_{m+n} \arrow[r] \arrow[d] & X_m \arrow[d] \\ X_n \arrow[r] & X_0 \end{tikzcd}$$

  is a homotopy pullback square.
- Let $X_\bullet$ be a Segal space, and let

$$\delta : X_0 \to X_1$$

  be the "degeneracy map" induced by the unique nondecreasing functor $\{0, 1\} \to \{0\}$. For every point x in X_0, the morphism $[\delta(x)]$ in the homotopy category $hX_\bullet$ coincides with the identity map $\mathrm{id}_x : x \to x$. In particular,

  $\delta(x)$ is invertible for each $x \in X_0$.
- Let $X_\bullet$ be a Segal space, and let $Z \subseteq X_1$ denote the subset consisting of the invertible elements (this is a union of path components in X_1; we will consider Z as endowed with the subspace topology). We will say that $X_\bullet$ is ***complete*** if the map

$$\delta : X_0 \to Z$$

  is a weak homotopy equivalence.
- ***$(\infty, 1)$-category*** is a complete Segal space.

An (∞, n)-category is defined by generalizing "complete Segal space"

An **$(\infty; n)$-category** is an n-fold complete Segal space.

Definition of a n-fold (complete) Segal space

Let $n > 0$, and let X be an n-fold simplicial space. Regard X as a simplicial object $X_\bullet$ in the category of $(n-1)$-fold simplicial spaces. X is said to be an ***n-fold Segal space*** if the following conditions are satisfied:

(A1) For every $0 \leq k \leq m$, the diagram

$$\begin{tikzcd} X_m \arrow[r] \arrow[d] & X_k \arrow[d] \\ X_{m-k} \arrow[r] & X_0 \end{tikzcd}$$

is a **homotopy pullback square** (of $(n-1)$-fold simplicial spaces).

(A2) The $(n-1)$-fold simplicial space X_0 is **essentially constant.**

(A3) Each of the $(n-1)$-uple simplicial spaces X_k is an $(n-1)$-fold Segal space.

We will say that an n-fold Segal space X is ***complete*** if it satisfies the following additional conditions:

(A4) Each of the $(n-1)$-dimensional Segal spaces X_n is complete (we regard this condition as vacuous when $n = 1$).

(A5) Let $Y_\bullet$ be the simplicial space described by the formula $Y_k = X_{k;0;\dots 0}$; note that condition (A3) guarantees that $Y_\bullet$ is a Segal space. Then $Y_\bullet$ is complete.

Completion

- If X is an n-fold Segal space, then there is a **universal example of a map**

$$X \to X'$$

in the homotopy category of n-fold simplicial spaces, such that X' is an n-fold complete Segal space, i.e. an (∞, n)-category.

We will refer to X' as the ***completion*** of X.

- Intuitively, the completion is a refinement of the composition:

simplicial $(\infty, n-1) \to$	$(\infty, n) \to$ simplicial $(\infty, n-1)$
$\mathcal{C}_\bullet \mapsto$	$\mathcal{C} \quad \mapsto \mathcal{C}'_\bullet$

- The completion may be interpreted as a

"***localization***"

in an appropriate sense.

4. What is a (symmetric monoidal) functor between (symmetric monoidal) (∞, n)-categories?

Just a summary!

- Let $\mathcal{C}$ and $\mathcal{D}$ be (∞, n)-categories. There exists another (∞, n)**-category**
$$\mathrm{Fun}(\mathcal{C}; \mathcal{D})$$
of functors from $\mathcal{C}$ to $\mathcal{D}$. The (∞, n)-category $\mathrm{Fun}(\mathcal{C}; \mathcal{D})$ is characterized up to equivalence by the following universal property: for every (∞, n)-categoy $\mathcal{C}'$, there is a bijection between
the set of isomorphism classes of functors
$$\mathcal{C}' \to \mathrm{Fun}(\mathcal{C}; \mathcal{D})$$
and
the set of isomorphism classes of functors
$$\mathcal{C}' \times \mathcal{C} \to \mathcal{D}.$$
- The collection of all (small) (∞, n)-categories can be organized into a **(large)** $(\infty, n+1)$**-category** $\mathcal{C}at_{(\infty;n)}$, with mapping objects given by
$$\mathrm{Map}_{\mathcal{C}at_{(\infty;n)}}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}(\mathcal{C}; \mathcal{D})$$
- Suppose that $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal (∞, n)-categories. Then we can also define an (∞, n)**-category**
$$\mathrm{Fun}^{\otimes}(\mathcal{C}; \mathcal{D})$$
of symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$.

5. What is the (∞, n)-category $\mathbf{Bord}_n^{(X,\zeta)}$?

We should first explain Atiyah's topological field theory...

The symmetric monoidal category **Cob**(n)

For $n \in \mathbb{N}$, define the ***symmetric monoidal category*** **Cob**(n) by:

(1) An ***object*** of **Cob**(n) is a closed oriented $(n-1)$-manifold M.

(2) Given a pair of objects $M, N \in \mathbf{Cob}(n)$, a ***morphism*** from M to N in **Cob**(n) is a bordism from M to N: that is, an oriented n-dimensional manifold B equipped with an orientation-preserving diffeomorphism

$$\partial B \simeq \overline{M} \coprod N.$$

Here $\overline{M}$ denotes the manifold M equipped with the opposite orientation.

We regard two bordisms B and B' as defining the same morphism in **Cob**(n) if there is an orientation-preserving diffeomorphism $B \simeq B'$ which extends the evident diffeomorphism $\partial B \simeq \overline{M} \coprod N \simeq \partial B'$ between their boundaries.

(3) For any object $M \in \mathbf{Cob}(n)$, ***the identity map*** id_M is represented by the product bordism

$$B = M \times [0;1].$$

(4) ***Composition of morphisms*** in **Cob**(n) is given by gluing bordisms together. More precisely, suppose we are given a triple of objects $M, M', M'' \in \mathbf{Cob}(n)$, and a pair of bordisms

$$B : M \to M', \ B' : M' \to M'',$$

the composition $B' \circ B$ is defined to be the morphism represented by the manifold

$$B \coprod_{M'} B'.$$

(5) ***The tensor product operation*** for the symmetric monoidal structure

$$\otimes : \mathbf{Cob}(n) \times \mathbf{Cob}(n) \to \mathbf{Cob}(n)$$

is given by the disjoint union of manifolds.

(6) ***The unit object*** for the symmetric monoidal structure of **Cob**(n) is

$$\emptyset,$$

the empty set (regarded as a manifold of dimension $(n-1)$).

Atiyah's topological field theory

Let k be a field. A ***topological field theory*** of dimension n is a symmetric monoidal functor

$$Z : \mathbf{Cob}(n) \to \mathbf{Vect}(k),$$

where $\mathbf{Vect}(k)$ is the usual symmetric monoidal category of vector spaces over k.

A couple of problems of the topological field theory

- ***Not interesting enough*** for those interested in $BDiff(M)$, especially those interested in the Mumford conjecture!
- ***Not easy to understand*** for large n, because the structure of a n-dimensional manifold tends to become complicated as n becomes larger!

For the first problem: encode the group of diffeos!

For $n \in \mathbb{N}$, define the ***topological symmetric monoidal category*** $\mathbf{Cob}_t(n)$ by ***topologically enriching*** $\mathbf{Cob}(n)$:

- The objects of $\mathbf{Cob}_t(n)$ are closed oriented manifolds of dimension $(n-1)$.
- Given a pair of objects $M, N \in \mathbf{Cob}_t(n)$, we let $\mathrm{Hom}_{\mathbf{Cob}_t(n)}(M, N)$ denote the classifying space $\mathcal{B}(M, N)$ of bordisms from M to N:

  $$\mathrm{Hom}_{\mathbf{Cob}_t(n)}(M, N) := \mathcal{B}(M, N) = BC,$$

  where C is a topological category s.t.
  - objects are oriented bordisms B from M to N,
  - for every pair of bordisms B and B', the collection of (orientation-preserving) diffeomorphisms

    $$\mathrm{Hom}_C(B; B')$$

    has a topology (the topology of uniform convergence of all derivatives) such that the composition maps are continuous.

  In particular,

  $$\pi_0 \,\mathrm{Hom}_{\mathbf{Cob}_t(n)}(M, N) = \mathrm{Hom}_{\mathbf{Cob}(n)}(M, N).$$

⋮

Now, for any closed oriented manifold M of dimension n,

$$BDiff(M) \xrightarrow{\text{connected component inclusion}} \mathcal{B}(\emptyset, \emptyset)$$
$$= \mathrm{Hom}_{\mathbf{Cob}_t(n)}(\emptyset, \emptyset)$$

does show up in $\mathbf{Cob}_t(n)$, and we may instead consider a topological symmetric monoidal functor

$$\overline{Z} : \mathbf{Cob}_t(n) \to \mathcal{C},$$

where $\mathcal{C}$ is some topological symmetric monoidal category...

For the second problem: take into account lower dim.!

Like a CW decmpotion of a CW complex or a pants decomposition of a surface, ***any n-dimensional manifold is composed of very simple $n-k$-dimensional manifolds*** $(0 \leq k \leq n)$

$\Longrightarrow$ Take into account lower dimensional manifolds:

Suppose given a pair of nonnegative integers $k \leq n$,

a ***k-category*** $\mathbf{Cob}_k(n)$ is given by

- The ***objects*** of $\mathbf{Cob}_k(n)$ are closed oriented $(n-k)$-manifolds
- Given a pair of objects $M, N \in \mathbf{Cob}_k(n)$, a ***1-morphism*** from M to N is a bordism from M to N: that is, a $(n-k+1)$-manifold B equipped with a diffeomorphism $\partial B \simeq \overline{M} \coprod N$.
- Given a pair of objects $M, N \in \mathbf{Cob}_k(n)$ and a pair of bordisms $B, B' : M \to N$, a ***2-morphism*** from B to B' is a bordism from B to B', which is required to be trivial along the boundary: in other words, a manifold with boundary

$$\overline{B} \coprod_{M \coprod \overline{N}} \left((\overline{M} \coprod N) \times [0,1] \right) \coprod_{\overline{M} \coprod N} B'$$

- $\cdots$
- A ***k-morphism*** in $\mathbf{Cob}_k(n)$ is an n-manifold X with corners, where the structure of ∂X is determined by the source and target of the morphism.

  Two n-manifolds with (specified) corners X and Y ***determine the same n-morphism*** in $\mathbf{Cob}_k(n)$ if they differ by an orientation-preserving diffeomorphism, relative to their boundaries.
- ***Composition of morphisms (at all levels)*** in $\mathbf{Cob}_k(n)$ is given by gluing of bordisms.

Then,

- $\mathbf{Cob}_0(n)$ may be identified with the set of of diffeomorphism classes of closed, oriented n-manifolds.
- $\mathbf{Cob}_1(n) = \mathbf{Cob}(n)$
- Objects and morphisms of $\mathbf{Cob}(n)$ can be regarded as $(n-1)$-morphisms and n-morphisms of $\mathbf{Cob}_n(n)$. We may therefore regard $\mathbf{Cob}_n(n)$ as an elaboration of $\mathbf{Cob}(n)$ obtained by considering also "lower"morphisms corresponding to manifolds of dimension $< n-1$.

So, we may instead consider a symmetric monoidal functor of n-categories

$$\overline{Z} : \mathbf{Cob}_n(n) \to \mathcal{C},$$

where $\mathcal{C}$ is some symmetric monoidal n-category...

A good news! **Bord**$_n$

- Using an (∞, n)-category $\mathbf{Bord}_n$, we may simultaneously elaborate both $\mathbf{Cob}_t(n)$ and $\mathbf{Cob}_n(n)$.
- Thus, $\mathbf{Bord}_n$ may solve the two problems:
  (1) Not interesting; (2) Not easy;
  of $\mathbf{Cob}(n)$.
- ***Symmetric monoidal*** $(\infty; n)$***-category*** $\mathbf{Bord}_n$ is described *informally* as follows:
  - The ***objects*** of $\mathbf{Bord}_n$ are 0-manifolds.
  - The 1***-morphisms*** of $\mathbf{Bord}_n$ are bordisms between 0-manifolds.
  - The 2***-morphisms*** of $\mathbf{Bord}_n$ are bordisms between bordisms between 0-manifolds.
  - ...
  - The n***-morphisms*** of $\mathbf{Bord}_n$ are bordisms between bordisms between ... between bordisms between 0-manifolds (in other words, n-manifolds with corners).
  - The $(n+1)$***-morphisms*** of $\mathbf{Bord}_n$ are ***diffeomorphisms*** (which reduce to the identity on the boundaries of the relevant manifolds).
  - The $(n+2)$***-morphisms*** of $\mathbf{Bord}_n$ are ***isotopies of diffeomorphisms.***
  - ...
  - The (∞, n)-category $\mathbf{Bord}_n$ is endowed with a ***symmetric monoidal structure***, given by disjoint unions of manifolds.
- $h_n(\mathbf{Bord}_n) = \mathbf{Cob}_n(n)$.
- The precise definition of $\mathbf{Bord}_n$ is given as
  the ***completion*** of
  a n-fold Segal space $P\mathbf{Bord}_n$,
  which is defined using more sophisticated geometric argument...

(X, ζ)-structure

- Let X be a topological space and
  let ζ be a real vector bundle on X of rank n.
- Let M be a manifold of dimension $m \leq n$.

An (X, ζ)***-structure*** on M consists of the following:

(1) A continuous map $f : M \to X$.
(2) An isomorphism of vector bundles
$$T_M \oplus \underline{\mathbb{R}}^{n-m} \simeq f^*\zeta.$$

$\mathbf{Bord}_n^{(X,\zeta)}$

- A n-fold Segal space $P\mathbf{Bord}_n^{(X,\zeta)}$ is constructed just like $P\mathbf{Bord}_n$, by using manifolds with (X,ζ)-structures..

$\mathbf{Bord}_n^{(X,\zeta)} \overset{\text{Definition}}{=\!:}$ the completion of
the n-fold Segal space $P\mathbf{Bord}_n^{(X,\zeta)}$

- Thus, $\mathbf{Bord}_n^{(X,\zeta)}$ is an (∞,n)-category, and we may consider a symmetric monoidal functor between (∞,n)-categories

$$\tilde{Z} : \mathbf{Bord}_n^{(X,\zeta)} \to \mathcal{C},$$

  or even better, the (∞,n)-category

$$\mathrm{Fun}^{\otimes}\left(\mathbf{Bord}_n^{(X,\zeta)}, \mathcal{C}\right),$$

  where $\mathcal{C}$ is some symmetric monoidal (∞,n)-category.
- When $\mathcal{C}$ is a symmetric monoidal (∞,n)-category ***with duals***, the cobordism hypthesis claims an equivalence of $(\infty,0)$-categories:

$$\mathrm{Fun}^{\otimes}\left(\mathbf{Bord}_n^{(X,\zeta)}, \mathcal{C}\right) \simeq \mathrm{Hom}_{O(n)}\left(\tilde{X}, T^{\sim}\right)$$

From Atiyah's TFT to (∞, n)-category

- For any symmetric monoidal functor $Z : \mathbf{Cob}(2) \to \mathbf{Vect}(k)$,
$$Z(S^1) \in \mathrm{Ob}\,(\mathbf{Vect}(k))$$
is always a ***finite dimensional*** k-vector space.
- This is a formal consequence of the fact that
$$X := Z(S^1), \quad X^\vee := Z(\overline{S^1})$$
are ***dual*** vector spaces of each other. To show this fact, use the obvious bordisms to produce the k-linear maps
$$ev_X : X \otimes X^\vee \to k$$
$$coev_X : k \to X^\vee \otimes X,$$
s.t.
$$\mathrm{id}_X : X \xrightarrow{\mathrm{id}_X \otimes coev_X} X \otimes X^\vee \otimes X \xrightarrow{ev_X \otimes \mathrm{id}_X} X$$
$$\mathrm{id}_{X^\vee} : X^\vee \xrightarrow{coev_X \otimes \mathrm{id}_{X^\vee}} X^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{id}_{X^\vee} \otimes ev_X} X^\vee$$
This characterization allows us to define the concept of ***have a dual*** for an object of a monoidal category.
- We would like to generalize the concept of "***finite dimensionality***" of $\mathrm{Ob}(\mathbf{Vect}(k))$ to an appropriate concept for an object of an (∞, n)-category, by defining an analogue of "***dual***" for an (∞, n)-category.
- For a monoidal category $\mathcal{C}$, define a category $B\mathcal{C}$ by
  - $\mathrm{Ob}(B\mathcal{C}) = \{*\}$.
  - $\mathrm{Hom}_{B\mathcal{C}}(*, *) = \mathrm{Ob}\,\mathcal{C}$
  - The composition
  $$\mathrm{Hom}_{B\mathcal{C}}(*, *) \times \mathrm{Hom}_{B\mathcal{C}}(*, *) \to \mathrm{Hom}_{B\mathcal{C}}(*, *)$$
  is induced by the monoidal structure of $\mathcal{C}$.

  Then,
  an object $x \in \mathrm{Ob}\,\mathcal{C}$ ***has a dual***
  $\iff$ ***a morphism*** $x \in \mathrm{Hom}_{B\mathcal{C}}(*, *)$ ***has an adjoint***
- For an (∞, n)-category $\mathcal{C}$, we can also define an $(\infty, n+1)$-category $B\mathcal{C}$ just as above. Then, we define an concept of "have adjoint"for an $(\infty, n+1)$-category s.t.
$$\mathcal{C} \text{ has duals} \overset{\text{def}}{\iff} B\mathcal{C} \text{ has adjoints}$$
- $\mathbf{Bord}_n^{(x,\zeta)}$ has duals given by the manifold with opposite orientation.

Formal reduction of $|\mathbf{Bord}_n^G| \simeq (QS^0)_{hG}$ from CH

Specialize to the case:

$\mathcal{C}$: a symmetric monoidal $(\infty, 0)$-category with duals

$\Longrightarrow$ $\exists$ a topological space T s.t.

($\because$ $\mathcal{C}$ **is** $(\infty, 0)$**-category):** $\mathcal{C} \cong \pi_{\leq\infty}T$;

($\because$ $\mathcal{C}$ **is symmetric monoidal):** T is a E_∞-space;

($\because$ $\mathcal{C}$ **with duals):** $\pi_0(T)$ is an abelian group.

$\Longrightarrow$ T is an infinite loop space, and as infinite loop space with $O(n)$-actions:

$$T \simeq \mathrm{Map}_{\text{infinite loop}}(QS^0, T),$$

where the $O(n)$-action is throuh Ω^n in $T \simeq \Omega^n T(n)$ on the left, and is ONLY through QS^0 on the right.

$\Longrightarrow$ For $\mathbf{Bord}_n^G := \mathbf{Bord}_n^{(BG, EG\times_G \mathbb{R}^n \to BG)}$
wtih $\widetilde{BG} = EG \times_G O(n)$,

the $(\infty, 0)$*-category* $|\mathbf{Bord}_n^G|$,

obtained from the symmetric monoidal (∞, n)-category with duals $\mathbf{Bord}_n^G$ by inverting all higher morphisms, also yields an infinite loop space, which we also denote by $|\mathbf{Bord}_n^G|$.

$\Longrightarrow$ For any infinite loop space T with $\mathcal{C} = \pi_{\leq\infty}T$,

$$\begin{aligned}
&\mathrm{Map}_{\text{infinite loop}}\left(|\mathbf{Bord}_n^G|, T\right)\\
&= \mathrm{Fun}^{\otimes}\left(|\mathbf{Bord}_n^G|, \mathcal{C}\right) \overset{\because\ \mathcal{C}:\ =(\infty,0)-\text{category}}{=} \mathrm{Fun}^{\otimes}\left(\mathbf{Bord}_n^G, \mathcal{C}\right)\\
&\overset{\because \text{CH}}{\simeq} \mathrm{Hom}_{O(n)}(EG \times_G O(n), T) = \mathrm{Hom}_G(EG, T)\\
&= \mathrm{Hom}_G\left(EG, \mathrm{Map}_{\text{infinite loop}}(QS^0, T)\right)\\
&= \mathrm{Map}_{\text{infinite loop},\ G}(EG \times QS^0, T)\\
&= \mathrm{Map}_{\text{infinite loop}}(EG \times_G QS^0, T)\\
&= \mathrm{Map}_{\text{infinite loop}}((QS^0)_{hG}, T)
\end{aligned}$$

$\Longrightarrow$ $|\mathbf{Bord}_n^G| \simeq (QS^0)_{hG}$