A BRIEF SURVEY ON STOCHASTIC CALCULUS IN MARKOV PROCESSES
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1 From A.N. Kolmogorov to P. Lévy and K. Itô


For a Markov process \((X_t, P_t)\), define \(P_t f(x) = E_x[f(X_t)]\).

Then \(u(t, x) = P_t f(x)\) satisfies the Kolmogorov equation
\[
\frac{\partial}{\partial t} u(t, x) = \mathcal{G}_x u(t, x),
\]
where \(\mathcal{G}\) is the infinitesimal generator of the transition semigroup \(\{P_t; t \geq 0\}\):
\[
\mathcal{G}v(x) = \lim_{t \downarrow 0} \frac{P_t v(x) - v(x)}{t}.
\]
In a special case, \(\mathcal{G}\) is a second order differential operator:
\[
\mathcal{G}v(x) = \frac{1}{2} a(x) v''(x) + b(x) v'(x).
\]


Lévy-Itô decomposition of the sample path \(X_t\) of a Lévy process as a sum of a Gaussian process \(X_t^{(1)}\) and an independent process \(X_t^{(2)}\) expressed using a Poisson random measure \(J\) with intensity \(ds \mid (dx)\) by
\[
X_t^{(2)} = \lim_{n \to \infty} \left( \int_{(0, t) \times (1/n, 1)} x J(d\xi) - t \int_{(1/n, 1)} x \nu(dx) \right) + \int_{(0, t) \times (1, \infty)} x J(d\xi).
\]

\(\Rightarrow\) Lévy-Khinchin formula of \(\varphi(z) = \log E[e^{i z X_1}]\):
\[
-\langle z, Az \rangle + i \langle \gamma, z \rangle + \int_{|z| < 1} (e^{i(z,x)} - 1 - i(z,x)) \nu(dx) + \int_{|x| \geq 1} (e^{i(z,x)} - 1) \nu(dx).
\]

SDE and Itô's formula


(in English) in Kiyosi Itô Selected Papers, 42-75, Springer-Verlag, 1986


Solution \(X_t\) of SDE \(dX_t = \sqrt{a(X_t)} dB_t + b(X_t) dt\) and Itô's formula
\[
v(X_t) - v(X_0) = \int_0^t \nu'(X_s) \sqrt{a(X_s)} dB_s + \int_0^t \left( \frac{1}{2} a v'' + b v' \right)(X_s) ds
\]
yields \(\mathcal{G}v = \frac{1}{2} av'' + bv'\).
2 Books on SDE


During the period 1955-1965, the Japanese probability school led by Itô was mostly concerned with the study of the one dimensional diffusion processes and its possible extensions to more general Markov processes, while the Russian probability school led by Dynkin was equally concerned with the theory of SDE initiated by Itô and Gihman-Skorohod. Notably the drift transformation by G.Maruyama 1954, M. Motoo 1960, I.V. Girsanov 1960

Contents of Cho's book
Chap.1 Basic concepts (measurable functions, conditional expectations, independence)
Chap.2 Stochastic integrals (based on Lévy processes)
Chap.3 Transformation formula for stochastic integrals
Chap.4 Existence theorems of the solutions of stochastic integral equations
(Uniqueness and existence, Markov property of the solution and its generator, continuity and differentiability of the solution with respect to the initial position)
Chap.5 Linear stochastic integral equations
Chap.6 Stability in stochastic equations

Excerpt from Preface of Cho's book (translated by Daehong Kim)

In dealing with random phenomena, a principal feature of stochastic equation is in that it describes the states of phenomena directly rather than describing them by means of probability distributions indirectly. This resembles the classical differential equations which give direct expressions of the changes of states of deterministic phenomena. Due to this common feature, stochastic equations can be actually regarded as extensions of differential equations.

In order to build up a theory of stochastic equations, we first need to introduce the concept of the stochastic integral and study its properties, which will be the basic contents of Chapters 2 and 3. In particular, the transformation formula of stochastic integrals derived in Chapter 3 is a new formula that can not be found in any other ordinary integration theory and will play very important roles throughout the first half of the present volume.

3 One dimensional diffusions and general Markov processes

(English translation by Yuji Ito) Essentials of Stochastic Processes, Translations of Mathematical
Intrinsic structures of the one-dimensional diffusion revealed by W. Feller

a general theory on a Markov process and the structures of its additive functionals

**Intrinsic generator of a one dimensional diffusion**

$X = (X_t, P_x)$: a diffusion (a continuous strong Markov process) on a one dimensional regular open interval $I$, quasi left continuous and of no killing inside. The generator $\mathcal{G}$ of $X$ admits the expression

$$\mathcal{G}u = \frac{d}{dm} \frac{d}{ds}u$$

where $s$ is a strictly increasing continuous function on $I$ and $m$ is a strictly increasing function on $I$ given by

$$P_x(\sigma_a < \sigma_b) = \frac{s(x) - s(a)}{s(b) - s(a)}, \quad a < x < b, \quad m_J(x) = -\frac{dE_x[\tau_J]}{ds(x)}; \quad x \in J, \quad \overline{J} \subset I.$$  

Feller's saying: $s$ indicates the road map and $m$ indicates the speed of the diffusion traveller $X$.

Itô-McKean [IM] legitimates Feller's saying in the following fashion:

Let $X = (X_t, P_x)$ be the Brownian motion on $R$. In this case, $s(x) = 2x, m(x) = x$. For a given strictly increasing function $m_0$, define

$$A_t = \int_R \ell(t, x)dm_0(x), \quad \tau_t = A_t^{-1} \quad (1)$$

where $\ell(t, x)$ is Lévy's local time of $X$ at $x \in R$. Then the time changed process $Y = (X_{\tau_t}, P_x)$ is a diffusion process on $R$ corresponding to $s(x) = 2x, m(x) = m_0(x)$, that is to say, a time change of $X$ amounts a replacement of the measure $m$ keeping the road map $s$ invariant.

The above defined functional $\{A_t; t \geq 0\}$ is the most general expression of the positive continuous additive functional (PCAF in abbreviation) of the one-dimensional Brownian motion. But such expression does not hold in general.

**Symmetry and Dirichlet form of a one dimensional diffusion**

Let $X$ be a diffusion on a regular open interval $I = (r_1, r_2)$ as before.

Then its generator is given by (1).

Since the resolvent $\{G_\alpha\}$ of $X$ is known to have a symmetric density kernel with respect to the speed measure $m$ ([I.6]), $X$ is $m$-symmetric. Define

$$\mathcal{E}^{(s)}(u, v) = \int_I \frac{du}{ds} \frac{dv}{ds} ds.$$
By making use of a detailed boundary behaviors of the resolvent proved in [I.6], an integration by parts gives
\[- \int \mathcal{G} u \cdot v dm = E^{(s)}(u, v), \quad u = G_\alpha f, \quad v = G_\alpha g, \quad f, g \in C_0(I)\]

Actually the Dirichlet form \((\mathcal{E}, \mathcal{F})\) of \(X\) on \(L^2(I; m)\) is given by
\[
\mathcal{F} = \{ u \in L^2(I; m) : \text{absolutely continuous in } s, \ E^{(s)}(u, u) < \infty, \ u(r_i) = 0, \text{ if } r_i \text{ is approachable}\},
\]
where \(r_i\) is said to be approachable if \(s\) has a finite limit there. \((\mathcal{E}, \mathcal{F})\) is a regular, strongly local Dirichlet form on \(L^2(I; rI)\). This identification is proved only recently in M. Fukushima, From one dimensional diffusions to symmetric Markov processes, a volume 'Tribute to Professor Kiyosi Itô' of SPA, to appear.

Thus the one dimensional absorbing diffusion on a regular open interval, its possible stochastic transformation and its possible symmetric extensions can be handled entirely in the framework of Dirichlet forms.

4 From one dimensional diffusions to symmetric Hunt processes

\(E\): a locally compact separable metric space

\(X = (X_t, P_x)\): a Hunt process (a right continuous strong Markov process with quasi left continuity) on \(E\)

An extended real valued function \(A_t(\omega)\) of \(t \geq 0, \omega \in \Omega\), is called an additive functional (AF for abbreviation) of \(X\) if it is right continuous in \(t\), has a left limit and \(A_{s+t}(\omega) = A_s(\omega) + A_t(\theta_s \omega)\). A \([0, \infty]\)-valued continuous AF is called a PCAF. The totality of PCAF’s of \(X\) is denoted by \(A^+_c\).

Let \(X\) be symmetric with respect to a positive Radon measure \(m\) on \(E\) of full support with the associated Dirichlet from \((\mathcal{E}, \mathcal{F})\) (\(\mathcal{F}\) is the domain of the form \(\mathcal{E}\)) being regular. The totality of the smooth measures (\(\sigma\)-finite positive measures charging no set of zero capacity) is denoted by \(S\).

There is a one-to-one correspondence between \(S\) and \(A^+_c\) characterized by the relation that \(\mu \in S\) is the Revuz measure of \(A \in A^+_c\) in the sense that
\[
\lim_{t \downarrow 0} \frac{1}{t} E_m \left[ \int_0^t f(X_s) dA_s \right] = \int_E f d\mu, \quad \forall f \in \mathcal{B}_+(E)
\]
The Revuz measure of \(A \in A^+_c\) will be denoted by \(\mu_A\).

Under the above setting, it was shown in
that a time change of \(X\) by a fully supported \(A \in A^+_c\) amounts to a replacement of \(m\) with \(\mu_A\) by keeping the (extended) Dirichlet form \(\mathcal{E}\) invariant, generalizing the Itô-McKean theorem for the one-dimensional diffusion.
5 Motoo-Watanabe theory on MAF of a Hunt process

For a general Hunt process $X$ on $E$, an AF $M_t$ of $X$ is said to be a martingale AF (MAF in abbreviation) if

$$E_x[M_t^2] < \infty, \quad E_x[M_t] = 0, \quad \forall t \geq 0, \ x \in E.$$  

The totality of MAF’s is denoted by $\mathcal{M}$. The structure of the space $\mathcal{M}$ was explored in two papers, which marked a starting point of the modern theory of stochastic calculus in Markov processes:


1. For any $M \in \mathcal{M}$, there exists a unique PCAF $\langle M \rangle \in \mathcal{A}_c^+$ such that $E_x[M_t^2] = E_x[\langle M \rangle_t]$ for any $t \geq 0, x \in E$. Let $\langle M, L \rangle = \frac{1}{2} \{\langle M + L \rangle - \langle M - L \rangle \}$ for $M, L \in \mathcal{M}$.

2. For any $M \in \mathcal{M}$ and any function $f$ on $E$ with $E_x \left[ \int_0^t |f(X_s)|d\langle M \rangle_s \right] < \infty$, there exists a unique $f \cdot M \in \mathcal{M}$ such that

$$\langle f \cdot M, L \rangle_t = \int_0^t f(X_s)d\langle M, L \rangle_s \quad \forall t \geq 0.$$  

(3)

$f \cdot M$ is called the stochastic integral.

3. Any $M \in \mathcal{M}$ admits a unique decomposition $M = M^c + M^d$, $M^c \in \mathcal{M}_c$, $M^d \in \mathcal{M}_d$, where

$$\mathcal{M}_c = \{ M \in \mathcal{M} : M_t \text{ is continuous}\}, \quad \mathcal{M}_d = \{ M \in \mathcal{M} : \langle M, L \rangle = 0 \ \forall L \in \mathcal{M}_c \}$$

4. A Lévy system for $X$ is a pair $(N, H)$ of a kernel $N(x, dy)$ on $(E_{\Delta}, \mathcal{B}(E_\Delta))$ and a PCAF $H \in \mathcal{A}_c^+$ such that

$$E_x \left[ \sum_{s \leq t} f(X_{s-}, X_s) \right] = E_x \left[ \int_0^t \left( \int_{E_\Delta} f(X_s, y)N(X_s, dy) \right) dH_s \right]$$  

(4)

for any non-negative Borel function $f$ on $E \times E$ vanishing on the diagonal.

A Lévy system exists.

Any $M \in \mathcal{M}_d$ can be represented using the Lévy system as a difference of functionals appearing in both sides of (4).

6 Stochastic calculus for semi-martingales

The space $\mathcal{M}$ of MAF's of a Hunt process is replaced by the space of general square integrable martingales $\mathcal{M}$. Due to the Doob-Meyer decomposition theorem of a submartingale, $\langle M \rangle$ is well defined as a predictable increasing process and the stochastic integral is defined analogously to (3). Itô's formula is established for a general semimartingale (local martingale + process of bounded variation). Semimartingale theory is further developed in [DM] C. Dellacherie and P. A. Meyer, \textit{Probabilités et Potentiel, Chap. I-IV, 1975; Chap. V-VIII, 1980; Chap. IX-XI, 1983; Chap. XII-XVI, 1987, Hermann, Paris.}

7 Decomposition of AF of a symmetric Hunt process

Let $X = (X_t, P_x)$ be a Hunt process on $E$ symmetric with respect to a fully supported positive Radon measure $m$ on $E$ with the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ being regular.

For $u \in \mathcal{F}$, the composite process $u(X_t)$ is not necessarily a semimartingale. Nevertheless, it admits a unique decomposition ([F])

$$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \quad t \geq 0$$

where $M^{[u]}$ is a MAF of finite energy and $N^{[u]}$ is a continuous AF of zero energy. A energy $e(A)$ of an AF is defined by $e(A) = \lim_{t \downarrow 0} \frac{1}{t} E_m[A_t^2]$. A CAF of zero energy is not necessarily of bounded variation but its quadratic variation is zero with respect to $P_m$.

8 Stochastic derivation of the Beurling-Deny formula


Any $A \in A_c^+$ admits its Revuz measure $\mu_A \in S$.

Let $(N, H)$ be a Lévy system of $X$. Using the Revuz measure $\mu_H$ of $H \in A_c^+$, define the jumping measure and the killing measure of $X$ by

$$J(dx, dy) = N(x, dy)\mu_H(dx) \quad \kappa(dx) = N(x, \{\triangle\})\mu_H(dx)$$

For any $u \in \mathcal{F}$, the following identity holds:

$$\mathcal{E}(u, u) = \frac{1}{2} \mu_{(M^{[u]c})}(E) + \frac{1}{2} \int_E u(x)^2 \kappa(dx).$$

Let $M^{[u]} = M^{[u],c} + M^{[u],d}$ be the Motoo-Watanabe decompostion of $M^{[u]} \in \mathcal{M}$ and $\mathcal{E}^{(c)}(u, u) = \frac{1}{2} \mu_{(M^{[u],c})}(E)$. [M] P. A. Meyer, Intégrals stochastiques (4 exposés), in Séminaire de Probabilités I, LNM 39, Springer, 1967, pp 72-162
A computation of $\frac{1}{2}\mu_{\langle M\rangle}(E)$ using the Lévy system formula (5) then yield
\[ \mathcal{E}(u, u) = \mathcal{E}^{(c)}(u, u) + \frac{1}{2} \int_{E \times E} (u(x) - u(y))^2 J(dx, dy) + \int_{E} u(x)^2 \kappa(dx). \] (5)
\[ \mathcal{E}^{(c)}(u, v) = \frac{1}{4} \left\{ \mathcal{E}^{(c)}(u + v, u + v) - \mathcal{E}^{(c)}(u - v, u - v) \right\} \]
has the strongly local property: if $u \cdot m$ has a compact support and $v$ is constant on a neighbourhood of it, then $\mathcal{E}(u, v) = 0$.

The formula (5) was first announced by

Lévy system is well defined for any Hunt process and special standard process as well. Revuz measure of a PCAF is well defined for any right process relative to any excessive measure. Any right process admits a weak dual moderate Markov process with respect to a given excessive measure.

has established the one-to-one Revuz correspondence between PCAF's and smooth measures ($\sigma$-finite measures charging no semi-polar sets).

Those suggest some possibility to extend the above calculus to non-symmetric Markov processes (Z. M. Ma et.al., G. Trutnau).