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On the Cross-Independence Theory of Turbulence and its Future Prospect

Tomomasa Tatsumi
Kyoto University (Professor Emeritus)

I. ABSTRACT

Statistical theory of turbulence based on the *cross-independence closurure hypothesis* has been developed by the author for homogeneous isotropic turbulence (Tatsumu (2001) and Tatsumi et al.(2004, 2007)) and now it is being extended to inhomogeneous turbulence (Tatsumi (2008)). At this stage, it may be appropriate to examine some fundamental questions raised on the theory in relation with *homogeous isotropic turbulence* and look for the prospect of the extension of the theory to general *inhomogeneous turbulence.*

II. CROSS-INDEPENDENCE CLOSURE HYPOTHESIS

When the theory has been presented so far in academic meetings demestic and international, the author has been asked several questions on the theory. Since some of them are concerned wth the fundamental structure of the theory, it seems appropriate to reexamine them in more detail and obtain right answers.

A. Validity of the Hypothesis

A question is concerned with the valilidity of the *cross-independence hypothesis.* Consider the components of two-point velocities of turbulence $u_1 = u(\mathbf{x}_1, t)$, $u_2 = u(\mathbf{x}_2, t)$ and their cross-velocities $u_+ = (u_1 + u_2)/2$, $u_- = (u_2 - u_1)/2$. Then, the third-order structure functions of them are expressed as

$$\begin{aligned}\langle u_1^3\rangle &= \left\langle (u_+ - u_-)^3\right\rangle = \langle u_+^3\rangle - 3\langle u_+^2 u_-\rangle + 3\langle u_+ u_-^2\rangle - \langle u_-^3\rangle,\\ \langle u_2^3\rangle &= \left\langle (u_+ + u_-)^3\right\rangle = \langle u_+^3\rangle + 3\langle u_+^2 u_-\rangle + 3\langle u_+ u_-^2\rangle + \langle u_-^3\rangle.\end{aligned}$$

If we take the sum and difference of the above equations, we obtain the following identities:

$$\begin{aligned}\langle u_1^3\rangle + \langle u_2^3\rangle &= 2\langle u_+^3\rangle + 6\langle u_+ u_-^2\rangle,\\ \langle u_2^3\rangle - \langle u_1^3\rangle &= 6\langle u_+^2 u_-\rangle + 2\langle u_-^3\rangle.\end{aligned}$$

For *homogeneous turbulence*, it holds that $\langle u_1^3\rangle = \langle u_2^3\rangle = 0$ and hence from the above identities that

$$\begin{aligned}\langle u_+^3\rangle &= -3\langle u_+ u_-^2\rangle,\\ \langle u_-^3\rangle &= -3\langle u_+^2 u_-\rangle.\end{aligned}$$

Under the "*cross-independence*" relation between the velocities u_+ and u_-, the above equations are written respectively as

$$\begin{aligned}\langle u_+^3\rangle &= -3\langle u_+\rangle\langle u_-^2\rangle,\\ \langle u_-^3\rangle &= -3\langle u_+^2\rangle\langle u_-\rangle.\end{aligned}$$

Further, in view of the conditions $\langle u_+\rangle = \langle u_-\rangle = 0$ due to the zero-mean conditions $\langle u_1\rangle = \langle u_2\rangle = 0$ for *homogeneous turbulence*, the above equations are reduced to that

$$\langle u_+^3\rangle = \langle u_-^3\rangle = 0. \tag{1}$$

Although the former half of Eq.(1) is admissible, the latter half is gainst the generally established result that the *skewness* of the *velocity-difference distribution* is negative in the *inertial subrange* of *homogeneous turbulence*:

$$S_{-} = \langle u_{-}^{3} \rangle / \langle u_{-}^{2} \rangle^{3/2} < 0. \tag{2}$$

Thus the "*cross-independence*" relation is not held in the *inertial subrange* or the region of *non-zero skewness*.

This argument is logically correct but it is not concerned with the "*cross-independence hypothesis*" which assumes the "*cross-independence*" relation only for either *large* distance $r = |\mathbf{x}_2 - \mathbf{x}_1| \rightarrow \infty$ or *small* distance $r \rightarrow 0$ (see Tatsumi (2001) and Tatsumi et al.(2004)). Then, it may still be asked if the "*cross-independence closure hypothesis*" actually has its region of validity either for large or small distance r out of the inertial subrange. This question will be answered positively in the next subsection.

B. Cross-Independence as the Closure Hypothesis

First, it should be noted that, in the Lundgren-Monin equations for the multi-point velocity distributions, the $(n+1)$-point velocity distributions in the equation for the n-point velocity distribution always appear in the *degenerate* forms associated with the vanishing distances $r_m = |\mathbf{x}_{n+1} - \mathbf{x}_m| \rightarrow 0$ $(m = 1, .., n)$.

It may easily be observed from the equation for the one-point velocity distribution $f(\mathbf{v}_1, \mathbf{x}_1, t)$,

$$\left[\frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{x}_1}\right] f(\mathbf{v}_1, \mathbf{x}_1, t) = \frac{1}{4\pi} \frac{\partial}{\partial \mathbf{x}_1} \cdot \frac{\partial}{\partial \mathbf{v}_1} \int\int |\mathbf{x}_2 - \mathbf{x}_1|^{-1} \left(\mathbf{v}_2 \cdot \frac{\partial}{\partial \mathbf{x}_2}\right)^2 f^{(2)}(\mathbf{v}_1, \mathbf{v}_2; \mathbf{x}_1, \mathbf{x}_2; t) d\mathbf{x}_2 d\mathbf{v}_2$$
$$- \nu \lim_{\mathbf{x}_2 \rightarrow \mathbf{x}_1} \left|\frac{\partial}{\partial \mathbf{x}_2}\right|^2 \frac{\partial}{\partial \mathbf{v}_1} \cdot \int \mathbf{v}^2 f^{(2)}(\mathbf{v}_1, \mathbf{v}_2; \mathbf{x}_1, \mathbf{x}_2; t) d\mathbf{v}_2, \tag{3}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ denote the probability variables corresponding to the velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ respectively. Actually, Eq(3) includes the the higher-order $f^{(2)}$ terms only as the integral with the dominant contribution from the region of $r = |\mathbf{x}_2 - \mathbf{x}_1| \rightarrow 0$ and also the limit of $r \rightarrow 0$.

The same is the case for the three-point velocity distributions $f^{(3)}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3; t)$ in the equation for the two-point velocity distribution $f^{(2)}(\mathbf{v}_1, \mathbf{v}_2; \mathbf{x}_1, \mathbf{x}_2; t)$, where the hypothesis is applied to the distances $r_1 = |\mathbf{x}_3 - \mathbf{x}_1|$ and $r_2 = |\mathbf{x}_3 - \mathbf{x}_2|$ which are eventually taken to be zero. Thus, the *small distances* as the ranges of validity for the hypothesis in the equation of the n-point velocity distribution are always supplied as the distances r_m $(m = 1, .., n)$.

Hence, the premise for the "*cross-independence closure hypothesis*" is perfectly satisfied and the failure of the "*cross-independence*" relation in the inertial subrange has nothing to do with the validity of the hypothesis itself. In fact, it can be shown that the longitudinal velocity-difference distribution permits the negative skewness in the inertial subrange. In.this sense, the hypothesis provides us with an ideal closure which can be said a "*natural closure*".

III. HOMOGENEOUS ISOTROPIC TURBULENCE

A. Inertial Similarity

Thanks to the cross-independence closure hypothesis, we obtain a closed set of equations governing *homogeneous isotropic turbulence*, which are the equations for the *one-point velocity distribution* $f(\mathbf{v}, \mathbf{x}, t)$ and the *two-point velocity distribution* $f^{(2)}(\mathbf{v}_1, \mathbf{v}_2; \mathbf{x}_1, \mathbf{x}_2; t)$, being equivalently expressed in terms of the *velocity-sum* and *-difference distributions* $g_{\pm}(\mathbf{v}_{\pm}, \mathbf{x}_1, \mathbf{x}_2; t)$. The eminent feature of these equations is the *inertial similarity*, or that they have only one parameter representing the *mean energy dissipation rate* of turbulence,

$$\varepsilon(t) = \nu \sum_{i,j=1}^{3} \left\langle \left(\frac{\partial u_i(\mathbf{x}, t)}{\partial x_j}\right)^2 \right\rangle. \tag{4}$$

The presence of the *finite energy-dissipation rate* $\varepsilon > 0$ in the *inviscid limit* $\nu \to 0$, or equivalently the *inviscid catastrophe* $\varepsilon/\nu \to \infty$ in this limit, is nothing but the *inertial similarity law* assumed by Kolmogorov (1941) in his theory of locally isotropic turbulence. Thus, it may be interesting to note that the *inertial similarity* of turbulence has not been assumed but proved here in the framework of the present theory.

B. Energy Decay Law

The first outcome of the *inertial similarity* of turbulence is its *energy decay law.* Since homogeneous isotropic turbulence has no genuin length-scale, its governing equations has only length-scale as the distance $r = |\mathbf{x}_2 - \mathbf{x}_1|$ between the two points .Then, taking the representative length- and time-scales as $L = r$ and $T = t$ respectively, we can express the dimensional relation of the energy dissipation rate ε defined by Eq.(4) as

$$[\varepsilon(t)] = [\nu] \frac{[L/T]^2}{[L]^2} = [\nu][T]^{-2}.$$

Hence, we obtain the decay law of the *energy dissipation rate* for homogeneous turbulence as

$$\varepsilon(t) = \varepsilon_0 t^{-2} \tag{5}$$

where $\varepsilon_0 = \varepsilon(t_0) t_0^2$ for a certain time $t = t_0$ represents a constant of $O(\nu)$.

On account of the equation for the *mean kinetic energy* $E(t) = \frac{1}{2}\left\langle |\mathbf{u}(\mathbf{x},t)|^2 \right\rangle$,

$$\frac{d}{dt} E(t) = -\varepsilon(t), \tag{6}$$

we can derive from Eq.(5) the following *decay law* of the *mean kinetic energy* for homogeneous turbulence:

$$E(t) = E_0 t^{-1}, \tag{7}$$

with $E_0 = E(t_0) t_0 = \varepsilon_0$.

The existing results of wind-tunnel experiments and numerical simulations, $E(t) \propto t^{-p}$, $p = 1 \sim 1.2$, are fairly close but not in complete agreement with Eq.(7). In view of the absolute accuracy of the theory, such small discrepancies seem to be attributed to any finiteness of the experimental and numerical environments. On the other hand, the existing theoretical results,

$$E(t) \propto t^{-p}, \quad p = \frac{2(a+1)}{a+3}, \quad -1 < a \leq 4,$$

seem to require serious consideration. Even in this case, it can be said that the arguments have mostly been made in terms of finite integral moments but this is not the case for the inviscid limit.

C. Time-Similarity of Velocity Distributions

The energy decay law (7) requires all velocity distributions in the present theory the time-evolutions compatible with such decay law. Thus, the equations for these velocity distributions must be solved under the time-similarity of the solutions in accordance with the energy decay. It seems fortunate that we can avoid the problems of ill-posedness of the initial-value problem and non-uniqueness of the solutions for the sake of such requirement.

D. Inertial Normal Velocity Distributions

If we apply the *cross-independence closure hypothesis* to the Lundgren-Monin system of equations, we can derive the closed equation of any order from the equation of the same order in the system. For instance, we obtain from Eq.(3) for the one-point velocity distribution the following very simple equation for homogeneous isotropic turbulence:

$$\left[\frac{\partial}{\partial t}+\alpha(t)\left|\frac{\partial}{\partial \mathbf{v}}\right|^2\right] f(\mathbf{v},t)=0, \tag{8}$$

$$\alpha(t)=\varepsilon(t)/3=\frac{2}{3}\nu\lim_{|\mathbf{r}|\to 0}\left|\frac{\partial}{\partial \mathbf{r}}\right|^2\int |\mathbf{v}_-|^2 g_-(\mathbf{v}_-,\mathbf{r},t)\,d\mathbf{v}_-. \tag{9}$$

where $\mathbf{r}=\mathbf{x}_2-\mathbf{x}_1$ and $g_-(\mathbf{v}_-,\mathbf{r},t)=\langle\delta(\mathbf{u}_-(\mathbf{x},\mathbf{x}+\mathbf{r},t)-\mathbf{v}_-)\rangle$ denotes the velocity-difference distribution. As seen from Eq.(9), the parameter $\alpha(t)$ is shown to be identical to the energy dissipation rare $\varepsilon(t)$ defined by Eq.(4).

In accordance to Eq.(5), the parameter $\alpha(t)$ must change in time according to the law,

$$\alpha(t)=\alpha_0 t^{-2}, \quad \alpha_0=\varepsilon_0/3. \tag{10}$$

Under the time-similarity (10) , the *velocity distributions* in *homogeneous isotropic turbulence* are obtained as follows.

One-Point Velocity Distribution (N1):

$$f(\mathbf{v},t)=f_0(\mathbf{v},t)=\left(\frac{t}{4\pi\alpha_0}\right)^{3/2}\exp\left[-\frac{|\mathbf{v}|^2 t}{4\alpha_0}\right], \tag{11}$$

Velocity-Sum and -Difference Distributions (N2):

$$g_\pm(\mathbf{v}_\pm,\mathbf{r},t)=g_0(\mathbf{v}_\pm,t)=\left(\frac{t}{2\pi\alpha_0}\right)^{3/2}\exp\left[-\frac{|\mathbf{v}_\pm|^2 t}{2\alpha_0}\right]. \tag{12}$$

It may clearly be observed that the velocity distributions are all *inertial normal distributions* and the velocity-sum and -difference distributions g_0, having a half of the variance of the one-point velocity distribution f_0, are equivalent to the convolution of the two independent one-point velocity distributions f_0's at arbitrary two points.

Although the velocity-sum and -difference distributions $g_\pm$ are expressed by g_0 of Eq.(12) for all distances $r>0$, they must satisfy the boundary conditions, $g_+\to f$ and $g_-\to\delta(\mathbf{r})$ (delta distribution) in the limit of $r\to 0$. These changes take place abruptly at $r=0$ in the ordinary coordinates, but the changes must be expressed as continuous ones in the *local range* under the local coordinates $\mathbf{x}^*=\mathbf{x}/\eta$ where $\eta=(\nu^3/\varepsilon)^{1/4}$ denotes Kolmogorov's length. Such behaviour of the distributions $g_\pm$ will be described in the next subsection.

E. Self-Energy and Self-Energy-Dissipation

In the local range, the velocity-sum distribution $g_+(\mathbf{v}_+^*,\mathbf{r}^*,t^*)$ and the lateral component $g_\perp(v_\perp^*,r^*,t^*)$ of the velocity-difference distribution $g_-(\mathbf{v}_-^*,\mathbf{r}^*,t^*)$ are expressed as the following *local inertial-normal dustributions.*

Local Velocity-Sum Distribution (N3):

$$g_+(\mathbf{v}_+^*,\mathbf{r}^*,t^*)=g_{+0}(\mathbf{v}_+^*,\mathbf{r}^*,t^*)=\left(\frac{t^*}{4\pi\alpha_{+0}^*(r^*)}\right)^{3/2}\exp\left[-\frac{|\mathbf{v}_+^*|^2 t^*}{4\alpha_{+0}^*(r^*)}\right], \tag{13}$$

$$\begin{aligned}\alpha_+^*(r^*,t^*) &= \varepsilon_+^*(r^*,t^*)/3=\frac{2}{3}\lim_{|\mathbf{r}^{*\prime}|\to 0}\left|\frac{\partial}{\partial \mathbf{r}^{*\prime}}\right|^2\int |\mathbf{v}_{+-}^{*\prime}|^2 g_{+-}(\mathbf{v}_{+-}^{*\prime};\mathbf{r}^*,\mathbf{r}^{*\prime};t^*)\,d\mathbf{v}_{+-}^{*\prime}\\ &= \alpha_{+0}^*(r^*)\,t^{*-2},\end{aligned} \tag{14}$$

where $\mathbf{v}^{*\prime}_{+-} = \left(\mathbf{v}^*_3 - \mathbf{v}^*_+\right)/2$ and $\mathbf{r}^{*\prime} = \mathbf{x}^*_3 - \mathbf{x}^*_1$. The parameter $\varepsilon^*_+ = 3\alpha^*_+$ represents the *self-energy-disspation rate* for the *velocity-sum* $\mathbf{u}^*_+ = (\mathbf{u}^*_1 + \mathbf{u}^*_2)/2$, which is related with the *self-energy* $E^*_+(r^*, t^*) = \left\langle |\mathbf{u}^*_+|^2 \right\rangle /2$ for the *velocity-sum* $\mathbf{u}^*_+$ by the energy equation,

$$\frac{d}{dt} E^*_+(r^*, t^*) = -\varepsilon^*_+(r^*, t^*), \quad E^*_+(r^*, t^*) = E^*_{+0}(r^*)\, t^{*-1}. \tag{15}$$

The parameter $\alpha^*_+(r^*, t^*)$ is a continuous function of r^* and it has to change from $\alpha^*_+ = \alpha$ at $r^* = 0$ to $\alpha^*_+ \to \alpha/2$ for $r^* \to \infty$ in accordance with the boundary conditions for the distribution $g_+\left(\mathbf{v}^*_+, \mathbf{r}^*, t^*\right)$ at $r^* = 0$ and $r^* \to \infty$.

Local Lateral Velocity-Difference Distribution (N4):

$$g_\perp(v^*_\perp, r^*, t^*) = g_{-0}(v^*_\perp, r^*, t^*) = \left(\frac{t^*}{4\pi\alpha^*_{-0}(r^*)}\right)^{1/2} \exp\left[-\frac{v^{*2}_\perp t^*}{4\alpha^*_{-0}(r^*)}\right], \tag{16}$$

$$\begin{aligned} \alpha^*_-(r^*, t^*) &= \varepsilon^*_-(r^*, t^*)/3 = \frac{2}{3}\lim_{|\mathbf{r}^{*\prime}|\to 0}\left|\frac{\partial}{\partial \mathbf{r}^{*\prime}}\right|^2 \int |\mathbf{v}^{*\prime}_{-\,-}|^2 g_{-\,-}\left(\mathbf{v}^{*\prime}_{-\,-}; \mathbf{r}^*, \mathbf{r}^{*\prime}; t^*\right) d\mathbf{v}^{*\prime}_{-\,-} \\ &= \alpha^*_{-0}(r^*)\, t^{*-2}, \end{aligned} \tag{17}$$

where $\mathbf{v}^{*\prime}_{-\,-} = \left(\mathbf{v}^*_3 + \mathbf{v}^*_-\right)/2$ and $\mathbf{r}^{*\prime} = \mathbf{x}^*_3 - \mathbf{x}^*_1$. The parameter $\varepsilon^*_- = 3\alpha^*_-$ represents the *self-energy-disspation rate* for the *velocity-difference* $\mathbf{u}^*_- = (\mathbf{u}^*_2 - \mathbf{u}^*_1)/2$, which is related with the *self-energy* $E^*_-(r^*, t^*) = \left\langle |\mathbf{u}^*_-|^2 \right\rangle /2$ for the velocity-difference $\mathbf{u}^*_-$ by the energy equation,

$$\frac{d}{dt} E^*_-(r^*, t^*) = -\varepsilon^*_-(r^*, t^*), \quad .E^*_-(r^*, t^*) = E^*_{-0}(r^*)\, t^{*-1}. \tag{18}$$

The parameter $\alpha^*_-(r^*, t^*)$ is a continuous function of r^* and it has to change from $\alpha^*_- = 0$ at $r^* = 0$ to $\alpha^*_- \to \alpha/2$ for $r^* \to \infty$ in accordance with the boundary conditions of the distribution $g_\perp(v^*_\perp, r^*, t^*)$ at $r^* = 0$ and $r^* \to \infty$.

Self-Energy and Self-Energy-Dissipation:

It has been established that the one and two-point velocity distributions in the local range are expressed in terms of the *inertial normal distributions* in the local variables except for the *longitudinal velocity-difference distribution* which will be discussed separately. The important feature of this result is that the local changes in the energy dissipation rate ε have been expressed as the *self-energy-dissipation rates* $\varepsilon^*_\pm(r^*, t^*)$ for the respective distributions $g_\pm\left(\mathbf{v}^*_\pm, \mathbf{r}^*, t^*\right)$. This result provides us with a clear image for the so-called *intermittency* effect of the energy dissipation in the local range.

The parameters $\alpha^*_\pm(r^*, t^*)$ have not yet been determined as functions of r^* since they are included as a whole in the respective distributions, but the following relation due to their definition should be noted:

$$\alpha^*_+(r^*, t^*) + \alpha^*_-(r^*, t^*) = \alpha^*(t^*) = \alpha(t)/\varepsilon(t_0). \tag{19}$$

If we obtain the parameter $\alpha^*_-(r^*, t^*)$ as a function of r^* from the analysis of the distribution g_-, we can derive the parameter $\alpha^*_+(r^*, t^*)$ from Eq.(19) and thus the complete knowledge of turbulence in the local range.

F. Longitudinal Velocity-Difference Distribution

An important exception from the prevailing *inertial normality* of the velocity distributions in *homogeneous isotropic turbulence* is provided by the *longitudinal component* $g_\parallel\left(v^*_\parallel, r^*, t^*\right)$ of the *velocity-difference distribution* $g_-\left(\mathbf{v}^*_-, \mathbf{r}^*, t^*\right)$ in the local range, which is expected to be an asymmetric non-normal distribution associated with the negative skewness in the inertial subrange. The reason for this is that although all other distributions depend upon the self-energy-dissipation rates $\alpha^*_\pm = \varepsilon^*_\pm/3$ as external parameters. the equation for the distribution $g_{\|}$ includes the self-energy-dissipation rate $\alpha_-^* = \varepsilon_-^*/3$ as an internal parameter related with the solution $g_{\|}$.

In order to decrease mathematical difficulty due to this intricacy. an isotropic simplification has been employed in the previous paper (Tatsumi & Yoshimura (2007)) with unsatiafactory outcome. The problem is now reattacked using no mathematical simplification and it is hoped to be able to attain complete results. Fortunately, this stalemate in homogeneous turbulence has no effect for the extension of the present approach to inhomogeneous turbulence.

IV. INHOMOGENEOUS TURBULENCE

A. Governing Equations for Inhomogeneous Turbulence

The theory of turbulence based on the cross-independence closure hypothesis has been extended to inhomogeneous turbulence. and the closed set of equations composed of the equations for the mean velocity $\overline{\mathbf{u}}(\mathbf{x},t) = \langle \mathbf{u}(\mathbf{x},t)\rangle$, the one- and two-point velocity distributions have been obtained (see Tatsumi (2008)). Prior to the systematic approach to *inhomogeneous turbulence*. let us consider the composition of complex turbulent flows by the superposition of simpler ones.

B. Inertial Bi-Normal Velocity Distribution

If we consider inhomogeneous turbulence without the mean velocity $\overline{\mathbf{u}}$, the closed equation for the one-point velocity distribution $f(\mathbf{v},\mathbf{x},t)$ is written as follows:

$$\left[\left\{\frac{\partial}{\partial t} - \nu\left|\frac{\partial}{\partial \mathbf{x}}\right|^2 + \alpha(\mathbf{x},t)\left|\frac{\partial}{\partial \mathbf{v}}\right|^2\right\} + \frac{\partial}{\partial \mathbf{x}}\cdot\left\{\mathbf{v} - \frac{\partial}{\partial \mathbf{v}}\left(\beta(\mathbf{v},\mathbf{x},t) + \gamma(\mathbf{v},\mathbf{x},t)\right)\right\}\right] f(\mathbf{v},\mathbf{x},t) = 0, \tag{20}$$

$$\alpha(\mathbf{x},t) = \varepsilon(\mathbf{x},t)/3 = \frac{2}{3}\nu \lim_{|r|\to 0}\left|\frac{\partial}{\partial \mathbf{r}}\right|^2 \int |\mathbf{v}_-|^2 g_-(\mathbf{v}_-,\mathbf{x},\mathbf{r},t)\, d\mathbf{v}_-, \tag{21}$$

where the equations for the parameters β and γ are omitted.

If we compare the equations (20) and (21) with the corresponding equations (8) and (9) for homogeneous isotropic turbulence, we find that whereas Eq.(20) for the distribution f is much more complex than Eq.(8). Eq.(21) for the parameter α is substantially the same as Eq.(9). This seems to justify Kolmogorov's hypothesis that the structure of small eddies responsible to the energy dissipation of turbulence is not affected by the variety of large eddies.

If we consider a spherical velocity distribution around the origin $\mathbf{x} = 0$, the distribution is governed by the isotropic part of Eq.(20),

$$\left[\frac{\partial}{\partial t} - \nu\left|\frac{\partial}{\partial \mathbf{x}}\right|^2 + \alpha(\mathbf{x},t)\left|\frac{\partial}{\partial \mathbf{v}}\right|^2\right] f(\mathbf{v},\mathbf{x},t) = 0, \tag{22}$$

In the case of homogeneous turbulence, the solution for the distribution f was given by f_0 of Eq.(11) and parameter $\alpha(\mathbf{x},t)$ by $\alpha(t)$ of Eqs.(9) and (10). For inhomogeneous turbulence, the distribution f may be expressed by f_0 where $\alpha(t)$ is replaced by $\alpha(\mathbf{x},t)$, and the $\mathbf{x}$-dependence of the distribution may be expressed through that of $\alpha(\mathbf{x},t)$. Then, in order that Eq.(22) is valid. the following two equations for $\alpha(\mathbf{x},t) = \alpha_0(\mathbf{x},t)\,t^{-2}$ must be satisfied.

$$\left[\left.\frac{\partial}{\partial t}\right|_{\alpha_0} + \alpha(\mathbf{x},t)\left|\frac{\partial}{\partial \mathbf{v}}\right|^2\right] f(\mathbf{v},\mathbf{x},t) = 0, \tag{23}$$

$$\left[\frac{\partial}{\partial t} - \nu\left|\frac{\partial}{\partial \mathbf{x}}\right|^2\right] \alpha_0(\mathbf{x},t) = 0. \tag{24}$$

The self-similar solutions of Eqs.(23) and (24) are given by

$$f(\mathbf{v},\mathbf{x},t) = f_0(\mathbf{v},\mathbf{x},t) = \left(\frac{t}{4\pi\alpha_0(\mathbf{x},t)}\right)^{3/2} \exp\left[-\frac{|\mathbf{v}|^2 t}{4\alpha_0(\mathbf{x},t)}\right], \tag{25}$$

$$\alpha_0(\mathbf{x},t) = \alpha_0 (4\pi\nu t)^{-3/2} \exp\left[-\frac{|\mathbf{x}|^2}{4\nu t}\right], \tag{26}$$

where $\alpha_0 = \int \alpha_0(\mathbf{x},t)\, d\mathbf{x}$ denotes the total magnitude of the enrgy dissipation $\alpha_0(\mathbf{x},t)$.

The distribution represented by Eqs.(25) and (26) expresses the normal distribution with respect to the variables $\mathbf{v}$ and $\mathbf{x}$, but its way of evolution in time is just opposit for $\mathbf{v}$ and $\mathbf{x}$. The velocity distribution f changes in time like the one-point velocity distribution (11) of homogeneous turbulence, starting from the uniform distribution with infinitesimal probability density at the initial time $t = 0$, contracting monotonically in time keeping its normal form, and eventually tending to the δ distribution for $t \to \infty$. On the other hand, the spatial distribution of the energy-dissipation $\alpha_0(\mathbf{x},t)$ changes in time, starting from the δ distribution at $t = 0$, expanding monotonically in time keeping its normal form, and eventually tending to the uniform distribution of infinitesimal probability density.

The spatial distribution of the energy $E(\mathbf{x},t)$ of this turbulent region changes in time in the same way as $\alpha_0(\mathbf{x},t)$ as

$$E(\mathbf{x},t) = \frac{1}{2}\left\langle |\mathbf{u}(\mathbf{x},t)|^2 \right\rangle = \varepsilon_0(\mathbf{x},t)\, t^{-1} = \varepsilon_0 t^{-1} (4\pi\nu t)^{-3/2} \exp\left[-\frac{|\mathbf{x}|^2}{4\nu t}\right], \tag{27}$$

and hence the total energy changes as

$$E(t) = \int E(\mathbf{x},t)\, d\mathbf{x} = \varepsilon_0 t^{-1} = E_0 t^{-1}, \tag{28}$$

in accordance with the energy decay law (7) of homogeneous isotropic turbulence.

Thus the *one-body inertial binormal velocity distribution* defined by Eqs.(25) and (26) actually represents a sherical turbulent region expanding in time from a single point to the whole space, maintaining the same velocity distribution as the *one-point inertial normal velocity distribution* in homogeneous isotropic turbulence. In this sense, the both distributions can be said to be the *canonical distributions* in respective turbulences.

C. Turbulent Wakes

According to the linearity of Eqs.(23) and (24), the *spherical turbulent region* represented by Eqs.(25) and (26) can be used for producing various inhomogeneous turbulent flows by making linear combination with other solutions.

If we place at a time $t = 0$ a spherical turbulent region at the point $\mathbf{x} = (x, y, z) = 0$ under a uniform stream of the velocity $\mathbf{U} = (U, 0, 0)$, the energy distribution of the turbulent region at the time t is derived from Eq.(28) as

$$\begin{aligned} E(\mathbf{x},t) &= E(x - Ut, y, z; t) \\ &= \varepsilon_0 t^{-1} (4\pi\nu t)^{-3/2} \exp\left[-\left\{(x - Ut)^2 + y^2 + z^2\right\} / (4\nu t)\right]. \end{aligned} \tag{29}$$

Now, let us consider turbulent wake formed by a spherical body in a uniform stream of large Reynolds numbers. Then, the turbulent region just behind the body may be replaced by the spherical turbulent region defined by Eq.(27) as a model for the inviscid limit. According to this model, the total kinetic energy $E(t) = \varepsilon_0 t^{-1}$ must be in balance with the work DU done by the body against the drag force D, so that $\varepsilon_0 t^{-1} = DU$.

In this case, the spatial distribution of the total turbulent region is obtained from Eq.(29) as

$$\begin{aligned} E(\mathbf{x}) &= \int_0^\infty E(\mathbf{x},t)\,dt = \int_0^\infty E(x-Ut,y,z;t)\,dt \\ &= DU\int_0^\infty (4\pi\nu t)^{-3/2}\exp\left[-\left\{(x-Ut)^2+y^2+z^2\right\}/(4\nu t)\right]dt. \end{aligned} \tag{30}$$

In the inviscid limit $\nu \to 0$, the integral of Eq.(30) is dominated by the contribution from the region near to $t = x/U$, the equation is written as

$$E(\mathbf{x}) = D(4\pi\nu x/U)^{-1/2}\exp\left[-\left(y^2+z^2\right)/(4\nu x/U)\right]. \tag{31}$$

Eq.(31) clearly shows that the *turbulent wake* due to a *spherical body* has the energy distribution in the form of the *axisymmetric parabolic surface* around the axis crossing the solid body along the direction of the uniform stream.

If we gather such axisymmetric turbulent wakes along the z axis uniformly, we obtain the *two-dimensional turbulent wake* due to a cylindrical body placed laterally to the uniform flow, whose energy distribution in the inviscid limit being obtained from Eq.(31) as

$$\begin{aligned} E(x,y) &= \int_{-\infty}^\infty E(\mathbf{x})\,dz \\ &= DU(4\pi\nu x/U)^{-1/2}\exp\left[-y^2/(4\nu x/U)\right]. \end{aligned} \tag{32}$$

Eq.(32) clearly shows that the *turbulent wake* due to *a cylindrical body* placed laterally to the uniform stream has the energy distribution in the form of the *two-dimensional parabolic surface* with the symmetric plane pallalel to both the direction of the uniform stream and the center line of the cylindrical body.

V. SCOPE TO SHEAR FLOW TURBULENCE

The discussions and results developed in the previous sections for *homogeneous* and *inhomogeneous turbulence* provide us with very bright scope for the extension of the present approach based on the *cross-independence closure hypothesis* to more general turbulent shear flows of practical importance. The eminent features of this theory, namely logical clarity, mathematical consistency and physical reality seem to guarantee rich prospects for further developments to various turbulent phenomena in plasma, quantum and relativistic fluids and those of environmental, global and celestial scales.

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